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Categorical accommodation of various notions in generalized multidimensional fuzzy sets

Jomal Josen [§]
Department of Mathematics
St. Thomas College
Palai 686574
Kerala
India

Sunil Jacob John [†]
Department of Mathematics
National Institute of Technology Calicut
Calicut 673601
Kerala
India

Jobish Vallikkavungal *
School of Engineering and Sciences
Tecnologico de Monterrey
Monterrey 64849
Nuevo Leon
Mexico

Abstract

This paper investigates the categorical framework for generalized multi-dimensional fuzzy sets, extending Zadeh's original theory. We introduce a generalized variant where the co-domain is a partially ordered set. Fundamental categorical notions such as initial and terminal objects, separators, co-separators, and various morphisms (e.g., mono-, epi-, iso-, section, and retraction) are explored. We prove the category is complete and co-complete

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[§] *ORCID-ID:* **0009-0009-5450-120X**
[†] *ORCID-ID:* **0000-0002-6333-2884**
* *ORCID-ID:* **0000-0003-4238-5829**

[§] *E-mail:* **jomaljosen@gmail.com**
[†] *E-mail:* **sunil@nitc.ac.in**
* *E-mail:* **jobish03@tec.mx (Corresponding Author)**

by identifying equalizers, co-equalizers, products, and co-products using new union and intersection operations for these sets.

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1. Introduction

Over recent decades, various mathematical tools have been introduced to handle uncertainty, with fuzzy sets [1] being the most prominent. Although fuzzy sets address many real-world scenarios, they often fall short in representing complex, ambiguous situations. This limitation has led to extensions such as Interval-Valued fuzzy sets [2], Hesitant fuzzy sets [3], Picture fuzzy sets [4], m-polar fuzzy sets [5], Intuitionistic fuzzy sets [6], and n-dimensional fuzzy sets [7], among others. These models are widely used in fields like medicine, engineering, finance, and data analysis. However, they generally fix the number of membership values per element, limiting their ability to represent varying degrees of ambiguity across individuals or attributes.

n-dimensional fuzzy sets assign to each element a vector in $\mathcal{J}_n([0,1])$, a sorted subset of $[0,1]^n$. This uniformity fails to reflect the differing uncertainties of elements. Real-world data often exhibits variability in the number of values required to model ambiguity, motivating the development of Multi-Dimensional Fuzzy Sets (MDFS), as introduced in [8]. In an MDFS, elements can have any number of membership values in $[0,1]$, sorted in increasing order, allowing a flexible representation of uncertainty. This adaptability makes MDFS suitable for applications such as academic assessments or medical diagnosis, where elements may require differing numbers of values to reflect their ambiguity.

Compared to other generalizations such as Type-2 fuzzy sets [9, 10], Genuine sets [11, 12], and Hesitant fuzzy sets [13], the MDFS framework is more intuitive and effective for modeling varied uncertainty. Its basis in the partially ordered set $\mathcal{J}_\infty([0,1])$ removes the rigid constraints on dimensionality found in n-dimensional or m-polar fuzzy sets. For instance, fixed-length sets disallow values like $(0.4,0.5,0.6)$ if $m = 2$, which can lead to information loss or forced simplification. MDFS avoids such issues by allowing element-specific cardinality.

Substantial studies have expanded the theory of MDFS. In [14], multidimensional t-norms, complements, and entropy measures are developed, along with similarity and distance measures useful for decision-making. The concept of a multidimensional topological measure space–combining topological structure with fuzzy distance–is also introduced, and further properties like continuity and homeomorphic isometry are analyzed. A novel integration of rough sets and MDFS for improved approximation has been presented in [15].

To better understand the structural behavior of MDFS, category theory provides a powerful lens. Formalized by Mac Lane and Eilenberg [16], category theory abstracts mathematical structures through objects and morphisms. It plays a vital role across mathematics and theoretical computer science, especially through constructs like natural transformations and adjoint functors. The categorization of MDFS is significant because many other fuzzy structures (e.g., interval-valued and n-dimensional fuzzy sets) can be interpreted as particular cases of MDFS.

This paper proposes an enriched formulation of MDFS by redefining its domain with a more general partially ordered set and introducing a refined partial order that extends beyond the natural order of $\mathcal{J}_\infty([0,1])$. All MDFS instances are then reclassified under this updated structure. We explore fundamental categorical constructs, including initial and terminal objects, zero objects, separators, co-separators, and specialized morphisms such as monomorphisms, epimorphisms, and isomorphisms. Furthermore, we establish the completeness and co-completeness of the resulting category through the identification of equalizers, co-equalizers, products, and co-products.

Earlier studies on the categorization of fuzzy sets by authors such as [17, 18] laid the groundwork, modeling fuzzy sets as pairs (R, G) with morphisms defined under specific order conditions. Later developments, including [19, 20, 21, 22, 23, 24, 25, 26], further expanded fuzzy category theory. Building on these, our study provides a categorical framework specifically tailored to MDFS, contributing a novel perspective to the broader theory of fuzzy structures.

2. Multidimensional fuzzy sets

Definition 1: Let V be a set, and given $q: V \to \mathbb{N}$. We define the set $\mathcal{M} = \{(r, \psi_{\mathcal{M}}^1(r), \dots, \psi_{\mathcal{M}}^{q(r)}(r)) | r \in V\}$ where $\psi_{\mathcal{M}}^i: V \to [0,1]$ satisfies $\psi_{\mathcal{M}}^1(r) \leq \cdots \leq \psi_{\mathcal{M}}^{q(r)}(r)$ for all $i = 1,2,\cdots,q(r)$ and for each $r \in V$. Then the set $\mathcal{M}$ is called a Multi-Dimensional Fuzzy Set (MDFS) on V.

