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External direct products of BG-algebras

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Abstract

The external direct product (EDP) of BG-algebras, a generalisation of the direct product (DP) in the sense of Widianto, Gemawati, and Kartini [10], is a term we present in this study. In addition, we present the idea of the weak direct product (WDP) of BG-algebras. Finally, in light of the EDP of BG-algebras, we investigate BG-homomorphism (BGH) and anti-BG-homomorphism (ABGH) theorems.

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Keywords: *BG-algebras, External direct product, Weak direct product, BG-homomorphism, Anti-BG-homomorphism.*

1. Introduction and Preliminaries

BCK-algebras and BCI-algebras are two types of abstract algebra that Imai and Iséki presented [1, 2]. Neggers and Kim created a brand-new algebraic structure in 2002 [3]. They adapted several characteristics from BCI and BCK-algebras to create B-algebra. In addition, Kim and Kim [4] created a new concept known as a BG-algebra, which is a generalisation of a B-algebra.

In the beginning, a DP [5] was defined in the group and was given certain characteristics. Senapati, Bhowmik, and Pal [6] presented and examined the DPs and T-products of T-fuzzy subalgebras of BG-algebras in 2015 and discovered

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several significant characteristics. Lingcong and Endam [7] discussed the idea of the DP of B-algebras in 2016 and discovered several related properties, one of which is a DP of two B-algebras that is also a B-algebra. Endam and Teves [8] developed and obtained related features for the DP of BF-algebras, 0-commutative BF-algebras, and BF-homomorphism. The idea of the finite DP of BRK-algebras was first suggested in 2018 by Abebe [9], who also demonstrated that the finite DP of BRK-algebras is a BRK-algebra. The DP of BG-algebras, 0-commutative BG-algebras, and BGH, as well as associated BG-algebra features, were defined by Widianto, Gemawati, and Kartini [10] in 2019. The DP of BP-algebras, which is identical to B-algebras, was defined by Setiani et al. [5] in 2020. The DP of GK-algebras was defined by Kavitha and Gowri [11] in 2021.

The EDP of BG-algebras, a generalisation of the DP in the sense of Widianto, Gemawati, and Kartini [10], is a term we present in this study. In addition, we present the idea of the WDP of BG-algebras. We conclude by discussing BGH and ABGH theorems in light of the EDP BG-algebras.

We will first review the definition and examples of BG-algebras and other definitions relevant to this study.

Definition 1.1: [4] A set $\mathcal{G} \neq \emptyset$ with a binary operation $\star$ and a constant $\wp$ that satisfies the following axioms is known as a *BG-algebra*.

$$(\forall g_1 \in \mathcal{G})(g_1 \star g_1 = \wp) \tag{BG-1}$$

$$(\forall g_1 \in \mathcal{G})(g_1 \star \wp = g_1) \tag{BG-2}$$

$$(\forall g_1, g_2 \in \mathcal{G})((g_1 \star g_2) \star (\wp \star g_2) = g_1) \tag{BG-3}$$

Hereafter, let $\mathcal{G} = (\mathcal{G}; \star, \wp)$ represent a BG-algebra.

Definition 1.2: [4] A subset $\mathcal{S} \neq \emptyset$ of $\mathcal{G}$ is called a *BG-subalgebra* of $\mathcal{G}$, we denote it with the symbol $\mathcal{S} \leq \mathcal{G}$, if

$$(\forall h_1, h_2 \in \mathcal{S})(h_1 \star h_2 \in \mathcal{S}).$$

Definition 1.3: [4] A subset $\mathcal{S} \neq \emptyset$ of $\mathcal{G}$ is said to be *normal* of $\mathcal{G}$, we denote it with the symbol $\mathcal{S} \trianglelefteq \mathcal{G}$, if

$$(\forall g_1, g_2, h_1, h_2 \in \mathcal{G})(g_1 \star g_2, h_1 \star h_2 \in \mathcal{S} \Rightarrow (g_1 \star h_1) \star (g_2 \star h_2) \in \mathcal{S}).$$

Theorem 1.4: [4] *Every normal subset of $\mathcal{G}$ is a BG-subalgebra.*

Definition 1.5: [12] A subset $\mathcal{S} \neq \emptyset$ of $\mathcal{G}$ is called a *BG-ideal* of $\mathcal{G}$, we denote it with the symbol $\mathcal{S} \sqsubseteq \mathcal{G}$, if

$$\wp \in \mathcal{S} \tag{BGI-1}$$

$$(\forall g_1 \in \mathcal{G}, \forall g_2 \in \mathcal{S})(g_1 \star g_2 \in \mathcal{S} \Rightarrow g_1 \in \mathcal{S}) \tag{BGI-2}$$

$$(\forall g_1 \in \mathcal{S}, \forall g_2 \in \mathcal{G})(g_1 \star g_2 \in \mathcal{S}) \tag{BGI-3}$$

By Theorem 1.4, we have the following.

Remark 1.6: Every normal subset of $\mathcal{G}$ satisfies the condition (BGI-1).

Definition 1.7: [4] $\mathcal{G}$ is said to be *commutative* if

$$(\forall g_1, g_2 \in \mathcal{G})(g_1 \star (\wp \star \wp_2) = g_2 \star (\wp \star g_1)).$$

The concept of BGHs was introduced by Kim and Kim in [4]. Let $\mathcal{G} = (\mathcal{G};\star_{\mathcal{G}}, \wp_{\mathcal{G}})$ and $\mathcal{G}' = (\mathcal{G}';\star_{\mathcal{G}'}, \wp_{\mathcal{G}'})$ be BG-algebras. A mapping $\varphi: \mathcal{G} \to \mathcal{G}'$ is called a *BG-homomorphism* (BGH) if $(\forall g_1, g_2 \in \mathcal{G})(\varphi(g_1 \star_{\mathcal{G}} g_2) = \varphi(g_1) \star_{\mathcal{G}'} \varphi(g_2))$, and an *anti-BG-homomorphism* (ABGH) if $(\forall g_1, g_2 \in \mathcal{G})(\varphi(g_1 \star_{\mathcal{G}} g_2) = \varphi(g_2) \star_{\mathcal{G}'} \varphi(g_1))$. The *kernel* of φ, denoted by $\ker \varphi$, is defined to be the set $\{g_1 \in \mathcal{G} | \varphi(g_1) = \wp_{\mathcal{G}'}\}$. A (A)BGH φ is called a (anti-) BG-monomorphism ((A) BGM), (anti-) BG-epimorphism ((A) BGE), or (anti-) BG-isomorphism ((A)BGI) if φ is one-to-one, onto, or a one-to-one correspondence, respectively.

Theorem 1.8: [4] *If $\varphi: \mathcal{G} \to \mathcal{G}'$ is a BGH, then $ker\ \varphi \trianglelefteq \mathcal{G}$.*

2. Main Results

Widianto, Gemawati, and Kartini [10] defined and studied the DP of BG-algebras in terms of the finite family of BG-algebras as follows:

Definition 2.1: [10] Let an algebra $\mathcal{G}_\iota = (\mathcal{G}_\iota;\star_{\mathcal{G}_\iota})$ $\forall \iota = 1,2,\ldots,k$. Define the *DP* of algebras $\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_k$ to be the algebra $(\prod_{\iota=1}^{k} \mathcal{G}_\iota;\otimes)$, where $\prod_{\iota=1}^{k} \mathcal{G}_\iota = \mathcal{G}_1 \times \mathcal{G}_2 \times \ldots \times \mathcal{G}_k = \{(g_1, g_2, \ldots, g_k) | g_\iota \in \mathcal{G}_\iota \ \ \forall \iota = 1,2,\ldots,k\}$ and whose operation $\otimes$ is given by $(\forall (g_1, g_2, \ldots, g_k), (h'_1, h'_2, \ldots, h'_k) \in \prod_{\iota=1}^{k} \mathcal{G}_\iota)((g_1, g_2, \ldots, g_k) \otimes (h'_1, h'_2, \ldots, h'_k) = (g_1 \star_{\mathcal{G}_1} h'_1, g_2 \star_{\mathcal{G}_2} h'_2, \ldots, g_k \star_{\mathcal{G}_k} h'_k))$.

