Journal of Interdisciplinary Mathematics
ISSN: 0972-0502 (Print), ISSN: 2169-012X (Online)
Vol. 28 (2025), No. 5, pp. 1847–1857
DOI : 10.47974/JIM-2141

Exploring fixed points in *b*-metric spaces using weakly contractive mappings

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Abstract

This paper is based on the idea of unique fixed points as the result of weakly contractive maps in the structure of asymmetric *b*–metric spaces. Moreover, provided to illustrate the broader implications of our results compared.

Subject Classification (2020): *47H10, 54H25.*

Keywords: *Weakly contractive mapping, b–metric spaces, Fixed point.*

1. Introduction & Preliminaries

Numerous Fixed Point (FP) results have been developed for mappings that satisfy various types of contractive inequalities. Banach's contraction principle is a foundational result in FP theory for contractive type mappings. Researchers have explored these inequalities, both with and without continuity constraints, across different abstract spaces, such as metric spaces(MS), convex MS, ordered MS, generalized MS, *b*−metric spaces(BMS), and more (see [4–14]). In 1989, Bakhtin [1] introduced BMS, where the "quadrilateral inequality" holds:

$$\rho(ß, ¶) \leq \rho(ß, \nu)+\rho(\nu, Ξ)+\rho(Ξ, ¶),\ \forall\ ß, ¶, \nu \text{ and } Ξ.$$

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Here, we explain a new concept of weakly contractive mappings and demonstrate several unique fixed points(UFP) theorems for these mappings within the framework of asymmetric BMS.

Definition 1.1: [3] Consider $\Omega \neq \emptyset$ and the map ρ: $\Omega \times \Omega \to R^+$ satisfies:

(1) $\rho(ß, ¶) = 0$ if and only if ß equal to ¶, for all ß, ¶ $\in \Omega$.

(2) there exist a real number $k \geq 1$ such that $\rho(ß, ¶) \leq k[\rho(ß, b) + \rho(b, ¶)]$, for all ß, ¶, $b \in \Omega$ then ρ is known as b-metric on Ω & (Ω, ρ) is called an asymmetric BMS with coefficient s.

Definition 1.2: [13] Let (Ω, ρ) be an asymmetric BMS, a sequence $\{ß_\alpha\}_{\alpha \in N}$ and ß $\in \Omega$.

(1) The sequence $\{ß_\alpha\}_{\alpha \in N}$ forward (backward) converges to ß
If $\lim_{\alpha \to +\infty} \rho(ß, ß_\alpha) = \lim_{\alpha \to +\infty} \rho(ß_\alpha, ß) = 0$.

(2) The sequence $\{ß_\alpha\}_{\alpha \in N}$ forward (back-ward) Cauchy
If $\lim_{\alpha,\beta \to +\infty} \rho(ß_\alpha, ß_\beta) = \lim_{\alpha,\beta \to +\infty} \rho(ß_\beta, ß_\alpha) = 0$.

Example 1.3: Consider $\Omega = E \cup F$, where $E = \{1,3\}$ and $F = \left\{\frac{1}{r}, r \in N^*\right\}$ and $\rho : \Omega \times \Omega \to R^+$ be defined by

$$\rho(ß, ¶) = \begin{cases} 2 \text{ if } ß = 1 \text{ and } ¶ = 3\,; \;\; ß = 3 \text{ and } ¶ = 1 \\ 1 \text{ if } ß = 1 \text{ and } ¶ = \frac{1}{r}\,; \;\; ß = \frac{1}{r} \text{ and } ¶ = 3 \\ \frac{1}{r} \text{ if } ß = \frac{1}{r} \text{ and } ¶ = 1\,; \;\; ß = 3 \text{ and } ¶ = \frac{1}{r} \\ 1 \text{ if } ß = \frac{1}{r} \text{ and } ¶ = \frac{1}{r} \; for \; r \neq m \end{cases}$$

then (Ω, ρ) is an asymmetric BMS. However, the below steps holds:

(1) (Ω, ρ) is not a BMS, since $\rho\left(\frac{1}{r}, 1\right) \neq \rho\left(1, \frac{1}{r}\right)$ $\forall\, r$ greater than one.

(2) (Ω, ρ) is not a asymmetric BMS, since $\rho(3,1) = 2 > 1 + \frac{1}{4} = \rho\left(3, \frac{1}{4}\right) + \rho\left(\frac{1}{4}, 1\right)$.

(3) (Ω, ρ) is not a rectangular BMS, since $\rho\left(\frac{1}{r}, 3\right) \neq \rho\left(3, \frac{1}{r}\right)$ $\forall\, r > 1$.

Result 1.4: [3] Consider (Ω, ρ) be an asymmetric BMS, the sequence $\{ß_\alpha\}_{\alpha \in N}$ be a forward(backward) Cauchy sequence with pairwise connected elements in Ω. If $\{ß_\alpha\}_{\alpha \in N}$ forward(backward) converges to ß $\in \Omega$ (¶ $\in \Omega$), then ß = ¶.

Definition 1.5: [3] Consider that (Ω, ρ) is an asymmetric BMS. Then Ω is referred as forward complete, provided if every forward Cauchy $\{ß_\alpha\}_{\alpha \in N} \in \Omega$ forward converges to ß $\in \Omega$.