To describe the MDFS as a function, let $\mathcal{J}_\infty([0,1])$=$\bigcup_{n=1}^{\infty} \mathcal{J}_n([0,1])$ where $\mathcal{J}_n([0,1]) = \{(r_1, \dots, r_n) \in [0,1]^n | r_1 \leq r_2 \leq \cdots \leq r_n\}$.Now, let $\mathcal{S}$ from $\mathcal{J}_\infty([0,1])$ denotes an arbitrary membership value of a MDFS , we define the cardinality of $\mathcal{S}$ for doing further studies on MDFS. If $\mathcal{S} \in \mathcal{J}_\infty([0,1])$ then $|\mathcal{S}|$ denote the natural number $n \in \mathbb{N}$ such that $\mathcal{S} \in \mathcal{J}_n([0,1])$ called cardinality of $\mathcal{S}$. Let $r \in V$ then for simplicity $\mathcal{M}_i(r)$ denotes the i^{th} component of $\mathcal{M}(r)$.

In order to give an ordering for MDFS a partial ordering of $\mathcal{J}_\infty([0,1])$ is required. The natural ordering (less than or equal) of $[0,1]$ is extended to $\mathcal{J}_n([0,1])$ as $\leq_n^p$ (product order) by: $\mathcal{S} \leq_n^p \mathcal{V} \Leftrightarrow s_i \leq v_i \ \forall i = 1,2...n$. where: $\mathcal{S} = \{s_1, s_2, \dots s_n\} \in \mathcal{J}_n([0,1])and\mathcal{V} = \{v_1, v_2, \dots v_n\} \in \mathcal{J}_n([0,1])$. This partial order on $\mathcal{J}_n([0,1])$ can be extended to $\mathcal{J}_\infty([0,1])$ by $\leq_\infty$ (say less than or equal) as: For $\mathcal{S}, \mathcal{V} \in \mathcal{J}_\infty([0,1])$ $\mathcal{S} \leq_\infty \mathcal{V} \Leftrightarrow |\mathcal{S}| = |\mathcal{V}| = n, \mathcal{S} \leq_n^p \mathcal{V}$. Now consider $\mathcal{M}_1$ and $\mathcal{M}_2$ are two MDFS, then we say that $\mathcal{M}_1 \subseteq \mathcal{M}_2$ ($\mathcal{M}_1$ subset $\mathcal{M}_2$)if $\mathcal{M}_1(r) \leq_\infty \mathcal{M}_2(r)$ for each r.

3. Category theory and morphisms

Definition 2: [27] A category is a structure of the form $\mathcal{C} = (\mathcal{O}, \mathbb{M}, \circ)$ which contains:

(i) $\mathcal{O}$, which is a class whose members are called objects.

(ii) For each pair P, Q of objects, a set $\overline{Mor(P,Q)} \subseteq \mathbb{M}$ whose members are called morphism from P to Q.

(iii) For each object P, a morphism $I_P \in \overline{Mor(P,P)}$ called the identity on P.

(iv) A composition operator connected with with each morphism $P \xrightarrow{\phi} Q$ and each morphism $Q \xrightarrow{\theta} R$, a morphism $P \xrightarrow{\phi \circ \theta} R$ called the composite of ϕ and θ, which satisfies the following:

  (a) For $P \xrightarrow{\phi} Q, Q \xrightarrow{\theta} R$ and $R \xrightarrow{\psi} S$, $\psi \circ (\theta \circ \phi) = (\psi \circ \theta) \circ \phi$

  (b) For morphism $P \xrightarrow{\phi} Q$ we have $I_Q \circ \phi = \phi$ and $\phi \circ I_P = \phi$

  (c) $\overline{Mor(P,Q)}$ are pairwise disjoint.

Definition 3: The initial object, denoted as P, is defined as an object where there exists precisely one morphism from P to any other object Q. Conversely, the terminal object, represented as R, is an object where there exists exactly one morphism from any other object S to R. A zero object is an object that fulfills both initial and terminal conditions simultaneously.

The notions of separators, co-separators, monomorphisms, epimorphisms, equalizers and co-equalizers used in this work are standard in category theory. We refer the reader to [28] for formal definitions and foundational discussion.

It is important to note that a regular monomorphism is a morphism that arises as the equalizer of a specific pair of morphisms. Likewise, a regular epimorphism is a morphism that arises as the coequalizer of some pair of morphisms.

The definitions of products, coproducts, pullbacks, and pushouts used in this work follow the standard categorical framework. For precise formulations, we refer the reader to [28]. The pullback has the property that for any object S' and morphisms $\psi': S' \to P$ and $\theta': S' \to Q$ satisfying $\phi \circ \psi' = \theta \circ \theta'$, there exists a unique morphism $f: S' \to S$ such that $\psi \circ f = \psi'$ and $\theta \circ f = \theta'$. Conversely, the pushout also satisfies the property that for any object S' and morphisms $\psi': P \to S'$ and $\theta': Q \to S'$ satisfying $\psi' \circ \phi = \theta' \circ \theta$, there exists a unique morphism $f: S \to S'$ such that $f \circ \psi = \psi'$ and $f \circ \theta = \theta'$.

3.1 *Category of fuzzy sets*

A fuzzy set on a crisp set P is viewed as a function $P \to [0,1]$ [1]. In fuzzy category theory, such sets are objects, and morphisms are functions linking them. Goguen [18] defined the fuzzy category $\mathbb{F}$ using a poset $(V, \leq)$, where objects are pairs (P, G), with $G: P \to V$. A morphism $\phi: P \to Q$ satisfies $G(p) \leq H(\phi(p))$ for all $p \in P$. Equalizers and co-equalizers were studied in [29]. If V

is a singleton, $\mathbb{F}$ reduces to the category of sets; if $V = [0,1]$, it becomes the classical fuzzy category. A detailed study appears in [30].

4. Generalized multidimensional fuzzy sets and their category

We present the Generalized Multi-Dimensional Fuzzy Sets (GMDFS) and their associated category to generalize MDFS's results. Instead of using $[0,1]$ and a new partial ordering for the membership values of GMDFS, we use an arbitrary lattice U. Consider a lattice $(U, \leq_U)$ with a least element $\dot{0}$ and a greatest element $\dot{1}$. Define $\mathcal{I}_n(U)$ and $\mathcal{I}_\infty(U)$ as: $\mathcal{I}_n(U) = \{(u_1, u_2, \dots u_n) \in U^n: u_1 \leq_U u_2 \leq_U \cdots \leq_U u_n\}$ and $\mathcal{I}_\infty(U) = \bigcup_{n=1}^{\infty} \mathcal{I}_n(U)$.

