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Certain subclasses of bi-univalent functions connected with polylogarithm function

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Abstract

We introduce and examine certain bi-univalent functions subclasses connected polylogarithm function. The convolution of two well-known differential operators define the new differential operator. By utilizing the new differential operator $Q^{\eta}_{\delta,\lambda}$ we provide some new bi-univalent subclasses and estimate the basic coefficients $|a_2|$ and $|a_3|$. Furthermore, specializing some parameters in our primary findings, we show that various fresh results have been obtained. The results of this study will fortify the field's theoretical foundations and open up fresh avenues for mathematical inquiry and application.

Subject Classification (2020): *30C45, 30C50.*

Keywords: *Bi-univalent functions, Coefficient estimates, Subordination, Polylogarithms function.*

1. Introduction

For the normalized analytic function, $\mathbb{A}$ represent the class of

$$f(\zeta) = \zeta + \sum_{k=2}^{\infty} a_k \zeta^k,$$

belongs to $\mathbb{U} = \{\zeta \in \mathbb{C}: |\zeta| < 1\}$, it is analytical.

Let $\mathcal{L}$ represent the subclass of all functions in $\mathbb{A}$, it also satisfies $f'(0) = 1$ in $\mathbb{U}$ and the following constraints $f(0) = 0$

Let $(f * g)(\zeta)$ be the convolution between subsequence functions $f(\zeta)$, $g(\zeta)$ is written by

$$(f * g)(\zeta) = \zeta + \sum_{k=2}^{\infty} a_k b_k \zeta^k,$$

where

$$g(\zeta) = \zeta + \sum_{k=2}^{\infty} b_k \zeta^k.$$

The polylogarithms function of $\gamma \in \mathbb{N}$, with $\gamma \geq 2$, defined as

$$Li_{\gamma}(\zeta) = \phi_{\gamma}(\zeta) = \sum_{k=1}^{\infty} \frac{\zeta^k}{(k)^{\gamma}}.$$

For $\lambda \in \mathbb{N}$, $\mathrm{Re}(\lambda) > 1$ and $\mathrm{Re}(c) > -1$, the λ^{th} order polylogarithms function defined by

$$\phi_{\lambda}(c;\zeta) = \sum_{k=0}^{\infty} \frac{\zeta^k}{(k+c)^{\lambda}}.$$

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For $f(\zeta) \in \mathcal{Q}$, Al-Shaqsi [3] introduced the following operator

$$\psi_\lambda(c;\zeta) = (1+c)^\lambda \phi_\lambda(c,\zeta) * f(\zeta) = \zeta + \sum_{k=2}^{\infty} \left(\frac{1+c}{k+c}\right)^\lambda a_k \zeta^k,$$

where $\lambda \in \mathbb{N}$, $\mathrm{Re}(\lambda) > 1$ and $\mathrm{Re}(c) > 0$.

For $f(\zeta) \in \mathcal{A}$, the operator has been defined as

$$\mathcal{Q}^n_{\lambda,\delta}(\psi_\lambda(c;\zeta)f(z)) = \mathcal{Q}_{\lambda,\delta}\left(\mathcal{Q}^{n-1}_{\lambda,\delta}\right) = \zeta + \sum_{k=2}^{\infty} [1+(k-1)\delta]^n \left(\frac{1+c}{k+c}\right)^\lambda a_k \zeta^k, \tag{1.1}$$

it is convolution of the operators of Al-Oboudi [1] and Al-Shaqsi [3].

Hence,

$$\begin{aligned}
\mathcal{Q}^0_{\lambda,\delta} &= \zeta + \sum_{k=2}^{\infty} \left(\frac{1+c}{k+c}\right)^\lambda a_k \zeta^k, \\
\mathcal{Q}^1_{\lambda,\delta} &= (1-\delta)\psi_\lambda(c,\zeta) + \delta\xi\left(\psi_\lambda(c,\zeta)\right) \\
&= \zeta + \sum_{k=2}^{\infty} [1+(k-1)\delta] \left(\frac{1+c}{k+c}\right)^\lambda a_k \zeta^k \\
&= \mathcal{Q}_{\lambda,\delta}, \\
\mathcal{Q}^2_{\lambda,\delta} &= \mathcal{Q}_{\lambda,\delta}\left(\mathcal{Q}_{\lambda,\delta}\right).
\end{aligned}$$