Definition 1.6: [2] An altering distance function τ: $R^+ \to R^+$ that satisfies the below properties:

(1) Non-decreasing: If $m \le r$, then $\tau(m) \le \tau(r)$.

(2) Positive: $\forall$ (ß > 0), (τ(ß) > 0).

Example 1.7: Consider $\tau(t) = t^3$, (1) Non-decreasing: If m ≤ r, then $m^3 \le r^3$. So, τ(m) = $m^3 \le r^3 = \tau$(r). (2) Positive: $\forall$ $t > 0$, $t^3 > 0$. So, $\tau(t) > 0$.

Definition 1.8: A mapping φ: $\Omega \to R^+$, where Ω be an asymmetric BMS is called lower semi-continuous if $\forall$ ß ∈ Ω and $ß_\alpha \in \Omega$ with

$$\lim_{\alpha\to\infty} ß_\alpha = ß,\ \varphi(ß) \le \lim_{\alpha\to\infty} inf\varphi(ß_\alpha)\ .$$

2. Main Results

Theorem 2.1: *Suppose that* Δ*:* $\Omega \to \Omega$ *be the function, and Consider (*Ω*,* ρ*) as an asymmetric BMS.*

$$\rho(\Delta ß, \Delta ¶) > 0 \Rightarrow \tau[\rho(\Delta ß, \Delta ¶)] \le \tau[\Theta(ß, ¶)] - \varphi[\Theta(ß, ¶)] \quad (2.1)$$

where $\Theta(ß, ¶) = max\left\{\rho(ß, ¶), \rho(ß, \Delta ß), \rho(¶, \Delta ¶), \frac{1}{2s}\left(\rho(ß, \Delta ¶) + \rho(¶, \Delta ß)\right)\right\}$

and ρ*(¶, β)* $\le \rho(\Delta^2 ¶, β)$ $\forall β, ¶ \in \Omega$*. Here* τ *be an altering distance function &* φ *be a lower semi continuous function with* $\varphi(t) = 0$ *if and only if* $t = 0$ *then* Δ *has a UFP.*

Proof: Consider an arbitrary point $ß_0 \in \Omega$. Then we define $\{ß_r\}$ by $\Delta ß_r = ß_{r+1}$, $\rho(ß_r, ß_{r+1}) = \rho^+ ß_r$, $\rho(ß_r, ß_{r-1}) = \rho^- ß_r$, $\forall\ r \in N$. If $\exists\ r_0 \in N$ such that $\rho^+ ß_{r0} = 0$ or $\rho^- ß_{r0+1} = 0$ then Δ has a FP $ß_{r0}$. Assume that $\rho(ß_r, ß_{r+1}) > 0$ & $\rho(ß_r, ß_{r-1}) > 0$.

Step 1: Verify that $\lim_{r\to\infty} \rho^+ ß_r = \lim_{r\to\infty} \rho^- ß_{r+1} = 0$.

Applying (2.1) with ß = $ß_r$ and ¶ = $ß_{r+1}$, we obtain

$\tau(\rho(\Delta ß_{r-1}, \Delta ß_r)) = \tau(\rho^+ ß_r) \le \tau(\max\{\rho^+ ß_{r-1}, \rho^+ ß_r\}) - \varphi(\Theta(\max\{\rho^+ ß_{r-1}, \rho^+ ß_r\})$.

If $\Theta(ß_{r-1}, ß_r) = \rho^+ ß_r$, then we have $\tau(\rho^+ ß_r) \le \tau(\rho^+ ß_r) - \varphi(\rho^+ ß_r)$. That is, $\varphi(\rho^+ ß_r) = 0$, i.e., $ß_r = ß_{r+1}$. This statement is false. Hence $\Theta(ß_{r-1}, ß_r) = \rho^+ ß_{r-1}$.

Thus $$\tau(\rho^+ ß_r) \le \tau(\rho^+ ß_{r-1}) - \varphi(\rho^+ ß_{r-1}). \quad (2.2)$$

So $\tau(\rho^+ ß_r) < \tau(\rho^+ ß_{r-1})$. Since τ is a continuous and non-increasing function, we deduced that

$$\rho^+ ßr < \rho^+ ß_{r-1} \quad (2.3)$$

Substitute in Equation (2.1) with ß = $ß_r$ & ¶ = $ß_{r-1}$,

$$\tau(\rho(\Delta ß_r, \Delta ß_{r-1})) \le \tau(\Theta(ß_r, ß_{r-1})) - \varphi(\Theta(ß_r, ß_{r-1}))$$

where $\Theta(ß_r, ß_{r-1}) = \max\{\rho^+ ß_{r-1}, \rho^+ ß_r\}$.

Suppose that $\rho^- ß_r \le \rho^- ß_{r+1}$ for some $r \in N$.

Case 1: If $\rho^- ß_r \ge \rho^+ ß_{r-1}$, then we get $\tau(\rho^- ß_r) \le \tau(\rho^- ß_r) - \varphi(\rho^- ß_r)$. Then $\varphi(\rho^- ß_r) = 0$. This statement is false. $\Theta(ß_r, ß_{r-1}) = \rho^- ß_r$. Thus $\tau(\rho^- ß_{r+1}) \le \tau(\rho^- ß_r) - \varphi(\rho^- ß_r)$.

Case 2: If $\rho^- ß_r < \rho^+ ß_{r-1}$, then we get $\tau(\rho^- ß_{r+1}) \le [\tau(\rho^+ ß_{r-1}) - \varphi(\rho^+ ß_{r-1})]$.