For now, let $\overline{1} = \{(\dot{1}, \dot{1}, \dots, \dot{1})_m, m \in \mathbb{N}\}$ and $\overline{0} = \{(\dot{0}, \dot{0}, \dots \dot{0})_m, m \in \mathbb{N}\}$. Since the membership values $(\dot{1}, \dot{1}, \dots, \dot{1})_m$ and $(\dot{1}, \dot{1}, \dots, \dot{1})_n$ for $n \neq m$ convey the same information, we denote the elements in the set $\overline{1}$ as $\hat{1}$ and those in $\overline{0}$ as $\hat{0}$.

Now, consider a new lattice $\mathcal{J}_\Delta(U)$ defined as: $\mathcal{J}_\Delta(U)\{\hat{0}, \hat{1}\} \cup \mathcal{I}_\infty(U) \backslash (\overline{0} \cup \overline{1})$. This lattice is equipped with a new partial ordering $\leq_\Delta$ as follows:

- If $P \in \mathcal{J}_\Delta(U)$, then $\hat{0} \leq_\Delta P$ and $P \leq_\Delta \hat{1}$.
- For $P, Q \in \mathcal{I}_\infty(U) \backslash (\overline{0} \cup \overline{1})$, $P \leq_\Delta Q$ if and only if $|P| \leq |Q|$ and $p_i \leq_U q_i$ for $i = 1,2 \dots n$, where $P = (p_1, p_2, \dots p_n)$ and $Q = (q_1, q_2, \dots q_m)$.

Thus, we can define the generalized multidimensional fuzzy set as follows:

Definition 4: Let S be a set. A Generalized Multi-Dimensional Fuzzy Set (GMDFS) is defined as a function $\mathcal{M}: S \to \mathcal{J}_\Delta(U)$, where U is a lattice with least and greatest elements.

Definition 5: Consider the lattice U, which includes both the least and greatest elements. Then, the category of GMDFS over U, denoted by $\mathcal{G}_{mfs}(U)$, can be described as a structure $(\mathcal{O}, \mathbb{M}, \circ)$, where $\mathcal{O}$ represents the class of all objects G_P (indicating a function $G: P \to \mathcal{J}_\Delta(U)$). For any two objects, G_P and H_Q, a multimorphism (representing morphism between objects here) from G_P to H_Q which is defined as a function $\phi: P \to Q$ with the condition $G(p) \leq_\Delta H(\phi(p))$ for every $p \in P$.

In this context, $\mathbb{M}$ represents the set of all multimorphisms between objects. When considering two multimorphisms $\phi: G_P \to H_Q$ and $\theta: H_Q \to I_R$, their composition, denoted as $\theta \circ \phi$, is defined following the conventional rules of function composition. The identity function, denoted as I, serves as the multimorphism that satisfies the properties: $\text{Forany}\phi: P \to Q, I_Q \circ \phi = \phi \text{and} \phi \circ I_P = \phi$ Given two multimorphisms $\phi: G_P \to H_Q$ and $\theta: H_Q \to I_R$, their composition $\theta \circ \phi$ is also a multimorphism, mapping from G_P to I_R. To illustrate this property: If $\phi: G_P \to H_Q$ and $\theta: H_Q \to I_R$ are two multimorphisms then we have: $G(p) \leq_{\Delta} H(\phi(p)), \forall p \in P$ and $H(q) \leq_{\Delta} I(\theta(q)), \forall q \in Q$ $H(\phi(p)) \leq_{\Delta} I(\theta(\phi(p))), \forall p \in P$

This leads to: $G(p) \leq_{\Delta} I(\theta(\phi(p))), \forall p \in P$ and hence $\theta \circ \phi$ is a multimorphism from G_P to I_R. The remaining conditions for $\mathcal{G}_{mfs}(U)$=($\mathcal{O}, \mathbb{M}$,∘) to be a category can be easily verified. Hence, the collection of all GMDFS, together with multimorphisms and their compositions, forms a category. This leads to Theorem 1.

Theorem 1: *The structure* $\mathcal{G}_{mfs}(U)$ =($\mathcal{O}, \mathbb{M}$,∘), *in Definition 5, is a category.*

An initial object within the category $\mathcal{G}_{mfs}(U)$ may not be unique. However, the following theorem establishes conditions for initial object:

Theorem 2: *An object G_P of $\mathcal{G}_{mfs}(U)$ is an initial object if and only if P is empty.*

Proof: Consider the scenario where P represents an empty set. In this case, the only possible function from P is the empty function. Consequently, this condition is sufficient for P to be considered an initial object.

Conversely, suppose there exists only one multimorphism from G_P to any other object H_Q. Let's assume P is non-empty for contradiction. Take H_Q as the object with $Q = \{q_1, q_2\}$ and $H(q_1) = H(q_1) = \hat{1}$. Now, for any multimorphism $\phi: P \to Q$, each $p \in P$ presents two choices. Consequently, there exist more than one multimorphism from G_P to H_Q. Therefore, P must indeed be empty.

Theorem 3: *An object H_Q of $\mathcal{G}_{MFS}(U)$ is a terminal object if and only if Q consists of a single element, denoted as $\{q\}$, and $H(q) = \hat{1}$.*

Proof: When Q is a singleton set, there exists only one function from P to Q for any object G_P. Consequently, there is precisely one multimorphism to H_Q, as any function to H_Q qualifies as a multimorphism.

Conversely, if Q is not a singleton set, consider the object G_P with the property that $G(p) = \hat{0}$ for every $p \in P$. By the definition of G_P, any function from G_P to H_Q is considered a multimorphism. Additionally, since each element p in P has two possible choices in Q, which leads to the existance of more than one multimorphism from G_P to H_Q.