Similarly,

$$\begin{aligned}
\mathcal{Q}^n_{\lambda,\delta} &= \mathcal{Q}_{\lambda,\delta}\left(\mathcal{Q}^{n-1}_{\lambda,\delta}\right) \\
&= \zeta + \sum_{k=2}^{\infty} [1+(k-1)\delta]^n \left(\frac{1+c}{k+c}\right)^\lambda a_k \zeta^k.
\end{aligned}$$

Remark 1.1

(i) For $\delta = 0$, and $c = 0$ in (1.1), we get the multiplier transformation $I^n f(\zeta)$ introduced by Flett [4].

(ii) For $\delta = 0, c = 0$ and $\lambda = -n, n \in \mathcal{N}_0$ in (1.1), we get the differential oprator $\mathcal{D}^n f(\zeta)$ introduced by Salagean [16].

(iii) For $\delta = 0, c = 1$ and $\lambda = n$ in (1.1), we get the differential oprator $\mathcal{I}^n f(\zeta)$ introduced by Uralegaddi and Somanatha [19].

(iv) For $\delta = 0, c = 1, \lambda = \delta$ in (1.1), we get the mutiplier transformation oprator $\mathcal{I}^\delta f(\zeta)$ was introduced by Jung, Kim, and Srivastava [9].

(v) For $\delta = 0, c = a-1\ (a > 0), \lambda = \delta$ in (1.1), we get the integral operator $\mathcal{I}^\delta_{a-1} f(\zeta)$ was analyzed and examined by Komatu [11].

In recent years, the researchers are fascinated by the differential and integral operators like Frasin and Aouf [5], Bansal and Sokól, [7], Joshi, Joshi, and Pawar [8], Lashin [12], Pall-Szabó and Oros [14], Srivastava, Mishra, and Gochhayat [17], Kassim, Reem, and Rana, [10], Al-Shbeil et al. [2], Hasan and Illafe [6], Kassim, Reem and Rana,[13], Yousef et al. [20] and Stalin et al. [18] investigated a number of bi-univalent function subclasses and came up with bounds for the fundamental coefficients $|a_2|$ as well $|a_3|$.

By the inspiration, we present some bi-univalent subclasses and compute the initial bounds $|a_2|$ and $|a_3|$.

In the sequence determine principal results, we really want to review the accompanying lemma.

Lemma 1.2: *[15] If $s \in P$, then $|s_n| \leq 2$ for every n, where s is the symbol for family containing all functions. which is analytic in $\mathbb{U}$ for which*

$$Re(s(\zeta)) > 0, s(\zeta) = 1 + s_1\zeta + s_2\zeta^2 + \cdots$$

for $\zeta \in \mathbb{U}$.

2. Coefficient Estimation related to the operator $\mathcal{Q}_{\delta,\lambda}^{n}$

2.1 *Coefficient Estimation for the class* $\mathcal{L}_{\delta,\lambda}^{\Sigma}(\phi)$

It is expected that the sequel will

$$\varphi(\zeta) = 1 + A_1\zeta + A_2\zeta^2 + A_3\zeta^3 + \cdots, A_1 > 0.$$

Let assume $u(\zeta)$ as well as $v(\zeta)$ are analytical in $\mathbb{U}$ with $u_1(0) = v_1(0) = 0$, $|u_1(\zeta)| < 1$, $|v_1(\tau)| < 1$, and

$$\psi(\zeta) = 1 + s_1\zeta + s_2\zeta^2 + s_3\zeta^3 + \cdots, (s_1 > 0),$$
$$\psi(\tau) = 1 + t_1\tau + t_2\tau^2 + t_3\tau^3 + \cdots, (t_1 > 0).$$