Theorem 2.2: [10] $\forall \iota = 1,2,\ldots,k$, *$(\mathcal{G}_\iota;\star_{\mathcal{G}_\iota}, \wp_\iota)$ is a BG-algebra if and only if $(\prod_{\iota=1}^{k} \mathcal{G}_\iota;\otimes, (\wp_1, \wp_2, \ldots, \wp_k))$ is a BG-algebra.*

We now present a few DP properties and extend the concept to a BG-algebra family that is infinite.

Definition 2.3: [13] Define the *EDP* of sets $\mathcal{G}_\iota \neq \emptyset$ $\forall \iota \in \mathfrak{I}$ to be the set $\prod_{\iota\in\mathfrak{I}} \mathcal{G}_\iota$, where $\prod_{\iota\in\mathfrak{I}} \mathcal{G}_\iota = \{\mathbf{f}: \mathfrak{I} \to \bigcup_{\iota\in\mathfrak{I}} \mathcal{G}_\iota | \mathbf{f}(\iota) \in \mathcal{G}_\iota \ \ \forall \iota \in \mathfrak{I}\}$. For ease of use, we create an element of $\prod_{\iota\in\mathfrak{I}} \mathcal{G}_\iota$ with a mapping $(g_\iota)_{\iota\in\mathfrak{I}}: I \to \bigcup_{\iota\in\mathfrak{I}} \mathcal{G}_\iota$, where $i \mapsto g_\iota \in \mathcal{G}_\iota$ $\forall \iota \in \mathfrak{I}$.

Definition 2.4: [14] Let an algebra $\mathcal{G}_\iota = (\mathcal{G}_\iota;\star_\iota)$ $\forall \iota \in \mathfrak{I}$. Following is a description of the binary operation $\otimes$ on the EDP $\prod_{\iota\in\mathfrak{I}} \mathcal{G}_\iota = (\prod_{\iota\in\mathfrak{I}} \mathcal{G}_\iota;\otimes)$:

$$(\forall (g_\iota)_{\iota\in\mathfrak{I}}, (h_\iota)_{\iota\in\mathfrak{I}} \in \textstyle\prod_{\iota\in\mathfrak{I}} \mathcal{G}_\iota)((g_\iota)_{\iota\in\mathfrak{I}} \otimes (h_\iota)_{\iota\in\mathfrak{I}} = (g_\iota \star_\iota h_\iota)_{\iota\in\mathfrak{I}}). \tag{2.1}$$

Let a BG-algebra $\mathcal{G}_\iota = (\mathcal{G}_\iota;\star_\iota, \wp_\iota)$ $\forall \iota \in \mathfrak{I}$. For $\iota \in \mathfrak{I}$, let $g_\iota \in \mathcal{G}_\iota$. Following is a description of the mapping $\mathbf{f}_{g_\iota}: I \to \bigcup_{\iota\in\mathfrak{I}} \mathcal{G}_\iota$:

$$(\forall \kappa \in \mathfrak{I})\left(\mathbf{f}_{g_\iota}(\kappa) = \begin{cases} g_\iota & \text{if } \kappa = \iota \\ \wp_\kappa & \text{otherwise} \end{cases}\right). \tag{2.2}$$

Hence, $\mathbf{f}_{g_\iota} \in \prod_{\iota\in\mathfrak{I}} \mathcal{G}_\iota$.

Lemma 2.5: Let a BG-algebra $\mathcal{G}_\iota = (\mathcal{G}_\iota; \star_\iota, \wp_\iota)$ $\forall \iota \in \mathfrak{I}$. For $\iota \in \mathfrak{I}$, let $g_\iota, h_\iota \in \mathcal{G}_\iota$. Then $\mathbf{f}_{g_\iota} \otimes \mathbf{f}_{h_\iota} = \mathbf{f}_{g_\iota \star_\iota h_\iota}$.

Proof: Now,

$$(\forall \kappa \in)\left((\mathbf{f}_{g_\iota} \otimes \mathbf{f}_{h_\iota})(\kappa) = \begin{cases} g_\iota \star_\iota h_\iota & \text{if } \kappa = \iota \\ \wp_\kappa \star_\kappa \wp_\kappa & \text{otherwise} \end{cases}\right).$$

By using (BG-1),

$$(\forall \kappa \in)\left((\mathbf{f}_{g_\iota} \otimes \mathbf{f}_{h_\iota})(\kappa) = \begin{cases} g_\iota \star_\iota h_\iota & \text{if } \kappa = \iota \\ \wp_\kappa & \text{otherwise} \end{cases}\right).$$

By using (2.2), $\mathbf{f}_{g_\iota} \otimes \mathbf{f}_{h_\iota} = \mathbf{f}_{g_\iota \star_\iota h_\iota}$.

Theorem 2.6: $\forall \iota \in \mathfrak{I}$, $\mathcal{G}_\iota = (\mathcal{G}_\iota; \star_\iota, \wp_\iota)$ is a BG-algebra if and only if $\prod_{\iota\in} \mathcal{G}_\iota = (\prod_{\iota\in} \mathcal{G}_\iota; \otimes, (\wp_\iota)_{\iota\in})$ is a BG-algebra.

Proof: Suppose $\mathcal{G}_\iota = (\mathcal{G}_\iota; \star_\iota, \wp_\iota)$ is a BG-algebra $\forall \iota \in \mathfrak{I}$.

(BG-1) Let $(g_\iota)_{\iota\in\mathfrak{I}} \in \prod_{\iota\in\mathfrak{I}} \mathcal{G}_\iota$. Since $\mathcal{G}_\iota$ satisfies (BG-1), $g_\iota \star_\iota g_\iota = \wp_\iota$ $\forall \iota \in \mathfrak{I}$. Thus $(g_\iota)_{\iota\in\mathfrak{I}} \otimes (g_\iota)_{\iota\in\mathfrak{I}} = (g_\iota \star_\iota g_\iota)_{\iota\in\mathfrak{I}} = (\wp_\iota)_{\iota\in\mathfrak{I}}$.

(BG-2) Let $(g_\iota)_{\iota\in\mathfrak{I}} \in \prod_{\iota\in\mathfrak{I}} \mathcal{G}_\iota$. Since $\mathcal{G}_\iota$ satisfies (BG-2), $g_\iota \star_\iota \wp_\iota = g_\iota$ $\forall \iota \in \mathfrak{I}$. Thus $(g_\iota)_{\iota\in\mathfrak{I}} \otimes (\wp_\iota)_{\iota\in\mathfrak{I}} = (g_\iota \star_\iota \wp_\iota)_{\iota\in\mathfrak{I}} = (g_\iota)_{\iota\in\mathfrak{I}}$.

(BG-3) Let $(g_\iota)_{\iota\in\mathfrak{I}}, (h_\iota)_{\iota\in\mathfrak{I}} \in \prod_{\iota\in\mathfrak{I}} \mathcal{G}_\iota$. Since $\mathcal{G}_\iota$ satisfies (BG-3), $(g_\iota \star_\iota h_\iota) \star_\iota (\wp_\iota \star_\iota h_\iota) = g_\iota$ $\forall \iota \in \mathfrak{I}$. Thus $((g_\iota)_{\iota\in\mathfrak{I}} \otimes (h_\iota)_{\iota\in\mathfrak{I}}) \otimes ((\wp_\iota)_{\iota\in\mathfrak{I}} \otimes (h_\iota)_{\iota\in\mathfrak{I}}) = (g_\iota \star_\iota h_\iota)_{\iota\in\mathfrak{I}} \otimes (\wp_\iota \star_\iota h_\iota)_{\iota\in\mathfrak{I}} = ((g_\iota \star_\iota h_\iota) \star_\iota (\wp_\iota \star_\iota h_\iota))_{\iota\in\mathfrak{I}} = (g_\iota)_{\iota\in\mathfrak{I}}$. Hence, $\prod_{\iota\in\mathfrak{I}} \mathcal{G}_\iota = (\prod_{\iota\in\mathfrak{I}} \mathcal{G}_\iota; \otimes, (\wp_\iota)_{\iota\in\mathfrak{I}})$ is a BG-algebra.