5. Properties of multimorphisms

A separator object O distinguishes morphisms $\phi, \theta: P \to Q$ via a morphism $s: O \to P$ such that $\phi \circ s \neq \theta \circ s$. A co-separator O_c similarly distinguishes morphisms using $c: Q \to O_c$ with $c \circ \phi \neq c \circ \theta$. se and cose state the necessary and sufficient conditions for objects to be separators and co-separators.

Theorem 4: *To define an object G_P as a separator, it is necessary and sufficient for the set P to be non-empty and for $G(p) = \hat{0}$, for every $p \in P$.*

Proof: Let H_Q and I_R represent two objects, and let θ and ψ be two distinct multimorphisms from H_Q to I_R. It follows that there exists an element $q \in Q$ such that $\theta(q) \neq \psi(q)$.

Consider the object G_P, and define a multimorphism $\phi: G_P \to H_Q$ as $\phi(p) = q$ for all $p \in P$. Clearly, ϕ is a multimorphism, and for every $p \in P$, we have $\theta \circ \phi(p) = \theta(q) \neq \psi(q) = \psi \circ \phi(p)$. Thus, ϕ serves as a separator.

Conversely, suppose G_P functions as a separator. If there exists an $p \in P$ such that $G(p) \neq \hat{0}$, then there can be no multimorphism from G_P to H_Q, where H is the GMDFS with $H(q) = \hat{1}$ for all $q \in Q$. Hence, we conclude that $G(p) = \hat{0}$ for every $p \in P$.

Theorem 5: *A co-separator in a category denoted as I_R, exists if and only if there are two distinct elements r_1 & r_2 in set R such that $I(r_1) = I(r_2) = \hat{1}$.*

Proof: Assuming object I_R possesses the aforementioned properties. Let $\phi, \theta: P_G \to Q_H$ represent two distinct multimorphisms, with $\phi(p) \neq \theta(p)$ for some $p \in P$. Define $\psi: H_Q \to I_R$ as : $\psi(q) = r_1$ if $q = \phi(p)$ and $\psi(q) = r_2$ otherwise. Given the definition of I_R, it's evident that ψ is a multimorphism between H_Q and I_R. Now, $\psi \circ \phi(p) = r_1 \neq r_2 = \theta \circ \phi(p)$, indicating that I_R is a co-separator. Conversely, assume I_R is a co-separator:

Case 1: If $R = \{\}$, then there are no multimorphisms from H_Q to I_R.

Case 2: If $R = \{r\}$, then $\psi \circ \phi(p) = \theta \circ \phi(p)$ for any multimorphisms $\phi, \theta: P_G \to Q_B$ and $\psi: H_Q \to I_R$.

Thus, R must have at least 2 elements.

Now, consider $\psi: H_Q \to I_R$ as a co-separator for $\phi, \theta: P_G \to Q_B$. Then, there exists $p \in P$ such that $\psi \circ \phi(p) = r_1 \neq r_2 = \theta \circ \phi(p)$, where $r_1, r_2 \in R$. If we define G_P such that $G(p) = \hat{1}$ for every p, it clearly implies that $I(r_1) = I(r_2) = \hat{1}$. Hence I_R is a co-separator if and only if there are $r_1 \neq r_2$ with $I(r_1) = I(r_2) = \hat{1}$.

From the Theorem 4 and Theorem 5, we can observe that there is no object which is a separator and co-separator simultaneously. Now, for the notions of section, retraction, and isomorphism employed here are standard in category theory. For formal definitions and further discussion, see [28]. We now shift our focus to $the\ inverse\ object$ of our study. The left and right inverses of $\mathcal{G}_{MFS}(U)$, with respect to multidimensional fuzzy composition, are detailed in Theorem 6 and Theorem 7, respectively.

Theorem 6: *A multimorphism $\phi: G_P \to H_Q$ qualifies as a section under the following conditions: (i) ϕ is injective; (ii) For every $p \in P$, $G(p) = H(\phi(p))$; (iii) For every $q \in Q$, there exists at least one $p \in P$ such that $H(q) \leq_\Delta G(p)$.*

Proof: Let $\phi: G_P \to H_Q$ be a section. Then, there exists a multimorphism $\theta: H_Q \to G_P$ such that $\theta \circ \phi = I_P$.

Firstly, we establish the injectivity of ϕ. Suppose $\phi(p_1) = \phi(p_2)$. Then, $\theta \circ \phi(p_1) = \theta \circ \phi(p_2) = I_P(p_1) = I_P(p_2)$, implying $p_1 = p_2$. Thus, ϕ is injective.

Now, we have: $G(p) \leq_\Delta H(\phi(p))$ and $H(q) \leq_\Delta G(\theta(q))$ from θ definition. $H(\phi(p)) \leq_\Delta G(\theta(\phi(p))) = G(p)$ (by definition of θ)

Combining, we conclude that $G(p) = H(\phi(p))$ for every $p \in P$. For the final condition, substituting $\theta(q) = p$ into above equation, which leads to : $H(q) \leq_\Delta G(p)$.

Conversely, assume that $\phi: G_P \to H_Q$ satisfies all the conditions of the theorem. Then, we define $\theta: H_Q \to G_P$ as: $\theta(q) = \phi^{-1}(q)$ if $q \in \phi(P)$ and $\theta(q) = p \in P: H(q) \leq_\Delta G(p)$ otherwise. The θ is well-defined by the first and third properties of ϕ. To prove that θ is a multimorphism, assume $q \in \phi(P)$, then $\theta(q) = p$ where p satisfies $\phi(p) = q$. Hence, $G(\theta(q)) = G(p) = H(\phi(p)) = H(q)$. If $q \notin \phi(P)$, then by the definition of θ, we have $H(q) \leq_\Delta G(p) = G(\theta(q))$.

Thus, for every $q \in Q$, we have $H(q) \leq_\Delta G(\theta(q))$. Also, by the definition of θ, $\theta \circ \phi = I_P$ is satisfied.