Since $\psi(0) = 1, \psi'(0) > 0$

$$u_1(\zeta) = \frac{\psi(\zeta)-1}{\psi(\zeta)+1} = \frac{1}{2}s_1\zeta + \frac{1}{2}(s_2 - s_1^2)\zeta^2 + \cdots,$$
$$v_1(\tau) = \frac{\psi(\tau)-1}{\psi(\tau)+1} = \frac{1}{2}t_1\tau + \frac{1}{2}(t_2 - t_1^2)\tau^2 + \cdots.$$

It is widely acknowledged that

$$|s_1| \leq 1, |s_2| \leq 1 - |s_1|^2, |t_1| \leq 1, |t_2| \leq 1 - |t_1|^2.$$

A brief computation yields the following:

$$\begin{aligned} \phi(\zeta) &= \varphi(u_1(\zeta)) \\ &= 1 + \frac{1}{2}A_1s_1\zeta + \left(\frac{1}{2}A_1(s_2 - \frac{s_1^2}{2}) + \frac{1}{4}A_2s_1^2\right)\zeta^2 + \cdots, \end{aligned} \tag{2.1}$$

and

$$\begin{aligned} \phi(\tau) &= \varphi(v_1(\tau)) \\ &= 1 + \frac{1}{2}A_1t_1\tau + \left(\frac{1}{2}A_1(t_2 - \frac{t_1^2}{2}) + \frac{1}{4}A_2t_1^2\right)\tau^2 + \cdots \end{aligned} \tag{2.2}$$

Definition 2.1: If a function $f \in \mathcal{L}_{\delta,\lambda}^{\Sigma}(\phi)$, then the subsequent subordinations are hold:

$$1 + \frac{1}{b}\left(\frac{\zeta(\mathcal{Q}_{\delta,\lambda}^{n}f(\zeta))'}{\mathcal{Q}_{\delta,\lambda}^{n}f(\zeta)} - 1\right) \prec \phi(\zeta) \tag{2.3}$$

and

$$1 + \frac{1}{b}\left(\frac{\tau(\mathcal{Q}_{\delta,\lambda}^{n}g(\tau))'}{\mathcal{Q}_{\delta,\lambda}^{n}g(\tau)} - 1\right) \prec \phi(\tau). \tag{2.4}$$

Theorem 2.2: *If $f(\zeta) \in \mathcal{L}_{\delta,\lambda}^{\Sigma}(\phi)$, then*

$$|a_2| \leq \sqrt{\frac{2|b|A_1(s_2+t_2)+|b|(A_2-A_1)(s_1^2+t_1^2)}{8[2M_1-M_2]}}$$

and

$$|a_3| \leq \frac{2|b|A_1(s_2+t_2)+|b|(A_2-A_1)(s_1^2+t_1^2)}{8[2M_1-M_2]} + \frac{2|b|A_1(s_2-t_2)+|b|(A_2-A_1)(s_1^2-t_1^2)}{16M_1},$$

where

$$M_1 = (1+2\delta)^n \left(\frac{1+c}{3+c}\right)^{\lambda},$$

$$M_2 = (1+\delta)^{2n} \left(\frac{1+c}{2+c}\right)^{2\lambda},$$

$$M_3 = (1+\delta)^n \left(\frac{1+c}{2+c}\right)^{\lambda}.$$

Proof: Let $f \in \mathcal{L}_{\delta,\lambda}^{\Sigma}(\phi)$. There are also analytical functions $u, v: \mathbb{U} \to \mathbb{U}$ given by (2.3) and (2.4) such that

$$1 + \frac{1}{b}\left(\frac{\zeta(\mathcal{Q}_{\delta,\lambda}^n f(\zeta))'}{\mathcal{Q}_{\delta,\lambda}^n f(\zeta)} - 1\right) = \phi(\zeta)$$

and

$$1 + \frac{1}{b}\left(\frac{\tau(\mathcal{Q}_{\delta,\lambda}^n g(\tau))'}{\mathcal{Q}_{\delta,\lambda}^n g(\tau)} - 1\right) = \phi(\tau).$$