Since $\rho(ß, ¶) \le \rho(\Delta^2 ¶, ß)$, $\rho^+ ß_{r-1} \le \rho^- ß_{r+1}$, which implies that

$\tau(\rho^{-}ß_r) \leq \tau(\rho^{-}ß_r) - \varphi(\rho^{-}ß_r)$. Thus $\varphi(\rho^{-}ß_r) = 0$ and so $\rho^{-}ß_r = 0$.

This statement is false.

Hence $$\tau(\rho^{-}ß_{r+1}) \leq [\tau(\rho^{-}ß_r) - \varphi(\rho^{-}ß_r)]. \tag{2.4}$$

Since τ is a continuous and non-increasing function, we deduce that

$$\rho^{-}ß_{r+1} < \rho^{-}ß_r. \tag{2.5}$$

From Equation (2.3), the sequence $\rho^{+}ß_r$ *for* $r \in \mathrm{N}$ is monotone non-increasing and thus bounded below. So $\exists\ \alpha \geq 0$ such that $\lim_{r\to\infty} \rho^{+} ß_r = \alpha$.

From Equation (2.4) and Equation (2.5), taking $\lim_{r\to\infty} sup$, we obtain

$$\tau(\lim_{r\to\infty} \sup \rho^{-} ß_r) \leq \tau(\lim_{r\to\infty} \sup \rho^{-} ß_r) - \lim_{r\to\infty} \inf \varphi(\lim_{r\to\infty} \rho^{-} ß_r).$$

Thus, we have $\tau(\alpha) \leq \tau(\alpha) - \varphi(\alpha)$, This implies that $\varphi(\alpha) = 0$. Therefore, by the property of φ, we conclude that $\alpha = 0$.

$$\lim_{r\to\infty} \rho^{+} ß_r = 0. \tag{2.6}$$

By Equation (2.5), $\rho^{-}ß_{r+1}$ *for* $r \in \mathrm{N}$ is monotone non-increasing and so bounded below. So $\exists\ \Xi \geq 0$ which implies $\lim_{r\to\infty} \rho^{+} ß_r = \Xi$.

From Equation (2.2), taking $\lim_{r\to\infty} sup$, we obtain

$$\tau(\lim_{r\to\infty} \sup \rho^{+} ß_{r-1}) \leq \tau(\lim_{r\to\infty} \sup \rho^{+} ß_{r-1}) - \lim_{r\to\infty} \inf \varphi(\lim_{r\to\infty} \rho^{+} ß_{r-1}).$$

Thus, we have $\tau(\Xi) \leq \tau(\Xi) - \varphi(\Xi)$, This implies that $\varphi(\Xi) = 0$.

Therefore, by the property of φ.

$$\lim_{r\to\infty} \rho^{-} ß_{r+1} = 0. \tag{2.7}$$

Step 2: Verify that $\lim_{r\to\infty} \rho(ß_r,\ ß_{r+2}) = 0$ & $\lim_{r\to\infty} \rho(ß_{r+2},\ ß_r) = 0$.

Suppose that $ß_r \neq ß_m\ \forall\ r, m \in \mathrm{N}$ with $r \neq m$. Assume that $ß_r = ß_m$ for some $r = m + k$ with $k > 0$. By Equation (2.3), we obtain $\rho^{+}ß_m = \rho^{+}ß_r < \rho^{-}ß_{r-1}$. Continuing this process, we can get that $\rho(ß_m, ß_{r+1}) = \rho^{+}ß_r < \rho^{+}ß_m$. This statement is false. So, $\rho(ß_r, ß_m) > 0$, $\forall\ r, m \in \mathrm{N}$ with $r \neq m$.

Putting $ß = ß_{r-1}$ & $¶ = ß_{r+1}$ in Equation (2.1), we obtain

$$\tau\big(\rho(ß_r, ß_{r+2})\big) \leq \tau(max\{\rho(ß_{r-1},\ ß_{r+1}), \rho^{+}\ ß_{r-1}\}) - \varphi(max\{\rho(ß_{r-1},\ ß_{r+1}), \rho^{+}\ ß_{r-1}\}) \tag{2.8}$$

Take $\Xi_r = \rho(ß_r, ß_{r+2})$ and $\beta_r = \rho^{+}ß_r$. From the Equation (2.8)

$$\tau(\Xi_r) \leq \tau(\max\{\Xi_{r-1}, \beta_{r-1}\}) - \varphi(\max\{\Xi_{r-1}, \beta_{r-1}\}). \tag{2.9}$$

We conclude that $\Xi_r \leq \max\{\Xi_{r-1}, \beta_{r-1}\}$. Repeat, from the Equation (2.2), $\beta_r \leq \beta_{r-1}$. So, $\max\{\Xi_r, \beta_r\} \leq \max\{\Xi_{r-1}, \beta_{r-1}\}\ \forall\ r \in \mathrm{N}$. The sequence $\max\{\Xi_r, \beta_r\}_{r\in\mathrm{N}}$ is monotone non-increasing & therefore converges to some $\pi \geq 0$.

From Equation (2.6), for $\pi > 0$, we obtain

$\lim_{r\to\infty} sup\ \Xi_r = \lim_{r\to\infty} \sup max\ \{\Xi_r, \beta_r\} = \lim_{r\to\infty} \sup max\ \{\Xi_{r-1}, \beta_{r-1}\} = \pi$.