Conversely, suppose $\prod_{\iota\in\mathfrak{I}} \mathcal{G}_\iota = (\prod_{\iota\in\mathfrak{I}} \mathcal{G}_\iota; \otimes, (\wp_\iota)_{\iota\in\mathfrak{I}})$ is a BG-algebra. Let $\iota \in \mathfrak{I}$.

(BG1) Let $g_\iota \in \mathcal{G}_\iota$. Then $\mathbf{f}_{g_\iota} \in \prod_{\iota\in} \mathcal{G}_\iota$. Since $\prod_{\iota\in\mathfrak{I}} \mathcal{G}_\iota$ satisfies (BG-1), $\mathbf{f}_{g_\iota} \otimes \mathbf{f}_{g_\iota} = (\wp_\iota)_{\iota\in\mathfrak{I}}$. Now,

$$(\forall \kappa \in)\left((\mathbf{f}_{g_\iota} \otimes \mathbf{f}_{g_\iota})(\kappa) = \begin{cases} g_\iota \star_\iota g_\iota & \text{if } \kappa = \iota \\ \wp_\kappa \star_\kappa \wp_\kappa & \text{otherwise} \end{cases}\right),$$

this suggests that $g_\iota \star_\iota g_\iota = \wp_\iota$.

(BG2) Let $g_\iota \in \mathcal{G}_\iota$. Then $\mathbf{f}_{g_\iota} \in \prod_{\iota\in\mathfrak{I}} \mathcal{G}_\iota$. Since $\prod_{\iota\in\mathfrak{I}} \mathcal{G}_\iota$ satisfies (BG-2), $\mathbf{f}_{g_\iota} \otimes (\wp_\iota)_{\iota\in\mathfrak{I}} = \mathbf{f}_{g_\iota}$. Now,

$$(\forall \kappa \in)\left((\mathbf{f}_{g_\iota} \otimes (\wp_\iota)_{\iota\in\mathfrak{I}})(\kappa) = \begin{cases} g_\iota \star_\iota \wp_\iota & \text{if } \kappa = \iota \\ \wp_\kappa \star_\kappa \wp_\kappa & \text{otherwise} \end{cases}\right),$$

this suggests that $g_\iota \star_\iota \wp_\iota = g_\iota$.

(BG-3) Let $g_\iota, h_\iota \in \mathcal{G}_\iota$. Then $\mathbf{f}_{g_\iota}, \mathbf{f}_{h_\iota} \in \prod_{\iota\in\mathfrak{I}} \mathcal{G}_\iota$. Since $\prod_{\iota\in\mathfrak{I}} \mathcal{G}_\iota$ satisfies (BG-3), $(\mathbf{f}_{g_\iota} \otimes \mathbf{f}_{h_\iota}) \otimes ((\wp_\iota)_{\iota\in\mathfrak{I}} \otimes \mathbf{f}_{h_\iota}) = \mathbf{f}_{g_\iota}$. Now,

$$(\forall\kappa \in \mathfrak{I})\left(((\mathbf{f}_{g_\iota} \otimes \mathbf{f}_{h_\iota}) \otimes ((\wp_\iota)_{\iota\in} \otimes \mathbf{f}_{h_\iota}))(\kappa) = \begin{cases}(g_\iota \star_\iota h_\iota) \star_\iota (\wp_\iota \star_\iota h_\iota) & \text{if } \kappa = \iota \\ (\wp_\kappa \star_\kappa \wp_\kappa) \star_\kappa (\wp_\kappa \star_\kappa \wp_\kappa) & \text{otherwise}\end{cases}\right),$$

this suggests that $(g_\iota \star_\iota h_\iota) \star_\iota (\wp_\iota \star_\iota h_\iota) = g_\iota$.

Hence, $\mathcal{G}_\iota = (\mathcal{G}_\iota;\star_\iota, \wp_\iota)$ is a BG-algebra $\forall\iota \in \mathfrak{I}$.

We call the BG-algebra $\prod_{\iota\in\mathfrak{I}} \mathcal{G}_\iota = (\prod_{\iota\in\mathfrak{I}} \mathcal{G}_\iota; \otimes, (\wp_\iota)_{\iota\in\mathfrak{I}})$ in Theorem 2.6 the EDP BG-algebra induced by a BG-algebra $\mathcal{G}_\iota = (\mathcal{G}_\iota;\star_\iota, \wp_\iota)$ $\forall\iota \in \mathfrak{I}$.

Theorem 2.7: $\forall\iota \in \mathfrak{I}$, *a BG-algebra* $\mathcal{G}_\iota = (\mathcal{G}_\iota;\star_\iota, \wp_\iota)$ *is commutative if and only if* $\prod_{\iota\in\mathfrak{I}} \mathcal{G}_\iota = (\prod_{\iota\in\mathfrak{I}} \mathcal{G}_\iota;\otimes, (\wp_\iota)_{\iota\in\mathfrak{I}})$ *is commutative.*

Proof: By Theorem 2.6, we have that $\mathcal{G}_\iota = (\mathcal{G}_\iota;\star_\iota, \wp_\iota)$ is a BG-algebra $\forall\iota \in \mathfrak{I}$ if and only if $\prod_{\iota\in\mathfrak{I}} \mathcal{G}_\iota = (\prod_{\iota\in\mathfrak{I}} \mathcal{G}_\iota;\otimes, (\wp_\iota)_{\iota\in\mathfrak{I}})$ is a BG-algebra.

Consider $\mathcal{G}_\iota$ to be commutative $\forall\iota \in \mathfrak{I}$. Let $(g_\iota)_{\iota\in\mathfrak{I}}, (h_\iota)_{\iota\in\mathfrak{I}} \in \prod_{\iota\in\mathfrak{I}} \mathcal{G}_\iota$. Since $\mathcal{G}_\iota$ is commutative $\forall\iota \in \mathfrak{I}$, $g_\iota \star_\iota (\wp_\iota \star_\iota h_\iota) = h_\iota \star_\iota (\wp_\iota \star_\iota g_\iota)$ $\forall\iota \in \mathfrak{I}$. Thus $(g_\iota)_{\iota\in\mathfrak{I}} \otimes ((\wp_\iota)_{\iota\in\mathfrak{I}} \otimes (h_\iota)_{\iota\in\mathfrak{I}}) = (g_\iota)_{\iota\in\mathfrak{I}} \otimes (\wp_\iota \star_\iota h_\iota)_{\iota\in\mathfrak{I}} = (g_\iota \star_\iota (\wp_\iota \star_\iota h_\iota))_{\iota\in\mathfrak{I}} = (h_\iota \star_\iota (\wp_\iota \star_\iota g_\iota))_{\iota\in\mathfrak{I}} = (h_\iota)_{\iota\in\mathfrak{I}} \otimes (\wp_\iota \star_\iota g_\iota)_{\iota\in\mathfrak{I}} = (h_\iota)_{\iota\in\mathfrak{I}} \otimes ((\wp_\iota)_{\iota\in\mathfrak{I}} \otimes (g_\iota)_{\iota\in\mathfrak{I}})$. Hence, $\prod_{\iota\in\mathfrak{I}} \mathcal{G}_\iota$ is commutative.