Since

$$1 + \frac{1}{b}\left(\frac{\zeta(\mathcal{Q}_{\delta,\lambda}^n f(\zeta))'}{\mathcal{Q}_{\delta,\lambda}^n f(\zeta)} - 1\right) = 1 + \frac{1}{b}[M_3 a_2]\zeta + \frac{1}{b}[M_1 a_3 - M_2 a_2^2]\zeta^2 + \cdots \tag{2.5}$$

and

$$1 + \frac{1}{b}\left(\frac{\tau(\mathcal{Q}_{\delta,\lambda}^n g(\tau))'}{\mathcal{Q}_{\delta,\lambda}^n g(\tau)} - 1\right) = 1 - \frac{1}{b}[M_3 a_2]\tau + \frac{1}{b}[-M_1 a_3]\tau^2 + \frac{1}{b}[(2M_1 - M_2)a_2^2]\tau^2 + \cdots. \tag{2.6}$$

It follows from (2.1), (2.2), (2.5) and (2.6), equating coefficient of ζ:

$$\frac{1}{b}[M_3]a_2 = \frac{1}{2}A_1 s_1. \tag{2.7}$$

Equating coefficient of ζ^2:

$$\frac{1}{b}[2M_1 a_3 - M_2 a_2^2] = \frac{1}{2}A_1\left(s_2 - \frac{s_1^2}{2}\right) + \frac{1}{4}A_2 s_1^2. \tag{2.8}$$

Equating coefficient of τ:

$$-\frac{1}{b}[M_3]a_2 = \frac{1}{2}A_1 t_1. \tag{2.9}$$

Equating coefficient of τ^2:

$$\frac{1}{b}[-2M_1 a_3 + 4M_1 - (1+\delta)^{2n} a_2^2] = \frac{1}{2}A_1\left(t_2 - \frac{t_1^2}{2}\right) + \frac{1}{4}A_2 t_1^2. \tag{2.10}$$

Adding (2.7) and (2.9), we have

$$s_1 = -t_1.$$

Squaring and adding the equations (2.7) and (2.9), we get

$$a_2^2 = \frac{b^2 A_1^2 (s_1^2 + t_1^2)}{8M_2}. \tag{2.11}$$

Substitute (2.8) in (2.10), we get

$$a_2^2 = \frac{2bA_1(s_2 + t_2) + b(A_2 - A_1)(s_1^2 + t_1^2)}{8[2M_1 - (1+\delta)^{2n}]}. \tag{2.12}$$

Subtracting (2.8) and (2.10), we obtain

$$a_3 = a_2^2 + \frac{2bA_1(s_2 - t_2) + b(A_2 - A_1)(s_1^2 - t_1^2)}{16M_1}. \tag{2.13}$$

Substitute (2.11) in (2.13)

$$a_3 = \frac{b^2 A_1^2 (s_1^2 + t_1^2)}{8M_2} + \frac{2bA_1(s_2 - t_2) + b(A_2 - A_1)(s_1^2 - t_1^2)}{16M_1}.$$

Substitute (2.12) in (2.13)

$$a_3 = \frac{2bA_1(s_2 + t_2) + b(A_2 - A_1)(s_1^2 + t_1^2)}{8[2M_1 - M_2]} + \frac{2bA_1(s_2 - t_2) + b(A_2 - A_1)(s_1^2 - t_1^2)}{16M_1}.$$

Hence

$$|a_2| \leq \sqrt{\frac{2|b|A_1(s_2 + t_2) + |b|(A_2 - A_1)(s_1^2 + t_1^2)}{8[2M_1 - M_2]}}$$

and

$$|a_3| \leq \frac{2|b|A_1(s_2 + t_2) + |b|(A_2 - A_1)(s_1^2 + t_1^2)}{8[2M_1 - M_2]} + \frac{2|b|A_1(s_2 - t_2) + |b|(A_2 - A_1)(s_1^2 - t_1^2)}{16M_1}.$$

Hence the proof is now completed.

2.2 *Coefficient Estimation for the class* $\mathcal{L}_{\delta,\lambda}^{\Sigma}(\varphi)$

If we set

$$\varphi(\zeta) = 1 + A_1\zeta + A_2\zeta^2 + A_3\zeta^3 + \cdots, (A_1 > 0).$$

Since $\varphi(0) = 1, \varphi'(0) > 0$.