Substitute $\lim_{r\to\infty} sup$ in Equation (2.9) & by the properties of τ & φ, we have

$$\tau(\pi) = \tau(\lim_{r\to\infty} sup\ \Xi_r) = \lim_{r\to\infty} sup\ \tau(\Xi_r)$$
$$\leq \tau(\lim_{r\to\infty} sup \max\{\Xi_{r-1}, \beta_{r-1}\}) - \varphi(\pi) = \tau(\pi) - \varphi(\pi).$$

We conclude that $\varphi(\pi) = 0$ & $\pi = 0$. This statement is false.

$$\lim_{r\to\infty} \rho(ß_r, ß_{r+2}) = 0.$$

Putting ß = $ß_r$ and ¶ = $ß_{r-1}$ from the Equation (2.1), we get

$$\tau(\rho(ß_{r+2}, ß_r)) \le \tau(\Theta(ß_{r+1}, ß_{r-1})) - \varphi(\Theta(ß_{r+1}, ß_{r-1})) \tag{2.10}$$

where $\Theta(ß_{r+1}, ß_{r-1}) \le \max\{\rho(ß_{r+1}, ß_{r-1}), \rho - ß_{r+1}\}$.

Take $\Xi_r = \rho(ß_{r+2}, ß_r)$ and $\beta_r = \rho^{-} ß_{r+1}$ from the Equation (2.10), we get

$$\tau(\Xi_r) = \tau(\max(\Xi_{r-1}, \beta_{r-1})) - \varphi(\max(\Xi_{r-1}, \beta_{r-1})).$$

We conclude that $\Xi_r \le \max(\Xi_{r-1}, \beta_{r-1})$.

Repeat, from Equation (2.5) $\beta_r \le \beta_{r-1}$. So, $\max(\Xi_r, \beta_r) \le \max(\Xi_{r-1}, \beta_{r-1})$ $\forall\ r \in N$. The sequence $\{\max(\Xi_r, \beta_r)\}_{r\in N}$ is monotone non-increasing, therefore converges to some $\pi \ge 0$. From Equation (2.7), for $\pi > 0$, we obtain

$$\lim_{r\to\infty} sup\ \Xi_r = \lim_{r\to\infty} sup \max(\Xi_{r-1}, \beta_{r-1}) = \pi.$$

Taking $\lim_{r\to\infty} sup$ in Equation (2.10) & by the properties of τ & φ, we have

$$\tau(\pi) = \tau(\lim_{r\to\infty} sup\ \Xi_r)$$

$$\le \tau(\lim_{r\to\infty} sup \max(\Xi_{r-1}, \beta_{r-1})) - \varphi(\pi) = \tau(\pi) - \varphi(\pi).$$

We conclude that $\varphi(\pi) = 0$ & $\pi = 0$. This statement is false. $\lim_{r\to\infty} \rho(ß_{r+2}, ß_r) = 0$.

Step 3: First, we prove $\{ß_r\}, \forall r \in N$ is a Cauchy sequence.

For $\pi > 0$, the sequences $\{ß_{p(k)}\}$ and $\{ß_{q(k)}\}$ so that, $\forall\ k > 0$,

$$\{ß_{p(k)}\} > \{ß_{q(k)}\} > k,$$

$$\rho(ß_{q(k)}, ß_{p(k)}) \ge \pi \text{ and } \rho(ß_{q(k)}, ß_{q(k)+1}) < \pi. \tag{2.11}$$

Using quadrilateral inequality, we have

$$\pi \le \rho(ß_{q(k)}, ß_{q(k)+1}) + \rho(ß_{q(k)+1}, ß_{p(k)+1}) + \rho(ß_{p(k)+1}, ß_{p(k)})$$

$$< \pi + \rho(ß_{q(k)+1}, ß_{p(k)+1}) + \rho(ß_{p(k)+1}, ß_{p(k)}).$$

Applying the limit as k tends to infinity, we obtain

$$\lim_{k\to\infty} \rho\left(ß_{q(k)}, ß_{p(k)}\right) = \pi. \tag{2.12}$$

Using quadrilateral inequality, we have

$$\rho(ß_{q(k)+1}, ß_{p(k)+1}) \le \rho(ß_{q(k)+1}, ß_{q(k)}) + \rho(ß_{q(k)}, ß_{p(k)}) + \rho(ß_{p(k)}, ß_{p(k)+1})$$

$$\rho(ßq(k), ßp(k)) \le \rho(ßq(k), ßq(k)+1) + \rho(ßq(k)+1, ßp(k)+1) + \rho(ßp(k)+1, ßp(k)).$$

Taking $k \to \infty$ and using Equation (2.11), we get

$$\lim_{k\to\infty} \rho\left(ß_{q(k)+1}, ß_{p(k)+1}\right) = \pi. \tag{2.13}$$

Let $\nu = \frac{\in}{2} > 0$. From Equation (2.13), $\exists\ r_0 \in N$ such that

$$|\rho(ß_{q(k)+1}, ß_{p(k)+1}) - \pi| \le \nu, \text{ for all } p \ge p_0,$$

we conclude that $\rho(ß_{q(k)+1}, ß_{p(k)+1}) \ge \nu$ *,for all* $p \ge p_0$.