Theorem 7: *A multimorphism $\phi: G_P \to H_Q$ is considered a retraction if and only if, for every $q \in Q$, there exists an $p \in P$ such that $G(p) = H(q)$, where $\phi(p) = q$.*

Proof: Let $\phi: G_P \to H_Q$ be a retraction. Consequently, there exists a multimorphism $\theta: H_Q \to G_P$ such that $\phi \circ \theta = I_Q$. This indicates that for any $q \in Q$, there exists $p = \theta(q)$ satisfying $\phi(p) = q$, thus demonstrating the surjectivity of ϕ. Utilizing the properties of ϕ and θ, we obtain:

$$G(p) \leq_\Delta H(\phi(p))\ \forall p \in P \text{ and } H(q) \leq_\Delta G(\theta(q))\ \forall q \in Q \tag{1}$$

By substituting $\phi(p) = q$ into the above equations, we derive: $G(p) \leq_\Delta H(q)$ and $H(q) \leq_\Delta G(\theta(\phi(p))) = G(p)$. Thus, G(p) = H(q).

Conversely, suppose ϕ possesses all the properties mentioned in the theorem. Then, define $\theta: Q \to P$ as $\theta(q) = p$, where $p \in P$ with $\phi(p) = q$ and $G(p) = H(q)$. Consequently, for any $q \in Q$, if $\theta(q) = p$, then $G(\theta(q)) = G(p) = H(q)$. From the definition of θ, it is evident that $\phi \circ \theta = I_P$, thus confirming that ϕ is a retraction.

Corollary 1: *A multimorphism $\phi: G_P \to H_Q$ is deemed an isomorphism if and only if ϕ is bijective, and for every $p \in P$, it holds that $G(p) = H(\phi(p))$.*

We already seen that, monomorphism and epimorphism play crucial roles in the context of canceling identical objects. The foundational definitions of monomorphism and epimorphism presented in ??. Termed as universal left and right cancelers in relation to composition, respectively, they offer valuable insights into morphism properties. Currently, our investigation focuses on the left and right cancelers within $\mathcal{G}_{MFS}(U)$. These cancelers correspond to the injective and surjective multimorphisms, as depicted in mo and ep.

Theorem 8: *A multimorphism, denoted as $\phi: G_P \rightarrow H_Q$, is deemed a monomorphism if and only if it is one-to-one.*

Proof: Assuming $\phi: G_P \rightarrow H_Q$ is a monomorphism, consider $R = \{r\}$ with $I(r) = \hat{0}$. If P is a singleton set, then ϕ is evidently injective. Now, suppose $p_1, p_2 \in P$ with $p_1 \neq p_2$. Define functions $\theta, \psi: R \rightarrow P$ such that $\theta(c) = p_1$ and $\psi(c) = p_2$. It is clear that θ and ψ are multimorphisms. We observe: $\phi(p_1) = \phi(p_2) \Rightarrow \phi(\theta(c)) = \phi(\psi(c)) \Rightarrow \theta(c) = \psi(c) \Rightarrow p_1 = p_2$. Thus, ϕ is injective.

Conversely, assume ϕ is injective. Let θ and ψ be two multimorphisms from R to P, such that $\phi \circ \theta = \phi \circ \psi$. Then, for $r \in R$, we have: $\phi(\theta(c)) = \phi(\psi(c)) \Rightarrow \theta(c) = \psi(c) \Rightarrow \theta = \psi$. Thus, ϕ is a monomorphism.

Theorem 9: *A multimorphism $\phi: G_P \rightarrow H_Q$ is considered an epimorphism if and only if ϕ is surjective, meaning it covers the entire co-domain.*

Proof: Let $\phi: G_P \rightarrow H_Q$ be an epimorphism, and let $R = \{r_1, r_2\}$ with $I(r) = \hat{1}$ for all $r \in R$. We define θ and ψ as mappings from Q to R as: $\theta(q) = r_1$ forall$q$, and $\psi(q) = r_1$if$q \in \phi(P)$, and $\psi(q) = r_2$ otherwise.

By the definition of I_R, both θ and ψ are multimorphisms to I_R. Furthermore, by the definition of θ and ψ, we have: $\phi \circ \theta = \phi \circ \psi \Rightarrow \theta = \psi \Rightarrow \theta(Q) = \psi(Q) \Rightarrow \{r_1\} = \theta(Q) = \psi(Q) = \psi(\phi(P)) \Rightarrow \phi(P) = Q$. Thus, ϕ is onto.

Conversely, assume ϕ is onto and $\phi \circ \theta = \phi \circ \psi$. For $q \in Q$, let $p \in P$ be such that $\phi(p) = q$. Thus, $\theta(q) = \theta(\phi(p)) = \psi(\phi(p)) = \psi(q)$. Hence, $\theta = \psi$, and ϕ is an epimorphism.

5.1 *Pullbacks and Pushouts in category of Generalized MDFS*

In this section, we focus on the construction of Pullbacks and Pushouts within the context of $\mathcal{G}_{MFS}(U)$. This construction relies on the standard multidimensional t-norm and t-conorm, along with associated properties [14].

Theorem 10: *Every multimorphism $\phi: G_P \rightarrow I_R$ and $\theta: H_Q \rightarrow I_R$ in the $\mathcal{G}_{MFS}(U)$ possesses a pullback.*

Proof: Consider two multimorphisms, $\phi: G_P \rightarrow I_R$, and $\theta: H_Q \rightarrow I_R$. Let $S = \{(p, q) \in P \times Q: \phi(p) = \theta(q)\}$, and define a GMDFS, denoted by J, on S as $J(p, q) = \overline{min}(G(p), H(q))$, where $\overline{min}$ represents the standard multidimensional t-norm [14]. Now, define two multimorphisms, $\psi: J_S \rightarrow G_P$, and $\theta: J_S \rightarrow H_Q$, as: $\psi(p, q) = p$ and $\theta(p, q) = q$.

Clearly, both ψ and θ are multimorphisms since: $J(p,q) = \overline{min}(G(p), H(q)) \leq_\Delta G(p) = G(\psi(p,q))$ and $J(p,q) = \overline{min}(G(p), H(q)) \leq_\Delta H(q) = H(\theta(p,q))$.