Let assume $u(\zeta)$ as well as $v(\zeta)$ are analytical about $\mathbb{U}$ with $u(0) = v(0) = 0$, $|u(\zeta)| < 1, |v(\tau)| < 1$, and

$$u(\zeta) = s_1\zeta + \sum_{k=2}^{\infty} s_k\zeta^k,$$
$$v(\tau) = t_1\tau + \sum_{k=2}^{\infty} q_k\tau^k.$$

It is widely acknowledged that

$$|s_1| \leq 1, |s_2| \leq 1 - |s_1|^2, |t_1| \leq 1, |t_2| \leq 1 - |t_1|^2.$$

A brief computation yields the following:

$$\varphi(u(\zeta)) = 1 + A_1 s_1\zeta + (A_1 s_2 + A_2 s_1^2)\zeta^2 + \cdots, |\zeta| < 1$$

and

$$\varphi(v(\tau)) = 1 + A_1 t_1\tau + (A_1 t_2 + A_2 t_1^2)\tau^2 + \cdots, |\tau| < 1.$$

Definition 2.3: If $f \in \mathcal{L}_{\delta,\lambda}^{\Sigma}(\varphi)$, then the subsequent subordinations are hold:

$$1+\frac{1}{b}\left(\frac{\zeta(\mathcal{Q}_{\delta,\lambda}^{n}f(\zeta))'}{\mathcal{Q}_{\delta,\lambda}^{n}f(\zeta)}-1\right)\prec\varphi(\zeta)$$

and

$$1+\frac{1}{b}\left(\frac{\tau(\mathcal{Q}_{\delta,\lambda}^{n}g(\tau))'}{\mathcal{Q}_{\delta,\lambda}^{n}g(\tau)}-1\right)\prec\varphi(\tau).$$

Theorem 2.4: *If* $f(\zeta)\in\mathcal{L}_{\delta,\lambda}^{\Sigma}(\varphi)$, *then*

$$|a_2|\leq\sqrt{\frac{|b|[A_1(s_2+t_2)+A_2(s_1^2+t_1^2)]}{2[2M_1-M_2]}}$$

and

$$|a_3|\leq\frac{|b|A_1(s_2+t_2)+|b|A_2(s_1^2+t_1^2)}{2[2M_1-M_2]}+\frac{|b|A_1(s_2-t_2)+|b|A_2(s_1^2-t_1^2)}{4M_1}.$$

2.3 *Coefficient Estimation for the class* $\mathcal{M}_{\delta,\lambda}^{\Sigma}(\psi)$

If we set

$$\psi(\zeta)=1+s_1\zeta+s_2\zeta^2+s_3\zeta^3+\cdots\ (s_1>0)$$

and

$$\psi(\tau)=1+t_1\tau+t_2\tau^2+t_3\tau^3+\cdots,(t_1>0).$$

Definition 2.5: If $f\in\mathcal{M}_{\delta,\lambda}^{\Sigma}(\psi)$, then the subsequent subordinations are hold:

$$1+\frac{1}{b}\left(\frac{\zeta(\mathcal{Q}_{\delta,\lambda}^{n}f(\zeta))'}{\mathcal{Q}_{\delta,\lambda}^{n}f(\zeta)}-1\right)\prec\psi(\zeta)$$

and

$$1+\frac{1}{b}\left(\frac{\tau(\mathcal{Q}_{\delta,\lambda}^{n}g(\tau))'}{\mathcal{Q}_{\delta,\lambda}^{n}g(\tau)}-1\right)\prec\psi(\tau).$$

Theorem 2.6: *If* $f(\zeta)\in\mathcal{M}_{\delta,\lambda}^{\Sigma}(\psi)$, *then*

$$|a_2|\leq\sqrt{\frac{|b|(s_2+t_2)}{2[2M_1-M_2]}}$$

and

$$|a_3|\leq\frac{|b|(s_2+t_2)}{2[2M_1-M_2]}+\frac{|b|(s_2-t_2)}{4M_1}.$$

2.4 *Coefficient Estimation for the class* $\mathcal{M}_{\delta,\lambda,\alpha}^{\Sigma}(\phi)$