Putting ß = $ß_{q(k)}$ and ¶ = $ß_{p(k)}$ in Equation (2.1), we have

$$\tau(\rho(ß_{q(k)}, ß_{p(k)})) \leq \tau(\Theta(ß_{q(k)}, ß_{p(k)}) - \varphi(\Theta(ß_{q(k)}, ß_{p(k)})) \tag{2.14}$$

where $\Theta(ß_{q(k)}, ß_{p(k)}) = \max\{\rho(ß_{q(k)}, ß_{p(k)}), \rho(ß_{p(k)}, ß_{p(k)+1}), \rho(ß_{q(k)}, ß_{q(k)+1})\}$.

Therefore by the Equations (2.6) and Equation (2.12), we have

$$\lim_{k\to\infty} \Theta\left(ß_{q(k)}, ß_{p(k)}\right) = \pi.$$

Taking $k \to \infty$ in Equation (2.14), we have $\tau(\pi) \leq \tau(\pi) - \varphi(\pi)$. This statement is false.

Therefore $\{ß_r\}$, $\forall r \in \mathbb{N}$ represents the Cauchy sequence in (Ω, ρ).

Hence, using the completeness of (Ω, ρ), $\exists\ \ell_1, \ell_2 \in \Omega$ such that

$$\lim_{r\to\infty} \rho(ß_r, \ell_1) = \lim_{r\to\infty} \rho(\ell_2, ß_r) = 0.$$

From the Result (1.4), we get $\ell_1 = \ell_2$,

$$\lim_{r\to\infty} \rho(ß_r, \ell_1) = \lim_{r\to\infty} \rho(\ell_1, ß_r) = 0.$$

Step 4: We start with the assumption $\ell_1 = \Delta\ell_1$, that is $\rho(\Delta\ell_1, \ell_1) = \rho(\ell_1, \Delta\ell_1) = 0$ Impossible conclusion, we consider that $\rho(\Delta\ell_1, \ell_1) > 0$ or $\rho(\ell_1, \Delta\ell_1) > 0$.

$$\rho(\Delta ß_r, \Delta\ell_1) \leq \rho(\Delta ß_r, ß_r) + \rho(ß_r, \ell_1) + \rho(\ell_1, \Delta\ell_1) \tag{2.15}$$

and

$$\rho(\ell_1, \Delta\ell_1) \leq \rho(\ell_1, ß_r) + \rho(ß_r, \Delta ß_r) + \rho(\Delta ß_r, \Delta\ell_1). \tag{2.16}$$

From the Equation (2.15) & Equation (2.16)

$$\lim_{r\to\infty} \rho(\Delta ß_r, \Delta\ell_1) = \rho(\ell_1, \Delta\ell_1).$$

Therefore $\exists\ r_0 \in \mathbb{N}$ such that $\rho(\Delta ß_r, \Delta\ell_1) \geq \rho(\ell_1, \Delta\ell_1) > 0$, for all $r \geq r_0$.

Substitute $ß = \Delta ß_r$ and $¶ = \Delta\ell_1$ in the Equation (2.1), we get

$$\tau(\rho(\Delta ß_r, \Delta\ell_1)) \leq \tau(\Theta(ß_r, \ell_1)) - \varphi(\Theta(ß_r, \ell_1)), \tag{2.17}$$

where $\Theta(ß_r, \ell_1) = \max\{\rho(ß_r, \Delta ß_r), \rho(\ell_1, \Delta\ell_1), \rho(ß_r, \ell_1)\}$.

Since $\lim_{r\to\infty} \rho^+ ß_r = \rho(ß_r, \ell_1) = 0$, we obtain that

$$\lim_{r\to\infty} \Theta(ß_r, \ell_1) = \rho(\ell_1, \Delta\ell_1).$$

Substitute r tends to infinity in Equation (2.17) and by the properties of τ and φ, we have $\tau(\rho(\Delta ß_r, \Delta\ell_1)) \leq \tau(\lim_{r\to\infty} \Theta(ß_r, \ell_1)) - \lim_{r\to\infty} sup\ \varphi(\Theta(ß_r, \ell_1))$

$$\leq \tau(\rho(\ell_1, \Delta\ell_1)) - \varphi(\rho(\ell_1, \Delta\ell_1)).$$

We conclude that $\varphi(\rho(\ell_1, \Delta\ell_1)) = 0$, therefore $\rho(\ell_1, \Delta\ell_1) = 0$, which is a contradiction. If $\rho(\Delta\ell_1, \ell_1) > 0$, similarly we have $\rho(\Delta\ell_1, \ell_1) = 0$.

So that, $\rho(\ell_1, \Delta\ell_1) = 0$ & $\rho(\Delta\ell_1, \ell_1) = 0$ and therefore $\ell_1 = \Delta\ell_1$.

Theorem 2.2. Let $\Delta: \Omega \to \Omega$ be the function, and Consider (Ω, ρ) as an asymmetric BMS. If $\exists\ \varphi \in \Delta$ such that $\forall\ ß, ¶ \in \Omega$,

$$\rho(\Delta ß, \Delta ¶) > 0 \Rightarrow \rho(\Delta ß, \Delta ¶) \leq \varphi[\Theta(ß, ¶)], \tag{2.18}$$

where $\Theta(ß, ¶) = \max\left\{\rho(ß, ¶), \rho(ß, \Delta ß), \rho(¶, \Delta ¶), \frac{1}{2s}\left(\rho(ß, \Delta ¶) + \rho(¶, \Delta ß)\right)\right\}$

and $\rho(¶, ß) \leq \rho(\Delta^2 ¶, ß)$. Then Δ has a UFP.