Conversely, consider $\prod_{\iota\in\mathfrak{I}} \mathcal{G}_\iota$ to be commutative. Let $\iota \in \mathfrak{I}$. Let $g_\iota, h_\iota \in \mathcal{G}_\iota$. Then $\mathbf{f}_{g_\iota}, \mathbf{f}_{h_\iota} \in \prod_{\iota\in} \mathcal{G}_\iota$. Since $\prod_{\iota\in\iota} \mathcal{G}$ is commutative, $\mathbf{f}_{g_\iota} \otimes ((\wp_\iota)_{\iota\in\mathfrak{I}} \otimes \mathbf{f}_{h_\iota})) = \mathbf{f}_{h_\iota} \otimes ((\wp_\iota)_{\iota\in} \otimes \mathbf{f}_{g_\iota}))$. Now,

$$(\forall\kappa \in)\left((\mathbf{f}_{g_\iota} \otimes ((\wp_\iota)_{\iota\in} \otimes \mathbf{f}_{h_\iota}))(\kappa) = \begin{cases} g_\iota \star_\iota (\wp_\iota \star_\iota h_\iota) & \text{if } \kappa = \iota \\ \wp_\kappa \star_\kappa (\wp_\kappa \star_\kappa \wp_\kappa) & \text{otherwise}\end{cases}\right)$$

and

$$(\forall\kappa \in)\left((\mathbf{f}_{h_\iota} \otimes ((\wp_\iota)_{\iota\in} \otimes \mathbf{f}_{g_\iota}))(\kappa) = \begin{cases} h_\iota \star_\iota (\wp_\iota \star_\iota g_\iota) & \text{if } \kappa = \iota \\ \wp_\kappa \star_\kappa (\wp_\kappa \star_\kappa \wp_\kappa) & \text{otherwise}\end{cases}\right),$$

this suggests that $g_\iota \star_\iota (\wp_\iota \star_\iota h_\iota) = h_\iota \star_\iota (\wp_\iota \star_\iota g_\iota)$. Hence, $\mathcal{G}_\iota$ is commutative $\forall\iota \in \mathfrak{I}$.

Following this, we introduce the notion of the WDP of BG-algebras:

Definition 2.8: Let a BG-algebra $\mathcal{G}_\iota = (\mathcal{G}_\iota;\star_\iota, \wp_\iota)$ $\forall\iota \in \mathfrak{I}$. Define the *WDP* of a BG-algebra $\mathcal{G}_\iota\, \forall\iota \in \mathfrak{I}$ to be the algebra $\prod^{\mathrm{w}}_{\iota\in\mathfrak{I}} \mathcal{G}_\iota = \left(\prod^{\mathrm{w}}_{\iota\in\mathfrak{I}} \mathcal{G}_\iota;\otimes\right)$, where $\prod^{\mathrm{w}}_{\iota\in\mathfrak{I}} \mathcal{G}_\iota = \{(g_\iota)_{\iota\in\mathfrak{I}} \in \prod_{\iota\in\mathfrak{I}} \mathcal{G}_\iota | g_\iota \neq \wp_\iota$, where the number of such i is finite$\}$. Then $(\wp_\iota)_{\iota\in\mathfrak{I}} \in \prod^{\mathrm{w}}_{\iota\in\mathfrak{I}} \mathcal{G}_\iota \subseteq \prod_{\iota\in\mathfrak{I}} \mathcal{G}_\iota$.

Theorem 2.9: *Let a BG-algebra* $\mathcal{G}_\iota = (\mathcal{G}_\iota;\star_\iota, \wp_\iota)$ $\forall\iota \in \mathfrak{I}$. *Then* $\prod^{\mathrm{w}}_{\iota\in\mathfrak{I}} \mathcal{G}_\iota \leq \prod_{\iota\in\mathfrak{I}} \mathcal{G}_\iota = (\prod_{\iota\in\mathfrak{I}} \mathcal{G}_\iota;\otimes, (\wp_\iota)_{\iota\in\mathfrak{I}})$.

Proof: Now, $(\wp_\iota)_{\iota\in\mathfrak{I}} \in \prod^{\mathrm{w}}_{\iota\in\mathfrak{I}} \mathcal{G}_\iota \neq \emptyset$. Let $(g_\iota)_{\iota\in\mathfrak{I}}, (h_\iota)_{\iota\in\mathfrak{I}} \in \prod^{\mathrm{w}}_{\iota\in\mathfrak{I}} \mathcal{G}_\iota$, where $\mathfrak{I}_1 = \{i \in \mid g_\iota \neq \wp_\iota\}$ and $\mathfrak{I}_2 = \{i \in \mid h_\iota \neq \wp_\iota\}$ are finite. So, $|\mathfrak{I}_1 \cup \mathfrak{I}_2|$ is finite. Hence,

$$(\forall\kappa \in \mathfrak{I})\left(((g_\iota)_{\iota\in\mathfrak{I}} \otimes (h_\iota)_{\iota\in\mathfrak{I}})(\kappa) = \begin{cases} g_\kappa \star_\kappa \wp_\kappa & \text{if } \kappa \in \mathfrak{I}_1 - \mathfrak{I}_2 \\ g_\kappa \star_\kappa h_\kappa & \text{if } \kappa \in \mathfrak{I}_1 \cap \mathfrak{I}_2 \\ \wp_\kappa \star_\kappa h_\kappa & \text{if } \kappa \in \mathfrak{I}_2 - \mathfrak{I}_1 \\ \wp_\kappa \star_\kappa \wp_\kappa & \text{otherwise} \end{cases}\right).$$

By using (BG-1) and (BG-2),

$$(\forall\kappa \in \mathfrak{I})\left(((g_\iota)_{\iota\in\mathfrak{I}} \otimes (h_\iota)_{\iota\in\mathfrak{I}})(\kappa) = \begin{cases} g_\kappa & \text{if } \kappa \in \mathfrak{I}_1 - \mathfrak{I}_2 \\ g_\kappa \star_\kappa h_\kappa & \text{if } \kappa \in \mathfrak{I}_1 \cap \mathfrak{I}_2 \\ \wp_\kappa \star_\kappa h_\kappa & \text{if } \kappa \in \mathfrak{I}_2 - \mathfrak{I}_1 \\ \wp_\kappa & \text{otherwise} \end{cases}\right).$$

This means that the number of such $((g_\iota)_{\iota\in\mathfrak{I}} \otimes (h_\iota)_{\iota\in\mathfrak{I}})(\kappa) \neq \wp_\kappa$ is not greater than $|\mathfrak{I}_1 \cup \mathfrak{I}_2|$, i.e. it is finite. Thus $(g_\iota)_{\iota\in\mathfrak{I}} \otimes (h_\iota)_{\iota\in\mathfrak{I}} \in \prod^{\mathrm{w}}_{\iota\in\mathfrak{I}} \mathcal{G}_\iota$. Hence, $\prod^{\mathrm{w}}_{\iota\in\mathfrak{I}} \mathcal{G}_\iota \leq \prod_{\iota\in\mathfrak{I}} \mathcal{G}_\iota$.

Theorem 2.10: *Let a BG-algebra* $\mathcal{G}_\iota = (\mathcal{G}_\iota; \star_\iota, \wp_\iota)$ *and* $\mathcal{H}_\iota \subseteq \mathcal{G}_\iota$ $\forall\iota \in \mathfrak{I}$. *Then* $\mathcal{H}_\iota \leq \mathcal{G}_\iota$ $\forall\iota \in \mathfrak{I}$ *if and only if* $\prod_{\iota\in\mathfrak{I}} \mathcal{H}_\iota \leq \prod_{\iota\in} \mathcal{G}_\iota = (\prod_{\iota\in\mathfrak{I}} \mathcal{G}_\iota; \otimes, (\wp_\iota)_{\iota\in\mathfrak{I}})$.

Proof: Suppose $\mathcal{H}_\iota \leq \mathcal{G}_\iota$ $\forall\iota \in \mathfrak{I}$. Then $\wp_\iota \in \mathcal{H}_\iota$ $\forall\iota \in \mathfrak{I}$, so $(\wp_\iota)_{\iota\in\mathfrak{I}} \in \prod_{\iota\in\mathfrak{I}} \mathcal{H}_\iota \neq \emptyset$. Let $(g_\iota)_{\iota\in\mathfrak{I}}, (h_\iota)_{\iota\in\mathfrak{I}} \in \prod_{\iota\in\mathfrak{I}} \mathcal{H}_\iota$. Then $g_\iota, h_\iota \in \mathcal{H}_\iota$ $\forall\iota \in \mathfrak{I}$. Thus $g_\iota \star_\iota h_\iota \in \mathcal{H}_\iota$ $\forall\iota \in \mathfrak{I}$, so $(g_\iota)_{\iota\in\mathfrak{I}} \otimes (h_\iota)_{\iota\in\mathfrak{I}} = (g_\iota \star_\iota h_\iota)_{\iota\in\mathfrak{I}} \in \prod_{\iota\in\mathfrak{I}} \mathcal{H}_\iota$.