If we set

$$\psi(\zeta)=1+s_1\zeta+s_2\zeta^2+s_3\zeta^3+\cdots,(s_1>0)$$

and

$$\begin{aligned}\psi(\tau)&=1+t_1\tau+t_2\tau^2+t_3\tau^3+\cdots,(t_1>0)\\ \phi(\zeta)&=(\psi(\zeta))^{\alpha}\\ &=1+\alpha s_1\zeta+\left(\alpha s_2+\frac{\alpha(\alpha-1)}{2}s_1^2\right)\zeta^2+\cdots\end{aligned}$$

$$\begin{aligned}\phi(\tau) &= (\psi(\tau))^{\alpha} \\ &= 1+\alpha t_1\tau+\left(\alpha t_2+\frac{\alpha(\alpha-1)}{2}t_1^2\right)w^2+\cdots\end{aligned}$$

Definition 2.7: If $f \in \mathcal{M}_{\delta,\lambda,\alpha}^{\Sigma}(\phi)$, then the subsequent subordinations are hold:

$$1+\frac{1}{b}\left(\frac{\zeta(\mathcal{Q}_{\delta,\lambda}^{n}f(\zeta))'}{\mathcal{Q}_{\delta,\lambda}^{n}f(\zeta)}-1\right) \prec \phi(\zeta)$$

and

$$1+\frac{1}{b}\left(\frac{\tau(\mathcal{Q}_{\delta,\lambda}^{n}g(\tau))'}{\mathcal{Q}_{\delta,\lambda}^{n}g(\tau)}-1\right) \prec \phi(\tau).$$

Theorem 2.8: *If* $f(\zeta) \in \mathcal{M}_{\delta,\lambda,\alpha}^{\Sigma}(\phi)$, *then*

$$|a_2| \leq \sqrt{\frac{2|b|\alpha(s_2+t_2)+|b|\alpha(\alpha-1)(s_1^2+t_1^2)}{4[2M_1-M_2]}}$$

and

$$|a_3| \leq \frac{2|b|\alpha(s_2+t_2)+|b|\alpha(\alpha-1)(s_1^2+t_1^2)}{4[2M_1-M_2]}+\frac{2|b|\alpha(s_2-t_2)+|b|\alpha(\alpha-1)(s_1^2-t_1^2)}{8M_1}.$$

2.5 *Coefficient Estimation for the class* $\mathcal{L}_{\delta,\lambda,\sigma}^{\Sigma}(\phi)$

If we set

$$\phi(\zeta)=\sigma+(1-\sigma)\psi(\zeta)=1+(1-\sigma)s_1\zeta+(1-\sigma)s_2\zeta^2+\cdots$$

and

$$\phi(\tau)=\sigma+(1-\sigma)\psi(\tau)=1+(1-\sigma)t_1\tau+(1-\sigma)t_2\tau^2+\cdots;\ 0\leq\sigma<1.$$

Definition 2.9: If $f \in \mathcal{L}_{\delta,\lambda,\sigma}^{\Sigma}(\phi)$, then the subsequent subordinations are hold:

$$1+\frac{1}{b}\left(\frac{\zeta(\mathcal{Q}_{\delta,\lambda}^{n}f(\zeta))'}{\mathcal{Q}_{\delta,\lambda}^{n}f(\zeta)}-1\right) \prec \phi(\zeta)$$

and

$$1+\frac{1}{b}\left(\frac{\tau(\mathcal{Q}_{\delta,\lambda}^{n}g(\tau))'}{\mathcal{Q}_{\delta,\lambda}^{n}g(\tau)}-1\right) \prec \phi(\tau).$$

Theorem 2.10: *If* $f(\zeta) \in \mathcal{L}_{\delta,\lambda,\sigma}^{\Sigma}(\phi)$, *then*

$$|a_2| \leq \sqrt{\frac{|b|(1-\sigma)(s_2+t_2)}{2[2M_1-M_2]}}$$

and

$$|a_3| \leq \frac{|b|(1-\sigma)(s_2+t_2)}{2[2M_1-M_2]}+\frac{|b|(1-\sigma)(s_2-t_2)}{4M_1}.$$