Proof: Consider an arbitrary point $ß_0 \in \Omega$. Then we define $\{ß_r\}$ by $ß_{r+1} = \Delta ß_r$, $\forall\, r \in \mathrm{N}$. If $\exists\, r_0 \in \mathrm{N}$ such that $\rho\,(ß_{r0},\, ß_{r0+1}) = 0$ or $\rho(ß_{r0+1},\, ß_{r0}) = 0$, then Δ maps $ß_{r0}$ to itself. Assume that $\rho^{+}ß_r > 0$ & $\rho^{-}ß_{r+1} > 0$.

Step 1: Verify that $\lim_{r\to\infty} \rho^{+}ß_r = \lim_{r\to\infty} \rho^{-}ß_{r+1} = 0$.

Applying Equation (2.18) with $ß = ß_r$ and $¶ = ß_{r+1}$, we have

$$\rho(\Delta ß_{r-1},\, \Delta ß_r) = \rho^{+}ß_r \le \varphi(\Theta(\max\{\rho^{+}ß_{r-1},\, \rho^{+}ß_r\})).$$

If $\Theta(ß_{r-1},\, ß_r) = \rho^{+}ß_r$, then we have $\rho^{+}ß_r \le \varphi(\rho^{+}ß_r) < \rho^{+}ß_r$.

This statement is false. Hence $\Theta(ß_{r-1},\, ß_r) = \rho^{+}ß_{r-1}$.

$$\text{Thus } \rho^{+}ß_r \le \varphi(\rho^{+}ß_{r-1}). \tag{2.19}$$

$$\text{So } \rho^{+}ß_r < \rho^{+}ß_{r-1}. \tag{2.20}$$

Letting $ß = ß_{r+1}$ and $¶ = ß_r$ in Equation (2.18), we obtain

$$\rho(ßr+1,\, ßm) = \rho - ßr+1 \le \varphi(\Theta(ßr,\, ßr-1)),$$

where $\Theta(ß_r,\, ß_{r-1}) = \max\{\rho^{+}ß_{r-1}, \rho^{-}ß_r\}$.

Suppose that $\rho^{-}ß_r \le \rho^{-}ß_{r+1}$ for some $r \in \mathrm{N}$.

Case 1: If $\rho^{-}ß_r \ge \rho^{+}ß_{r-1}$, then we get $\rho^{-}ß_r \le \varphi(\rho^{-}ß_r) < \rho^{-}ß_r$.

This statement is false.

Case 2: If $\rho^{-}ß_r < \rho(ß_{r-1},\, ß_r)$, then we get $\rho^{-}ß_{r+1} \le \varphi(\rho^{+}ß_{r-1}) < \rho^{+}ß_{r-1}$, since $\rho(¶,\, ß) \le (\Delta^{2}¶,\, ß)$, $\rho(ß_{r-1},\, ß_r) \le \rho^{-}ß_{r+1}$, which implies that $\rho^{-}ß_r \le \varphi(\rho^{-}ß_r) < \rho^{-}ß_r$. This statement is false.

$$\text{Therefore, } \rho^{-}ß_{r+1} \le \varphi(\rho^{+}ß_{r-1}) < \rho^{+}ß_{r-1}. \tag{2.21}$$

From Equation (2.20), the sequence $\rho^{+}ß_r \in \mathrm{N}$ is monotone non-increasing and bounded below. So $\exists\, ß \ge 0$ such that $\lim_{r\to\infty} \rho^{+}ß_r = ß$.

By induction, Equation (2.19) yields $\rho^{+}ß_r \le \varphi^{r}\rho^{+}ß_0$, $\forall r \in \mathrm{N}$. By the property of φ, clearly

$$\lim_{r\to\infty} \rho^{+}\, ß_r = 0. \tag{2.22}$$

From Equation (2.21), the sequence $\{\rho^{-}ß_{r+1}\}_{r\in\mathrm{N}}$ is monotone non-increasing and so bounded below. So $\exists\, \delta \ge 0$ such that $\lim_{r\to\infty} \rho^{+}ß_r = \delta$.

By induction, Equation (2.21) yields. $\rho^{-}ß_{r+1} \le \varphi^{r}\rho^{-}ß_1$, $\forall r \in \mathrm{N}$. using the property of φ, it is clear that $\lim_{r\to\infty} \rho^{-}\, ß_{r+1} = 0$.

Step 2: Verify that $\lim_{r\to\infty} \rho(ß_r,\, ß_{r+2}) = \lim_{r\to\infty} \rho(ß_{r+2},\, ß_r) = 0$.