For any $(p,q) \in S$, we have: $\phi(\psi(p,q)) = \phi(p) = \theta(q) = \theta(\theta(p,q))$. Let's assume that $\psi': E_V \to G_P$ and $\theta': E_V \to H_Q$ are two multimorphisms such that $\phi \circ \psi' = \theta \circ \theta'$. Then, define $f: V \to S$ by: $f(v) = (\psi'(v), \theta'(v))$. Clearly, $f(v) \in S$ since $\phi(\psi'(v)) = \theta(\theta'(v))$. To prove that f is a multimorphism between E_V and J_S, we utilize the result from [14] that: $E(v) \leq_\Delta G(\psi'(v))$ and $E(v) \leq_\Delta H(\theta'(v))$ $E(v) \leq_\Delta \overline{min}(\psi'(v), \theta'(v))$

Thus, $E(v) \leq_\Delta J(f(v))$ and hence f is a multimorphism between E_V and J_S. Now, we have: $\psi'(v) = \psi(\psi'(v), \theta'(v)) = \psi(f(v))$ and $\theta'(v) = \theta(\psi'(v), \theta'(v)) = \theta(f(v))$. Thus, J_S, ψ, θ is a pullback for the multimorphisms $\phi: G_P \to I_R$ and $\theta: H_Q \to I_R$.

We now turn our attention to understanding pushouts within the same framework. To achieve this, we must first establish the notion of a union in the category of generalized multi-dimensional fuzzy sets as follows:

Definition 6: Consider an indexed family of GMDFS denoted by G_j, where $j \in J$, defined on the set P over a complete lattice U. The union of these GMDFS is denoted as G_U (or $\sup A$) on P, with $|G_U(p)| = \max\{|G_j(p)|, j \in J\}$, provided such a maximum exists.

If $\max\{|G_j(p)|, j \in J\}$ exists, let $G_U(p) = (G_U^1(p), G_U^2(p), \dots, G_U^n(p))$ where $|G_U(p)| = n$. Then, $G_U^r(p)$ is defined as $G_U^r(p) = \sup_{j \in J}\{G_j^r(p)\}$, and $\hat{1}$ if no such maximum exists.

It is noteworthy that, for an indexed family of GMDFS $G_j, j \in J$, it holds that $G_j(p) \leq_\Delta G_U(p)$ for all $p \in P$.

Theorem 11: *Each multimorphism, denoted as $\phi: I_R \to G_P$ and $\theta: I_R \to H_Q$, within the category $\mathcal{G}_{MFS}(U)$, possesses pushouts.*

Proof: Consider two multimorphisms, denoted as $\phi: I_R \to G_P$ and $\theta: I_R \to H_Q$. Let S denote the collection of all equivalence classes obtained from the equivalence relation on $P \cup Q$ generated by the relation $\{(\phi(r), \theta(r)): r \in R\}$.

Define a GMDFS, J on S as: Let $[s]$ denotes an arbitrary element from S, then define $J([s]) = \overline{max}\left(\sup_{p\in[s]} G(p), \sup_{q\in[s]} H(q)\right)$ where $\overline{max}$ denotes the standard multidimensional t-conorm and sup denotes the arbitrary union as defined in Definition 6.

Now, define two functions: $\psi: P \to S$ and $\theta: Q \to S$ as: $\psi(p) = [p]$ and $\theta(q) = [q]$. It is evident that both ψ and θ are multimorphisms because: $G(p) \leq_\Delta \sup_{p\in[p]} G(p) \leq_\Delta \overline{max}\left(\sup_{p\in[p]} G(p), \sup_{q\in[p]} H(q)\right) = J([p]) = J(\psi(p))$. and similarly, $H(q) \leq_\Delta J(\theta(q))$. If $\phi(r) = p$, then there exists $q \in Q$ such that

$q = \theta(r)$, thus: $\psi(\phi(r)) = \psi(p) = [p] = [q] = \theta(q) = \theta(\theta(r))$. Hence, $\psi \circ \phi = \theta \circ \theta$.

Assume that $\psi': G_P \to E_V$ and $\theta': H_Q \to E_V$ are two multimorphisms such that $\psi' \circ \phi = \theta' \circ \theta$. Then define $f: S \to V$ as: $f([s]) = \psi'(p)$ if $[s] = [p], p \in P$ and $f([s]) = \theta'(q)$ if $[s] = [q], q \in Q$. Note that if $[s] = [p] = [q]$, then there exists $r \in R$ such that $\phi(r) = p, \theta(r) = q$, implying $\psi(\phi'(r)) = \theta(\theta'(r))$, and consequently $\psi'(p) = \theta'(q)$.

Since $\psi': G_P \to E_V$ and $\theta': H_Q \to E_V$ are multimorphisms, we have for $p \in [s]$ and for $q \in [s]$: $G(p) \leq_\Delta E(\psi'(p)) = E(f([s]))$ and $H(q) \leq_\Delta E(\theta'(q)) = E(f([s]))$. Which leads to: $J([s]) = \overline{max}(\sup_{p\in[s]} G(p), \sup_{q\in[s]} H(q)) \leq_\Delta E(f([s]))$.

Hence, $f: S \to V$ is a multimorphism and we have: $f(\psi(p)) = f([p]) = \psi'(p), f(\theta(q)) = f([q]) = \theta'(q)$, so that $f \circ \psi = \psi'$ and $f \circ \theta = \theta'$. Thus, J_S together with ψ and θ form a pullout for ϕ and θ.

6. Completness and co-completeness in category of Generalized MDFS

Definition 7: Consider an indexed family of GMDFS denoted by G_j, where $j \in J$, operating on the set P within a complete lattice U. The intersection also referred to as the infimum, of these GMDFS s is defined as the GMDFS G_I (or $\inf A$) on P, satisfying $|G_I(p)| = \min\{|G_j(p)|, j \in J\}$.

Now, suppose $G_I(p) = (G_I^1(p), G_I^2(p), \dots, G_I^n(p))$ (where $|G_I(p)| = n$), then $G_I^r(p)$ is defined as: $G_I^r(p) = \inf_{j\in J}\{G_j^r(p)\}$. Here, $G_I^r(p)$ represents the r-th component of $G_I(p)$, and it is obtained by taking the infimum over all $G_j^r(p)$ for j belonging to the index set J.