Hence, $\prod_{\iota\in} \mathcal{H}_\iota \leq \prod_{\iota\in} \mathcal{G}_\iota$.

Conversely, suppose $\prod_{\iota\in\mathfrak{I}} \mathcal{H}_\iota \leq \prod_{\iota\in\mathfrak{I}} \mathcal{G}_\iota$. Then $(\wp_\iota)_{\iota\in\mathfrak{I}} \in \prod_{\iota\in\mathfrak{I}} \mathcal{H}_\iota$, so $\wp_\iota \in \mathcal{H}_\iota \neq \emptyset$ $\forall\iota \in \mathfrak{I}$. Let $\iota \in \mathfrak{I}$ and let $g_\iota, h_\iota \in \mathcal{H}_\iota$. Then $\mathbf{f}_{g_\iota}, \mathbf{f}_{h_\iota} \in \prod_{\iota\in\mathfrak{I}} \mathcal{H}_\iota$. Since $\prod_{\iota\in\mathfrak{I}} \mathcal{H}_\iota \leq \prod_{\iota\in\mathfrak{I}} \mathcal{G}_\iota$, $\mathbf{f}_{g_\iota} \otimes \mathbf{f}_{h_\iota} \in \prod_{\iota\in\mathfrak{I}} \mathcal{H}_\iota$. Now,

$$(\forall\kappa \in \mathfrak{I})\left((\mathbf{f}_{g_\iota} \otimes \mathbf{f}_{h_\iota})(\kappa) = \begin{pmatrix} g_\iota \star_\iota h_\iota & \text{if } \kappa = \iota \\ \wp_\kappa \star_\kappa \wp_\kappa & \text{otherwise} \end{pmatrix}\right),$$

this suggests that $g_\iota \star_\iota h_\iota \in \mathcal{H}_\iota$.

Hence, $\mathcal{H}_\iota \leq \mathcal{G}_\iota$ $\forall\iota \in \mathfrak{I}$.

Theorem 2.11: *Let a BG-algebra* $\mathcal{G}_\iota = (\mathcal{G}_\iota; \star_\iota, \wp_\iota)$ *and* $\mathcal{H}_\iota \subseteq \mathcal{G}_\iota$ $\forall\iota \in \mathfrak{I}$. *Then* $\mathcal{H}_\iota (\mathcal{G}_\iota$ $\forall\iota \in \mathfrak{I}$ *if and only if* $\prod_{\iota\in\mathfrak{I}} \mathcal{H}_\iota \trianglelefteq \prod_{\iota\in\mathfrak{I}} \mathcal{G}_\iota = (\prod_{\iota\in\mathfrak{I}} \mathcal{G}_\iota; \otimes, (\wp_\iota)_{\iota\in\mathfrak{I}})$.

Proof: Consider $\mathcal{H}_\iota \trianglelefteq \mathcal{G}_\iota$ $\forall\iota \in \mathfrak{I}$. Then $\mathcal{H}_\iota \neq \emptyset$ $\forall\iota \in \mathfrak{I}$, so $\prod_{\iota\in\mathfrak{I}} \mathcal{H}_\iota \neq \emptyset$. Let $(g_\iota)_{\iota\in\mathfrak{I}}, (g'_\iota)_{\iota\in\mathfrak{I}}, (h_\iota)_{\iota\in\mathfrak{I}}, (h'_\iota)_{\iota\in\mathfrak{I}} \in \prod_{\iota\in\mathfrak{I}} \mathcal{G}_\iota$ be such that $(g_\iota)_{\iota\in\mathfrak{I}} \otimes (h_\iota)_{\iota\in}\mathfrak{I}$, $(g'_\iota)_{\iota\in\mathfrak{I}} \otimes (h'_\iota)_{\iota\in\mathfrak{I}} \in \prod_{\iota\in\mathfrak{I}} \mathcal{H}_\iota$. Then $g_\iota \star_\iota h_\iota, g'_\iota \star_\iota h'_\iota \in \mathcal{H}_\iota$ $\forall\iota \in \mathfrak{I}$. Since $\mathcal{H}_\iota \trianglelefteq \mathcal{G}_\iota$, $(g_\iota \star_\iota g'_\iota) \star_\iota (h_\iota \star_\iota h'_\iota) \in \mathcal{H}_\iota$ $\forall\iota \in \mathfrak{I}$. Thus $((g_\iota)_{\iota\in} \otimes (g'_\iota)_{\iota\in\mathfrak{I}}) \otimes ((h_\iota)_{\iota\in\mathfrak{I}} \otimes (h'_\iota)_{\iota\in\mathfrak{I}}) = (g_\iota \star_\iota g'_\iota)_{\iota\in\mathfrak{I}} \otimes (h_\iota \star_\iota h'_\iota)_{\iota\in\mathfrak{I}} = ((g_\iota \star_\iota g'_\iota) \star_\iota (h_\iota \star_\iota h'_\iota))_{\iota\in\mathfrak{I}} \in \prod_{\iota\in\mathfrak{I}} \mathcal{H}_\iota$. Hence, $\prod_{\iota\in} \mathcal{H}_\iota \trianglelefteq \prod_{\iota\in} \mathcal{G}_\iota$.

Conversely, consider $\prod_{\iota\in\mathfrak{J}}\mathcal{H}_\iota \trianglelefteq \prod_{\iota\in\mathfrak{J}}\mathcal{G}_\iota$. Then $\prod_{\iota\in\mathfrak{J}}\mathcal{H}_\iota \neq \emptyset$, so $\mathcal{H}_\iota \neq \emptyset$ $\forall\iota\in\mathfrak{J}$. By Remark 1.6, $(\wp_\iota)_{\iota\in\mathfrak{J}} \in \prod_{\iota\in\mathfrak{J}}\mathcal{H}_\iota$. Thus $\wp_\iota \in \mathcal{H}_\iota$ $\forall\iota\in\mathfrak{J}$. Let $\iota\in\mathfrak{J}$ and let $g_\iota, h_\iota, g'_\iota, h'_\iota \in \mathcal{G}_\iota$ be such that $g_\iota \star_\iota h_\iota, g'_\iota \star_\iota h'_\iota \in \mathcal{H}_\iota$. Then $\mathbf{f}_{g_\iota}, \mathbf{f}_{h_\iota}, \mathbf{f}_{g'_\iota}, \mathbf{f}_{h'_\iota} \in \prod_{\iota\in\mathfrak{J}}\mathcal{G}_\iota$. Now,

$$(\forall\kappa\in\mathfrak{J})\left((\mathbf{f}_{g_\iota}\otimes\mathbf{f}_{h_\iota})(\kappa) = \begin{pmatrix} g_\iota \star_\iota h_\iota & \text{if } \kappa=\iota \\ \wp_\kappa \star_\kappa \wp_\kappa & \text{otherwise}\end{pmatrix}\right).$$