Letting $ß = ß_r$ and $¶ = ß_{r+2}$ in Equation (2.18), we obtain

$$\rho(ß_r,\, ß_{r+2}) \le \varphi(\max\{\rho(ß_{r-1},\, ß_{r+1}),\, \rho^{+}ß_{r-1}\}). \tag{2.23}$$

Take $a_r = \rho(ß_r,\, ß_{r+2})$ and $\delta_r = \rho^{+}ß_r$. Then, from Equation (2.23), we have

$$a_r \le \varphi(\max\{a_{r-1},\, \delta_{r-1}\}), \qquad a_r \le \max\{a_{r-1},\, \delta_{r-1}\}. \tag{2.24}$$

Again, by Equation (2.22), $\delta_r \le \delta_{r-1}$. Therefore, $\max\{a_r,\, \delta_r\} \le \max\{a_{r-1},\, \delta_{r-1}\}$ $\forall r \in \mathrm{N}$. Then the sequence $\{\max\{a_r, \delta_r\}\}_{r\in\mathrm{N}}$ is monotone non-increasing, and so it converges to some $l \ge 0$. By Equation (2.21), for $l > 0$, we have

$$\lim_{r\to\infty} sup\ a_r = \lim_{r\to\infty} \sup max\ \{a_r, \delta_r\} = \lim_{r\to\infty} \sup max\ \{a_{r-1}, \delta_{r-1}\} = l.$$

Taking $\lim_{r\to\infty} sup$ in Equation (2.23) and the properties of φ, we obtain

$$l = \lim_{r\to\infty} sup\ a_r \le \lim_{r\to\infty} \sup max\ \{a_{r-1}, \delta_{r-1}\}$$

$$\leq \varphi(\lim_{r\to\infty} \sup max\ \{a_{r-1}, \delta_{r-1}\}) = \varphi(l) < l.$$

This statement is false. Thus

$$\lim_{r\to\infty} \rho(ß_r, ß_{r+2}) = 0. \tag{2.25}$$

Letting ß = $ß_r$ and ¶ = $ß_{r+2}$ Equation (2.18), we obtain

$$\rho(ß_{r+2}, ß_r) \leq \varphi(\max\{\rho(ß_{r+1}, ß_{r-1}), \rho^{-}ß_r\}). \tag{2.26}$$

Take $\omega = \rho(ß_{r+2}, ß_r)$ and $\pi_r = \rho^{-}ß_{r+1}$. Then, from Equation (2.26), we have

$$\omega_r \leq \varphi(\max\{\omega_{r-1}, \pi_{r-1}\}), \quad \omega_r \leq \max\{\omega_{r-1}, \pi_{r-1}\}. \tag{2.27}$$

So we have $\pi_r \leq \pi_{r-1}$. Therefore $\max\{\omega_r, \pi_r\} \leq \max\{\omega_{r-1}, \pi_{r-1}\}\ \forall\ r \in N$.

Then the sequence $\{\max\{\omega_r, \pi_r\}\}_{r\in N}$ is monotone non-increasing, and so it converges to some $l \geq 0$. By Equation (2.21), for $l > 0$,

$$\lim_{r\to\infty} sup\ \omega_r = \lim_{r\to\infty} \sup max\ \{\omega_r, \pi_r\}$$
$$= \lim_{r\to\infty} \sup max\ \{\omega_{r-1}, \pi_{r-1}\} = l.$$

Taking $\lim_{r\to\infty} sup$ in Equation (2.27) and the properties of φ, we obtain

$$l = \lim_{r\to\infty} sup\ \omega_r \leq \varphi(\lim_{r\to\infty} max\ \{\omega_{r-1}, \pi_{r-1}\}) = \varphi(h) < h.$$

This statement is false. Thus $\lim_{r\to\infty}\rho(ß_{r+2}, ß_r) = 0$.

Step 3: First, we prove $\{ß_r\}_{r\in N}$ is a Cauchy sequence,

$$\lim_{r\to\infty}\rho(ß_r, ß_{r+a}) = 0, \forall a \in N$$

by the Equation (2.22) & Equation (2.25), the cases for $a = 1$ and $a = 2$, are proved. For $a \geq 3$,

Case I: Assume that $a = 2m + 1$, where $m \geq 1$ then by the quadrilateral inequality, we get $\rho(ß_r, ß_{r+a}) \leq \rho^{+}ß_r + \rho^{+}ß_{r+1} + \cdots + \rho(ß_{r+2m}, ß_{r+2m+1})$

$$\leq \sum_{p=r}^{r+2m} \varphi^p(\rho^{+}ß_0) \leq \sum_{p=r}^{\infty} \varphi^p(\rho^{+}ß_0) \to 0\ as\ r \to \infty.$$

Case II: Assume that $a = 2m$, where $m \geq 2$.

By Equation (2.24) & by the quadrilateral inequality, we get

$$\rho(ß_r, ß_{r+a}) \leq \rho(ß_r, ß_{r+2m})$$
$$\leq \sum_{p=r+2}^{r+2m-1} \varphi^p(\rho^{+}ß_0) \leq \sum_{p=r+2}^{\infty} \varphi^p(\rho^{+}ß_0) \to 0\ as\ r \to \infty.$$

By merging the expressions, we get $\lim_{r\to\infty} sup\ \rho(ßr, ßr + a)=0, \forall\ a \in N$. Therefore, $\{ß_r\}$ is a Cauchy sequence in (Ω, ρ). Hence, by completeness of (Ω, ρ), $\exists\ \ell_1, \ell_2 \in \Omega$ such that

$$\lim_{r\to\infty} \rho(ß_r, \ell_1) = \lim_{r\to\infty} \rho(\ell_2, ß_r) = 0.$$

From Results (1.4), put $\ell_1 = \ell_2$

$$\lim_{r\to\infty} \rho(ß_r, \ell_1) = \lim_{r\to\infty} \rho(\ell_1, ß_r) = 0$$