It is evident that for an indexed family of GMDFS denoted by $G_j, j \in J$, the inequality $G_I(p) \leq_\Delta G_j(p)$ holds for all $p \in P$.

Theorem 12: $\mathcal{G}_{MFS}(U)$ offers products provided that U, a complete lattice, is also complete.

Proof: Consider the category $\mathcal{G}_{mfs}(U)$ with U, a lattice, being complete. Let $G_{j_{P_j}}, j \in J$ denote an arbitrary family of indexed objects from $\mathcal{G}_{MFS}(U)$.

Define $P = \prod_{j\in J} P_j$ and a GMDFS on P as: $G(r) = G(\{r_j\}) = \inf_{j\in J}(G_{j_{P_j}}(r_j))$, that is $G(r)$ represents the $infimum$ of $G_{j_{P_j}}(r_j)$ and $r = \{r_j\} \in P$.

Let $f_j: P \to P_j$ denote the usual projection functions. Then, $G(r) \leq_\Delta G_j(f_j(p)) = G_j(r_j)$, implying that f_j are multimorphisms. Now, suppose $g_j: H_Q \to G_{j_{P_j}}$ is a family of multimorphisms. Then, there exists a unique function $h: Q \to P$ defined by $h(q) = (g_j(q)), j \in J$, satisfying: $f_j \circ h = g_j$.

Now, $H(q) \leq_{\Delta} G_j(g_j(q)), \forall j$, $H(q) \leq_{\Delta} G(h(q)), \forall q$. Thus, from the definition of h, it follows that h is a multimorphism and is unique if it satisfies equation $f_j \circ h = g_j$. Hence, $\mathcal{G}_{MFS}(U)$ has products if U is complete.

Theorem 13: *The category of generalized multidimensional fuzzy set,* $\mathcal{G}_{MFS}(U)$*, has co-products.*

Proof: Consider an arbitrary family indexed by $j \in J$ of objects $G_{j_{P_j}}$ drawn from $\mathcal{G}_{MFS}(U)$. Let P denote the disjoint union of P_j s, and define a multimorphism G on P as: $G(p) = G_j(p)$, where $p \in P_j$.

Let $f_i: P_j \to P$ represent the inclusion maps. Consequently, we observe that: $G_j(p) \leq_{\Delta} G(p) = G(f_j(p))$. This implies that $f_j s$ are indeed multimorphisms.

Next, consider multimorphisms $g_j: G_{j_{P_j}} \to H_Q$. This implies the existence of a function $h: P \to Q$, defined as $h(p) = g_j(p)$, where j is the specific index such that $p \in P_j$, and satisfying $h \circ f_j = g_j$.

Given that $f_j s$ represents inclusion maps, the uniqueness of h is evident. Furthermore, $G_j(p) \leq_{\Delta} H(g_j(p))$ for all p, which leads to: $G(p) = G_j(p) \leq_{\Delta} H(g_j(p)) = H(h(p))$. Consequently, h also qualifies as a multimorphism. Thus, $\mathcal{G}_{MFS}(U)$ possesses co-products.

Now, the proof of completeness can be accomplished through the equalizers and co-equalizers of $\mathcal{G}_{mfs}(U)$ which are given by the following two theorems.

Theorem 14: *Consider* G_P *and* H_Q *as two objects in* $\mathcal{G}_{mfs}(U)$ *and let* $\phi: G_P \to H_Q$ *and* $\theta: G_P \to H_Q$ *denote two multimorphisms. Then, the equalizer of* ϕ *and* θ *is a multimorphism* $\psi: I_R \to G_P$*, where* $R = \{p \in P: \phi(p) = \theta(p)\}$*,* I *represents the restriction function of* G *to* R*, and* ψ *signifies the inclusion function.*

Proof: Given that I represents the restriction of G to R, and ψ denotes the inclusion from R to P, it follows that $I(r) = G(r) = G(\psi(r))$. Consequently, ψ qualifies as a multimorphism, and $\phi \circ \psi = \theta \circ \psi$ due to ϕ and θ having identical behavior on R with $\psi(R) = R$.

Now, suppose $\theta: J_S \to G_P$ serves as a multimorphism satisfying $\phi \circ \theta = \theta \circ \theta$ $\theta(S) \subseteq R$. Hence, we can define $f: S \to R$ by $f(s) = \theta(s)$ for all $s \in S$. As $\theta: J_S \to G_P$ is a multimorphism, and by the definition of f, we have: $J(s) \leq_{\Delta} G(\theta(s)) J(s) \leq_{\Delta} I(f(s))$ for all $s \in S$. Thus, f also qualifies as a multimorphism and $\psi \circ f = \theta$. If $g: S \to R$ represents any other multimorphism such that $\psi \circ g = \theta$, then $\psi \circ f = \psi \circ g$. The injectivity of the inclusion map ψ implies $f = g$, demonstrating the uniqueness of f. Hence, f exists uniquely.

Theorem 15: *Consider G_P and H_Q as two objects within the category $\mathcal{G}_{MFS}(U)$, and let $\phi: G_P \to H_Q$ and $\theta: G_P \to H_Q$ denote two multimorphisms. Then, a co-equalizer of ϕ and θ is represented by a multimorphism $\psi: H_Q \to I_R$, where R consists of the equivalence classes generated by the set $\{(\phi(p), \theta(p)) | p \in P\}$.*

Proof: Let R denote the set of all equivalence classes formed under the equivalence relation generated by $\{(\phi(p), \theta(p)): p \in P\}$.

We define a multimorphism I on R as: $I([r']) = \sup\{H(q): q \in [r']\}$ where sup signifies $I_U(r)$ for $r \in [r']$ (15).

Let $\psi: H_Q \to I_R$ be defined as $\psi(q) = [q]$. Then, we observe that: $H(q) \leq_\Delta \sup\{H(q'): q' \in [q]\} = I(\psi(q))$ and $\psi(\phi(p)) = [\phi(p)] = [\theta(p)] = \psi(\theta(p))\ \forall p \in P$. Thus, ψ is a multimorphism, and $\psi \circ \phi = \psi \circ \theta$.