By using (BG-1),

$$(\forall\kappa\in\mathfrak{J})\left((\mathbf{f}_{g_\iota}\otimes\mathbf{f}_{h_\iota})(\kappa) = \begin{pmatrix} g_\iota \star_\iota h_\iota & \text{if } \kappa=\iota \\ \wp_\kappa & \text{otherwise}\end{pmatrix}\right),$$

this suggests that $\mathbf{f}_{g_\iota}\otimes\mathbf{f}_{h_\iota} \in \prod_{\iota\in\mathfrak{J}}\mathcal{H}_\iota$. Similarly, $\mathbf{f}_{g'_\iota}\otimes\mathbf{f}_{h'_\iota} \in \prod_{\iota\in}\mathcal{H}_\iota$. Since $\prod_{\iota\in\mathfrak{J}}\mathcal{H}_\iota \trianglelefteq \prod_{\iota\in\mathfrak{J}}\mathcal{G}_\iota$, $(\mathbf{f}_{g_\iota}\otimes\mathbf{f}_{g'_\iota})\otimes(\mathbf{f}_{h_\iota}\otimes\mathbf{f}_{h'_\iota}) \in \prod_{\iota\in}\mathcal{H}_\iota$. Now,

$$(\forall\kappa\in\mathfrak{J})\Big(((\mathbf{f}_{g_\iota}\otimes\mathbf{f}_{g'_\iota})\otimes(\mathbf{f}_{h_\iota}\otimes\mathbf{f}_{h'_\iota}))(\kappa) = \begin{pmatrix} (g_\iota \star_\iota g'_\iota)\star_\iota(h_\iota \star_\iota h'_\iota) & \text{if } \kappa=\iota \\ (\wp_\kappa \star_\kappa \wp_\kappa)\star_\kappa(\wp_\kappa \star_\kappa \wp_\kappa) & \text{otherwise}\end{pmatrix}\Big),$$

this suggests that $(g_\iota \star_\iota g'_\iota)\star_\iota(h_\iota \star_\iota h'_\iota) \in \mathcal{H}_\iota$.

Hence, $\mathcal{H}_\iota \trianglelefteq \mathcal{G}_\iota$ $\forall\iota\in\mathfrak{J}$.

Theorem 2.12: *Let a BG-algebra* $\mathcal{G}_\iota = (\mathcal{G}_\iota;\star_\iota,\wp_i)$ *and* $\mathcal{H}_\iota \subseteq \mathcal{G}_\iota$ $\forall\iota\in\mathfrak{J}$. *Then* $\mathcal{H}_\iota \sqsubseteq \mathcal{G}_\iota$ $\forall\iota\in\mathfrak{J}$ *if and only if* $\prod_{\iota\in\mathfrak{J}}\mathcal{H}_\iota \sqsubseteq \prod_{\iota\in\mathfrak{J}}\mathcal{G}_\iota = (\prod_{\iota\in\mathfrak{J}}\mathcal{G}_\iota;\otimes,(\wp_\iota)_{\iota\in\mathfrak{J}})$.

Proof: Suppose $\mathcal{H}_\iota \sqsubseteq \mathcal{G}_\iota$ $\forall\iota\in\mathfrak{J}$.

(BGI-1) By using (BGI-1), $\wp_\iota \in \mathcal{H}_\iota$ $\forall\iota\in\mathfrak{J}$. Then $(\wp_\iota)_{\iota\in\mathfrak{J}} \in \prod_{\iota\in\mathfrak{J}}\mathcal{H}_\iota$.

(BGI-2) Let $(g_\iota)_{\iota\in\mathfrak{J}}, (h_\iota)_{\iota\in\mathfrak{J}} \in \prod_{\iota\in\mathfrak{J}}\mathcal{G}_\iota$ be such that $(g_\iota)_{\iota\in\mathfrak{J}} \otimes (h_\iota)_{\iota\in\mathfrak{J}} \in \prod_{\iota\in\mathfrak{J}}\mathcal{H}_\iota$ and $(h_\iota)_{\iota\in\mathfrak{J}} \in \prod_{\iota\in\mathfrak{J}}\mathcal{H}_\iota$. Then $(g_\iota \star_\iota h_\iota)_{\iota\in\mathfrak{J}} \in \prod_{\iota\in\mathfrak{J}}\mathcal{H}_\iota$. Thus $g_\iota \star_\iota h_\iota \in \mathcal{H}_\iota$ and $h_\iota \in \mathcal{H}_\iota$, it follows from (BGI-2) that $g_\iota \in \mathcal{H}_\iota$ $\forall\iota\in\mathfrak{J}$. Thus $(g_\iota)_{\iota\in\mathfrak{J}} \in \prod_{\iota\in\mathfrak{J}}\mathcal{H}_\iota$.

(BGI-3) Let $(h_\iota)_{\iota\in\mathfrak{J}} \in \prod_{\iota\in\mathfrak{J}}\mathcal{H}_\iota$ and $(g_\iota)_{\iota\in\mathfrak{J}} \in \prod_{\iota\in\mathfrak{J}}\mathcal{G}_\iota$. Then $h_\iota \in \mathcal{H}_\iota$ and $g_\iota \in \mathcal{G}_\iota$ $\forall\iota\in\mathfrak{J}$, it follows from (BGI-3) that $h_\iota \star_\iota g_\iota \in \mathcal{H}_\iota$ $\forall\iota\in\mathfrak{J}$. Thus $(h_\iota)_{\iota\in\mathfrak{J}} \otimes (g_\iota)_{\iota\in\mathfrak{J}} = (h_\iota \star_\iota g_\iota)_{\iota\in\mathfrak{J}} \in \prod_{\iota\in\mathfrak{J}}\mathcal{H}_\iota$.

Hence, $\prod_{\iota\in\mathfrak{J}}\mathcal{H}_\iota \sqsubseteq \prod_{\iota\in\mathfrak{J}}\mathcal{G}_\iota$.

Conversely, suppose $\prod_{\iota\in\mathfrak{J}}\mathcal{H}_\iota \sqsubseteq \prod_{\iota\in\mathfrak{J}}\mathcal{G}_\iota$. Then $\prod_{\iota\in\mathfrak{J}}\mathcal{H}_\iota \neq \emptyset$, so $\mathcal{H}_\iota \neq \emptyset$ $\forall\iota\in\mathfrak{J}$. Let $\iota\in\mathfrak{J}$.

(BGI-1) By using (BGI-1), $(\wp_\iota)_{\iota\in\mathfrak{J}} \in \prod_{\iota\in\mathfrak{J}}\mathcal{H}_\iota$. Then $\wp_\iota \in \mathcal{H}_\iota$.

(BGI-2) Let $g_\iota, h_\iota \in \mathcal{G}_\iota$ be such that $g_\iota \star_\iota h_\iota \in \mathcal{H}_\iota$ and $h_\iota \in \mathcal{H}_\iota$. By using (BGI-1), $\wp_\iota \in \mathcal{H}_\iota$ $\forall\iota\in\mathfrak{J}$. Then $\mathbf{f}_{g_\iota} \in \prod_{\iota\in\mathfrak{J}}\mathcal{G}_\iota$ and $\mathbf{f}_{g_\iota\star_\iota h_\iota}, \mathbf{f}_{h_\iota} \in \prod_{\iota\in\mathfrak{J}}\mathcal{H}_\iota$. By Lemma 2.5, $\mathbf{f}_{g_\iota}\otimes\mathbf{f}_{h_\iota} = \mathbf{f}_{g_\iota\star_\iota h_\iota} \in \prod_{\iota\in\mathfrak{J}}\mathcal{H}_\iota$. By using (BGI-2), $\mathbf{f}_{g_\iota} \in \prod_{\iota\in\mathfrak{J}}\mathcal{H}_\iota$. By using (2.2), $g_\iota \in \mathcal{H}_\iota$.

(BGI-3) Let $h_\iota \in \mathcal{H}_\iota$ and $g_\iota \in \mathcal{G}_\iota$. By using (BGI-1), $\wp_\iota \in \mathcal{H}_\iota$ $\forall\iota\in\mathfrak{J}$. Then $\mathbf{f}_{h_\iota} \in \prod_{\iota\in\mathfrak{J}}\mathcal{H}_\iota$ and $\mathbf{f}_{g_\iota} \in \prod_{\iota\in}\mathcal{G}_\iota$. By using (BGI-3) and Lemma 2.5, $\mathbf{f}_{h_\iota\star_\iota g_\iota} = \mathbf{f}_{h_\iota}\otimes\mathbf{f}_{g_\iota} \in \prod_{\iota\in\mathfrak{J}}\mathcal{H}_\iota$. By using (2.2), $h_\iota \star_\iota g_\iota \in \mathcal{H}_\iota$.