Step 4: We start with the assumption $\ell_1 = \Delta\ell_1$, that is, $\rho(\Delta\ell_1, \ell_1) = 0$ & $\rho(\ell_1, \Delta\ell_1) = 0$. Impossible conclusion, we consider that $\rho(\Delta\ell_1, \ell_1) > 0$ or $\rho(\ell_1, \Delta\ell_1) > 0$. Suppose that $\rho(\Delta\ell_1, \ell_1) > 0$. By Theorem (2.1), we obtain

$$\lim_{r\to+\infty}\rho(\Delta ß_r,\ \Delta\ell_1) = \rho(\ell_1,\ \Delta\ell_1). \tag{2.28}$$

So $\exists\ r_2 \in \mathrm{N}$ such that $\rho(\Delta ß_r,\ \Delta\ell_1) \geq \rho(\ell_1,\ \Delta\ell_1) > 0,\ \forall\ r \geq r_2$.

Putting $ß = \Delta ß_r$ & $¶ = \Delta\ell_1$ in Equation (2.18), we get $\rho(\Delta ß_r,\ \Delta\ell_1)) \leq \varphi(\Theta(ß_r,\ \ell_1))$ where $\lim_{r\to\infty}\Theta(ß_r,\ \ell_1) = \rho(\ell_1,\ \Delta\ell_1)$.

Letting $\lim_{r\to\infty} sup$ in Equation (2.28) and by the properties of φ, we conclude that

$$\rho(\ell_1, \Delta\ell_1) = \lim_{r\to\infty}\rho(\Delta ß_r,\ \Delta\ell_1) \leq \lim_{r\to+\infty} \sup \varphi(\ \Theta(ß_r,\ \ell_1) \leq \varphi(\lim_{r\to\infty}\Theta(ß_r,\ \ell_1)).$$

This statement is false. If $\rho(\Delta\ell_1,\ \ell_1) > 0$, then by similar method, which is a contradiction. So $\rho(\ell_1,\ \Delta\ell_1) = 0$ and $\rho(\Delta\ell_1,\ \ell_1) = 0$ & therefore $\ell_1 = \Delta\ell_1$.

Example 2.3: Let a mapping $\Delta: \Omega \to \Omega$ and $\Omega = \mathrm{R}^+$. Explain $\rho: \Omega\times\Omega \to \mathrm{R}^+$ by $\rho(ß,¶) = \max\{¶-ß,\ 0\}$. Then $(\Omega,\ \rho)$ is a complete BMS.

$$\Delta(ß) = \ln\left(\frac{ß}{3}+1\right),\quad \forall ß \in \Omega.$$

Let the mapping $\varphi,\ \tau : \mathrm{R}^+ \to \mathrm{R}^+$ defined by $\tau(\Xi) = 2\Xi, \forall \Xi \in R^+, \varphi(\Xi) = \frac{\Xi}{3}$, $\forall\ \Xi \in \mathrm{R}^+,\ \forall\ (ß,\ ¶) \in \Omega^2$, we have $\rho(\Delta^2¶,\ ß) = \max\{ß - \Delta^2¶, 0\}$ and $\Delta^2¶ = \ln\left(\frac{1}{3}\ln\left(\frac{¶}{3}+1\right)+1\right)$. Hence

$$max\{ß - ¶,\ 0\} \leq max\left\{ß - \ln\left(\frac{1}{3}\ln\left(\frac{¶}{3}+1\right)+1\right), 0\right\}.\ \text{Thus}$$

$\rho(¶,ß) \leq \rho(\Delta^2¶,ß)$. Otherwise, $\rho(\Delta ß, \Delta¶) = \max\left\{\ln\left(\frac{¶}{3}+1\right) - \ln\left(\frac{ß}{3}+1\right), 0\right\}$

$$\Theta(ß, ¶) = \max\left\{max\{¶ - ß, 0\}, max\left\{\ln\left(\frac{ß}{3}+1\right), 0\right\}, max\left\{\ln\left(\frac{¶}{3}+1\right) - ¶, 0\right\}\right\}.$$

1. If $ß \geq ¶$, then we have $\rho(\Delta ß,\Delta¶) = 0,\ \Theta(ß,¶) = 0$. So $\tau(\rho(\Delta ß,\Delta¶)) = \tau(\Theta(ß,¶)) - \varphi(\Theta(ß,¶))$.
2. If $¶ > ß$, then we have $(\Delta ß, \Delta¶) = \ln\left(\frac{¶}{3}+1\right) - \ln\left(\frac{ß}{3}+1\right), \Theta(ß, ¶) = max\{¶ - ß, 0,0\} = ¶ - ß$. So $\tau\left(\rho(\Delta ß, \Delta¶)\right) = 2\ln\left(\frac{¶}{3}+1\right) - 2\ln\left(\frac{ß}{3}+1\right)$, $\tau\left(\Theta(ß, ¶)\right) = 2(¶ - ß),\quad \varphi\left(\Theta(ß, ¶)\right) = \frac{¶-ß}{3}$.

Thus $\tau(\rho(\Delta ß,\Delta¶)) \leq \tau(\Theta(ß,¶)) - \varphi(\max(\rho(ß,¶), \rho(¶,\Delta¶)))$.
Therefore zero is a UFP of Δ.

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Received December, 2024