2.6 *Coefficient Estimation for the class* $\mathcal{M}_{\delta,\lambda,\sigma}^{\Sigma}(\phi)$

If we set

$$\phi(\zeta)=\frac{1+(1-2\sigma)\zeta}{1-\zeta}=1+2(1-\sigma)\zeta+2(1-\sigma)^2\zeta^2+\cdots$$

and

$$\phi(\tau) = \frac{1+(1-2\sigma)\tau}{1-\tau} = 1 + 2(1-\sigma)\tau + 2(1-\sigma)^2\tau^2 + \cdots;\ 0 \le \sigma < 1.$$

Definition 2.11: If a function $f \in \mathcal{M}^{\Sigma}_{\delta,\lambda,\sigma}(\phi)$, then the subsequent subordinations are hold:

$$1 + \frac{1}{b}\left(\frac{\zeta(\mathcal{Q}^n_{\delta,\lambda}f(\zeta))'}{\mathcal{Q}^n_{\delta,\lambda}f(\zeta)} - 1\right) \prec \phi(\zeta)$$

and

$$1 + \frac{1}{b}\left(\frac{\tau(\mathcal{Q}^n_{\delta,\lambda}g(\tau))'}{\mathcal{Q}^n_{\delta,\lambda}g(\tau)} - 1\right) \prec \phi(\tau).$$

Theorem 2.12: *If* $f(\zeta) \in \mathcal{M}^{\Sigma}_{\delta,\lambda,\sigma}(\phi)$, *then*

$$|a_2| \le \sqrt{\frac{2|b|(1-\sigma)^2}{2M_1 - M_2}}$$

and

$$|a_3| \le \frac{2|b|(1-\sigma)^2}{2M_1 - M_2}.$$

2.7 *Coefficient Estimation for the class* $\mathcal{L}^{\Sigma}_{\delta,\lambda,\alpha}(\phi)$

If we set

$$\phi(\zeta) = \left(\frac{1+\zeta}{1-\zeta}\right)^{\alpha} = 1 + 2\alpha\zeta + 2\alpha^2\zeta^2 + \cdots$$

and

$$\phi(\tau) = \left(\frac{1+\tau}{1-\tau}\right)^{\alpha} = 1 + 2\alpha\tau + 2\alpha^2\tau^2 + \cdots;\ 0 < \alpha \le 1.$$

Definition 2.13: If $f \in \mathcal{L}^{\Sigma}_{\delta,\lambda,\alpha}(\phi)$, then the subsequent subordinations are hold:

$$1 + \frac{1}{b}\left(\frac{\zeta(\mathcal{Q}^n_{\delta,\lambda}f(\zeta))'}{\mathcal{Q}^n_{\delta,\lambda}f(\zeta)} - 1\right) \prec \phi(\zeta)$$

and

$$1 + \frac{1}{b}\left(\frac{\tau(\mathcal{Q}^n_{\delta,\lambda}g(\tau))'}{\mathcal{Q}^n_{\delta,\lambda}g(\tau)} - 1\right) \prec \phi(\tau).$$

Theorem 2.14: *If* $f(\zeta) \in \mathcal{L}^{\Sigma}_{\delta,\lambda,\alpha}(\phi)$, *then*

$$|a_2| \le \sqrt{\frac{2|b|\alpha^2}{2M_1 - M_2}}$$

and

$$|a_3| \le \frac{2|b|\alpha^2}{2M_1 - M_2}.$$

3. Conclusion

In this current paper two initial coefficients $|a_2|$ and $|a_3|$ are estimated for the sub-class of bi-univalent function. Furthermore, results of our work motivates to extend certain new subclasses of meromorphic functions related with q-calculus of bi-univalent functions. Also the researchers to extend into the famous inequalities like second, third Hankel determinants and Fekete szego inequality . It has many uses in science, engineering, and other related fields like control theory as well as signal processing, of course.

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Conflict of interest

There are no conflicts of interest, according to the authors.

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