Hence, $\mathcal{H}_\iota \sqsubseteq \mathcal{G}_\iota$ $\forall\iota\in\mathfrak{J}$.

We examine BGH theorems in the context of EDP BG-algebras.

Definition 2.13: Let algebras $\mathcal{G}_\iota = (\mathcal{G}_\iota; \star_\iota)$ and $\mathcal{H}_\iota = (\mathcal{H}_\iota; \circ_\iota)$ and a mapping $\varphi_\iota : \mathcal{G}_\iota \to \mathcal{H}_\iota\ \forall \iota \in \mathfrak{I}$. Following is a description of the mapping $\Phi: \prod_{\iota\in\mathfrak{I}} \mathcal{G}_\iota \to \prod_{\iota\in\mathfrak{I}} \mathcal{H}_\iota$:

$$(\forall (g_\iota)_{\iota\in\mathfrak{I}} \in \textstyle\prod_{\iota\in\mathfrak{I}} \mathcal{G}_\iota)(\Phi(g_\iota)_{\iota\in\mathfrak{I}} = (\varphi_\iota(g_\iota))_{\iota\in\mathfrak{I}}). \tag{2.3}$$

Then $\Phi: \prod_{\iota\in\mathfrak{I}} \mathcal{G}_\iota \to \prod_{\iota\in\mathfrak{I}} \mathcal{H}_\iota$ is a mapping (see [13]).

Theorem 2.14: [13] *Let algebras $\mathcal{G}_\iota = (\mathcal{G}_\iota; \star_\iota)$ and $\mathcal{H}_\iota = (\mathcal{H}_\iota; \circ_\iota)$ and a mapping $\varphi_\iota : \mathcal{G}_\iota \to \mathcal{H}_\iota\ \forall \iota \in \mathfrak{I}$.*

(1) *φ_ι is one-to-one $\forall \iota \in \mathfrak{I}$ if and only if Φ is one-to-one,*
(2) *φ_ι is onto $\forall \iota \in \mathfrak{I}$ if and only if Φ is onto,*
(3) *φ_ι is a one-to-one correspondence $\forall \iota \in \mathfrak{I}$ if and only if Φ is a one-to-one correspondence.*

Theorem 2.15: *Let BG-algebras $\mathcal{G}_\iota = (\mathcal{G}_\iota; \star_\iota, \wp_\iota)$ and $\mathcal{H}_\iota = (\mathcal{H}_\iota; \circ_\iota, 1_\iota)$ and a mapping $\varphi_\iota : \mathcal{G}_\iota \to \mathcal{H}_\iota\ \forall \iota \in \mathfrak{I}$. Then*

(1) *φ_ι is a BGH $\forall \iota \in \mathfrak{I}$ if and only if Φ is a BGH,*
(2) *φ_ι is a BGM $\forall \iota \in \mathfrak{I}$ if and only if Φ is a BGM,*
(3) *φ_ι is a BGE $\forall \iota \in \mathfrak{I}$ if and only if Φ is a BGE,*
(4) *φ_ι is a BGI $\forall \iota \in \mathfrak{I}$ if and only if Φ is a BGI,*
(5) *$ker\Phi = \prod_{\iota\in\mathfrak{I}} ker\varphi_\iota$ and $\Phi(\prod_{\iota\in\mathfrak{I}} \mathcal{G}_\iota) = \prod_{\iota\in\mathfrak{I}} \varphi_\iota(\mathcal{G}_\iota)$.*

Proof: (1) Suppose φ_ι is a BGH $\forall \iota \in \mathfrak{I}$. Let $(g_\iota)_{\iota\in\mathfrak{I}}, (g'_\iota)_{\iota\in\mathfrak{I}} \in \prod_{\iota\in\mathfrak{I}} \mathcal{G}_\iota$. Then $\Phi((g_\iota)_{\iota\in\mathfrak{I}} \otimes (g'_\iota)_{\iota\in\mathfrak{I}}) = \Phi(g_\iota \star_\iota g'_\iota)_{\iota\in\mathfrak{I}} = (\varphi_\iota(g_\iota \star_\iota g'_\iota))_{\iota\in\mathfrak{I}} = (\varphi_\iota(g_\iota) \star_\iota \varphi_\iota (g'_\iota))_{\iota\in\mathfrak{I}} = (\varphi_\iota(g_\iota))_{\iota\in\mathfrak{I}} \otimes (\varphi_\iota(g'_\iota))_{\iota\in\mathfrak{I}} = \Phi(g_\iota)_{\iota\in\mathfrak{I}} \otimes \Phi(g'_\iota)_{\iota\in\mathfrak{I}}$. Hence, Φ is a BGH.

Conversely, suppose Φ is a BGH. Let $\iota \in \mathfrak{I}$. Let $g_\iota, h_\iota \in \mathcal{G}_\iota$. Then $\mathbf{f}_{g_\iota}, \mathbf{f}_{h_\iota} \in \prod_{\iota\in\mathfrak{I}} \mathcal{G}_\iota$. Since Φ is a BGH, $\Phi(\mathbf{f}_{g_\iota} \otimes \mathbf{f}_{h_\iota}) = \Phi(\mathbf{f}_{g_\iota}) \otimes \Phi(\mathbf{f}_{h_\iota})$. Since

$$(\forall \kappa \in \mathfrak{I})\left((\mathbf{f}_{g_\iota} \otimes \mathbf{f}_{h_\iota})(\kappa) = \begin{cases} g_\iota \star_\iota h_\iota & \text{if } \kappa = \iota \\ \wp_\kappa \star_\kappa \wp_\kappa & \text{otherwise} \end{cases}\right),$$

$$(\forall \kappa \in \mathfrak{I})\left(\Phi(\mathbf{f}_{g_\iota} \otimes \mathbf{f}_{h_\iota})(\kappa) = \begin{cases} \varphi_\iota(g_\iota \star_\iota h_\iota) & \text{if } \kappa = \iota \\ \varphi_\kappa(\wp_\kappa \star_\kappa \wp_\kappa) & \text{otherwise} \end{cases}\right). \tag{2.4}$$

Since

$$(\forall \kappa \in \mathfrak{I})\left(\Phi(\mathbf{f}_{g_\iota})(\kappa) = \begin{cases} \varphi_\iota(g_\iota) & \text{if } \kappa = \iota \\ \varphi_\kappa(\wp_\kappa) & \text{otherwise} \end{cases}\right)$$

and

$$(\forall \kappa \in \mathfrak{I})\left(\Phi(\mathbf{f}_{h_\iota})(\kappa) = \begin{cases} \varphi_\iota(h_\iota) & \text{if } \kappa = \iota \\ \varphi_\kappa(\wp_\kappa) & \text{otherwise} \end{cases}\right),$$

$$(\forall\kappa\in\mathfrak{I})\left((\Phi(\mathbf{f}_{g_\iota})\otimes\Phi(\mathbf{f}_{h_\iota}))(\kappa)=\begin{cases}\varphi_\iota(g_\iota)\circ_\iota\varphi_\iota(h_\iota) & \text{if } \kappa=\iota\\ \varphi_\kappa(\wp_\kappa)\circ_\kappa\varphi_\kappa(\wp_\kappa) & \text{otherwise}\end{cases}\right). \quad (2.5)$$

By using (2.4) and (2.5), $\varphi_\iota(g_\iota\star_\iota h_\iota)=\varphi_\iota(g_\iota)\circ_\iota\varphi_\iota(h_\iota)$. Hence, φ_ι is a BGH $\forall\iota\in\mathfrak{I}$.

(2) (resp., (3), (4)) is obtained as a result of (1) and Theorem 2.14 (1) (resp., (2), (3)).