Let $\theta: H_Q \to J_S$ be any other multimorphism satisfying $\theta \circ \phi = \theta \circ \theta$. Define $f: R \to S$ by $f([q]) = \theta(q)$. Then, $f \circ \psi = \theta$, and since θ is a multimorphism, we have $H(q) \leq_\Delta J(\theta(q))$ for each q. Consider an arbitrary $[r'] \in R$ and any $q \in [r']$, which leads to: $H(q) \leq_\Delta J(\theta(q)) = J(f \circ \psi(q)) = J(f([r']))$. Hence, we establish:

$$I([r']) = \sup\{H(q): q \in [r']\} \leq_\Delta J(f([r'])) \tag{2}$$

Consequently, f is a multimorphism. Furthermore, if $g: R \to S$ is any other multimorphism such that $g \circ \psi = \theta$, then: $f \circ \psi(q) = g \circ \psi(q), \forall q \in Q \Longrightarrow f([q]) = g([q]), [q] \in R$. Thus, ϕ, θ have co-equalizers, proves the theorem.

It is evident that the arbitrary union $\sup\{H(q): q \in [r']\}$ in Equation (2) cannot yield $\hat{1}$ unless $J(f([r']))$ also yields $\hat{1}$, owing to the condition $|H(q)| \leq |J(f([r']))|$ for all $q \in [r']$. In light of the aforementioned theorem, the subsequent corollaries can be formulated as follows:

Corollary 2: *The category $\mathcal{G}_{MFS}(U)$ is both complete and co-complete if U, a complete lattice.*

A general category theorem on the existence and co-limit of small diagrams is given by the following theorem, which is immediately followed by the next corollary:

Theorem 16: [28] *A category is complete if and only if each small diagram in it has a limit. Similarly, a category is co-complete if and only if each small diagram in it has a co-limit.*

Corollary 3*:* *In $\mathcal{G}_{MFS}(U)$ each small diagram has limits and co-limits if U, a complete lattice.*

Regular monomorphism and epimorphism are some kinds of weak monomorphism and epimorphism respectively. Monomorphism and epimorphism can directly identify the same morphisms whereas their regular forms identify the same morphisms up to the composition of another morphism. The following theorems provide sufficient conditions for multimorphisms to be a regular monomorphism and epimorphism respectively.

Theorem 17: *A multimorphism $\psi: I_R \to G_P$ is deemed a regular monomorphism if it satisfies the condition of being injective, denoted as one-to-one, and if $I = G \circ \psi$.*

Proof: Assuming ψ is a one-to-one function and $I = G \circ \psi$, let $Q = \{q_1, q_2\}$ and define $H(q) = \hat{1}, \forall q \in Q$. We then define $\phi, \theta: G_P \to H_Q$ as: $\phi(p) = q_1$ if $p \in \psi(R)$, $\phi(p) = q_2$, otherwise. And $\theta(p) = q_1, \forall p \in P$.

It is evident that ϕ and θ are multimorphisms to H_Q, as per the definition of H_Q. Furthermore, we observe that $\phi \circ \psi = \theta \circ \psi$. Now, assuming the existence of a multimorphism $\theta: J_S \to I_R$ such that $\phi \circ \theta = \theta \circ \theta$, we assert the existence of a unique function $\lambda: S \to R$ defined by $\lambda = \psi^{-1} \circ \theta$ such that $\psi \circ \lambda = \theta$.

Then, we have: $J(s) \leq_{\Delta} G(\theta(s)) = G(\psi(\psi^{-1}\theta(s))) = I(\psi^{-1}(\theta(s))) = I(\lambda(s))$

Thus, λ is a multimorphism, implying that ψ is a regular monomorphism.

Theorem 18: *Consider a bijective multimorphism $\psi: H_Q \to I_R$, where $H(q) = I(\psi(q))$ for all $q \in Q$. Then, ψ is a regular epimorphism.*

Proof: Let G_P be a GMDFS with $G(p) = \hat{0}$ for all $p \in P$. Consider two multimorphisms f and g defined as $f, g: G_P \to H_Q$, where $f(p) = g(p) = q, \in Q$. It is evident that $\psi \circ f = \psi \circ g$.

Now, suppose $\theta: H_Q \to J_S$ is another multimorphism such that $\theta \circ f = \theta \circ g$. Then, there exists a unique function $h: I_R \to J_S$ defined as $h = \theta \circ \psi^{-1}$, which satisfies $h \circ \psi = \theta$. Since θ is a multimorphism, we have: $H(q) \leq_{\Delta} J(\theta(q)), \forall q \in Q$. Thus, we obtain: $I(r) \leq_{\Delta} I(\psi(q)) = H(q) \leq_{\Delta} J(\theta(q)) = J(\theta(\psi^{-1}(r))) = J(h(r)), \forall r \in R$. Hence, h is a multimorphism and ψ is a regular epimorphism.

As a corollary to this theorem, it can be asserted that an isomorphism $\phi: G_P \to H_Q$ exhibits properties of both a regular monomorphism and a regular epimorphism.

Conclusions and future directions

This paper presents an investigation into the categorical structure and properties of generalized multidimensional fuzzy sets, which represent the most comprehensive form of fuzzy sets. Therefore, delving into the categorical accommodation of GMDFS offers a comprehensive understanding of categorical structures across various extensions of fuzzy sets. Key categorical properties including separators, co-separators, sections, retractions, monomorphisms, epimorphisms, and isomorphisms are analyzed, leading to significant characterizations. Furthermore, pullbacks and pushouts of $\mathcal{G}_{MFS}(U)$ are identified. Additionally, the investigation extends to the characterization of equalizers, co-equalizers, products, and co-products, with findings on the completeness and co-completeness of $\mathcal{G}_{MFS}(U)$. Attention is given to the study of regular monomorphisms and regular epimorphisms as well. Looking ahead, one can expand this research into areas such as functors, concreteness, and topos structures within the generalized multidimensional fuzzy set category - $\mathcal{G}_{MFS}(U)$.

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