(5) Let $(g_\iota)_{\iota\in\mathfrak{I}}\in\prod_{\iota\in\mathfrak{I}}\mathcal{G}_\iota$. Then $(g_\iota)_{\iota\in\mathfrak{I}}\in\ker\Phi\Leftrightarrow\Phi(g_\iota)_{\iota\in\mathfrak{I}}=(1_\iota)_{\iota\in\mathfrak{I}}\Leftrightarrow(\varphi_\iota(g_\iota))_{\iota\in\mathfrak{I}}=(1_\iota)_{\iota\in\mathfrak{I}}\Leftrightarrow\varphi_\iota(g_\iota)=1_\iota\ \forall\iota\in\Leftrightarrow g_\iota\in\ker\varphi_\iota\ \forall\iota\in\Leftrightarrow(g_\iota)_{\iota\in\mathfrak{I}}\in\prod_{\iota\in\mathfrak{I}}\ker\varphi_\iota$. Hence, $\ker\Phi=\prod_{\iota\in\mathfrak{I}}\ker\varphi_\iota$. Now, $(h_\iota)_{\iota\in\mathfrak{I}}\in\Phi(\prod_{\iota\in}\mathcal{G}_\iota)\Leftrightarrow\exists(g_\iota)_{\iota\in\mathfrak{I}}\in\prod_{\iota\in\mathfrak{I}}\mathcal{G}_\iota$ s.t. $(h_\iota)_{\iota\in\mathfrak{I}}=\Phi(g_\iota)_{\iota\in\mathfrak{I}}\Leftrightarrow\exists(g_\iota)_{\iota\in\mathfrak{I}}\in\prod_{\iota\in\mathfrak{I}}\mathcal{G}_\iota$ s.t. $(h_\iota)_{\iota\in\mathfrak{I}}=(\varphi_\iota(g_\iota))_{\iota\in\mathfrak{I}}\Leftrightarrow\exists g_\iota\in\mathcal{G}_\iota$ s.t. $h_\iota=\varphi_\iota(g_\iota)\in\Phi(\mathcal{G}_\iota)\ \forall\iota\in\Leftrightarrow(h_\iota)_{\iota\in\mathfrak{I}}\in\prod_{\iota\in\mathfrak{I}}\varphi_\iota(\mathcal{G}_\iota)$. Hence, $\Phi(\prod_{\iota\in\mathfrak{I}}\mathcal{G}_\iota)=\prod_{\iota\in\mathfrak{I}}\varphi_\iota(\mathcal{G}_\iota)$.

In addition, we discuss ABGH theorems in the context of the EDP BG-algebras.

Theorem 2.16: *Let BG-algebras $\mathcal{G}_\iota=(\mathcal{G}_\iota;\star_\iota,\wp_\iota)$ and $\mathcal{H}_\iota=(\mathcal{H}_\iota;\circ_\iota,1_\iota)$ and a mapping $\varphi_\iota:\mathcal{G}_\iota\to\mathcal{H}_\iota$ $\forall\iota\in\mathfrak{I}$. Then*

(1) *φ_ι is an ABGH $\forall\iota\in\mathfrak{I}$ if and only if Φ is an ABGH,*

(2) *φ_ι is an ABGM $\forall\iota\in\mathfrak{I}$ if and only if Φ is an ABGM,*

(3) *φ_ι is an ABGE $\forall\iota\in\mathfrak{I}$ if and only if Φ is an ABGE,*

(4) *φ_ι is an ABGI $\forall\iota\in\mathfrak{I}$ if and only if Φ is an ABGI.*

Proof: (1) Suppose φ_ι is an ABGH $\forall\iota\in\mathfrak{I}$. Let $(g_\iota)_{\iota\in\mathfrak{I}},(g'_\iota)_{\iota\in\mathfrak{I}}\in\prod_{\iota\in\mathfrak{I}}\mathcal{G}_\iota$. Then $\Phi((g_\iota)_{\iota\in\mathfrak{I}}\otimes(g'_\iota)_{\iota\in\mathfrak{I}})=\Phi(g_\iota\star_\iota g'_\iota)_{\iota\in\mathfrak{I}}=(\varphi_\iota(g_\iota\star_\iota g'_\iota))_{\iota\in\mathfrak{I}}=(\varphi_\iota(g'_\iota)\star_\iota\varphi_\iota(g_\iota))_{\iota\in\mathfrak{I}}=(\varphi_\iota(g'_\iota))_{\iota\in\mathfrak{I}}\otimes(\varphi_\iota(g_\iota))_{\iota\in\mathfrak{I}}=\Phi(g'_\iota)_{\iota\in\mathfrak{I}}\otimes\Phi(g_\iota)_{\iota\in\mathfrak{I}}$. Hence, Φ is an ABGH.

Conversely, suppose Φ is an ABGH. Let $\iota\in\mathfrak{I}$. Let $g_\iota,h_\iota\in\mathcal{G}_\iota$. Then $\mathbf{f}_{g_\iota},\mathbf{f}_{h_\iota}\in\prod_{\iota\in\mathfrak{I}}\mathcal{G}_\iota$. Since Φ is an ABGH, $\Phi(\mathbf{f}_{g_\iota}\otimes\mathbf{f}_{h_\iota})=\Phi(\mathbf{f}_{h_\iota})\otimes\Phi(\mathbf{f}_{g_\iota})$. Since

$$(\forall\kappa\in\mathfrak{I})\left((\mathbf{f}_{g_\iota}\otimes\mathbf{f}_{h_\iota})(\kappa)=\begin{pmatrix}g_\iota\star_\iota h_\iota & \text{if } \kappa=\iota\\ \wp_\kappa\star_\kappa\wp_\kappa & \text{otherwise}\end{pmatrix}\right),$$

$$(\forall\kappa\in\mathfrak{I})\left(\Phi(\mathbf{f}_{g_\iota}\otimes\mathbf{f}_{h_\iota})(\kappa)=\begin{pmatrix}\varphi_\iota(g_\iota\star_\iota h_\iota) & \text{if } \kappa=\iota\\ \varphi_\kappa(\wp_\kappa\star_\kappa\wp_\kappa) & \text{otherwise}\end{pmatrix}\right). \quad (2.6)$$

Since

$$(\forall\kappa\in\mathfrak{I})\left(\Phi(\mathbf{f}_{h_\iota})(\kappa)=\begin{pmatrix}\varphi_\iota(h_\iota) & \text{if } \kappa=\iota\\ \varphi_\kappa(\wp_\kappa) & \text{otherwise}\end{pmatrix}\right)$$

and

$$(\forall\kappa\in\mathfrak{I})\left(\Phi(\mathbf{f}_{g_\iota})(\kappa)=\begin{pmatrix}\varphi_\iota(g_\iota) & \text{if } \kappa=\iota\\ \varphi_\kappa(\wp_\kappa) & \text{otherwise}\end{pmatrix}\right),$$

$$(\forall\kappa\in\mathfrak{I})\left((\Phi(\mathbf{f}_{h_\iota})\otimes\Phi(\mathbf{f}_{g_\iota}))(\kappa)=\begin{pmatrix}\varphi_\iota(h_\iota)\circ_\iota\varphi_\iota(g_\iota) & \text{if } \kappa=\iota\\ \varphi_\kappa(\wp_\kappa)\circ_\kappa\varphi_\kappa(\wp_\kappa) & \text{otherwise}\end{pmatrix}\right). \quad (2.7)$$

By using (2.6) and (2.7), $\varphi_\iota(g_\iota \star_\iota h_\iota) = \varphi_\iota(h_\iota) \circ_\iota \varphi_\iota(g_\iota)$. Hence, φ_ι is an ABGH $\forall \iota \in \mathfrak{J}$.

(2) (resp., (3), (4)) is obtained as a result of (1) and Theorem 2.14(1) (resp., (2), (3)).

Conclusion

This paper introduces the EDP of BG-algebras, a generalization of DP in the sense of Widianto, Gemawati, and Kartini [10]. We demonstrate that the DP of BG-algebras, defined by an infinite family of BG-algebras, is also a BG-algebra. In addition, we have introduced the WDP of BG-algebras. We demonstrate that the WDP of BG-algebras is a subalgebra of EDP BG-algebras. The BGH and ABGH theorems have been discussed in light of the EDP BG-algebras.

This article's discussion of the EDP of BG-algebras can be applied to studying the EDP in other algebraic systems. B-algebras and BCC-algebras will be studied in the future for EDP and WDP research.

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