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Almost (*K, L*)-continuous functions in ideal bitopological spaces

Carpintero Carlos *
Facultad de Ciencias Básicas
Corporación Universitaria del Caribe CECAR
Sincelejo 700009
Colombia

Rajesh Neelamegarajan †
Department of Mathematics
Rajah Serfoji Govt. College
Thanjavur 613005
Tamil Nadu
India

Rosas Ennis §
Departamento de Ciencias Naturales y Exactas
Universidad de la Costa
Barranquilla 080001
Colombia

Sanabria José ‡
Departamento de Matemáticas
Facultad de Educación y Ciencias
Universidad de Sucre
Sincelejo 700009
Colombia

* *ORCID-ID:* **0000-0003-3831-952X**
† *ORCID-ID:* **0000-0003-0921-6054**
§ *ORCID-ID:* **0000-0001-8123-9344**
‡ *ORCID-ID:* **0000-0002-9749-4099**

* *E-mail:* **carpintero.carlos@gmail.com (Corresponding Author)**
† *E-mail:* **nrajesh_topology@yahoo.co.in**
§ *E-mail:* **ennisrafael@gmail.com**
‡ *E-mail:* **jesanabri@gmail.com**

Abstract

Using the notion of ideal bitopological space we introduce the (*k, l*)-almost (*K, L*)-continuous functions and obtain several properties of this functions and finally we obtain some relationships with another type of continuity.

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1. Introduction

In 1963, Kelly [1] introduced the concept of bitopological space. Following Kelly's work, many topologists have focused on generalizing various concepts from single topological spaces within the framework of bitopological spaces. We can see, for example, the recent works of Wasi and Rajihy [2]. Generalized open sets are highly significant in General Topology and have now become a key area of research for topologists worldwide. Currently, leveraging the concept of an ideal topological, various types of continuous functions have been investigated [3]-[6]. The study of ideal topological space was proposed and studied by Kuratowski [7] whereas Vaidyanathaswamy [8] introduced the local function $S^*(\tau, K)$ associated to each subset $S \subseteq Z$ in a structure (Z, τ, K), with τ a topology and K an ideal considered on Z. Atiya, Ibrahem, and Adeeb [9], defined K-openness in a structure (Z, τ, K). In particular, S is $\{\emptyset\}$-open (resp. semi-$\{\emptyset\}$-open) iff S is preopen (resp. S is semi-$\{\emptyset\}$-open). In this paper $(Z, \tau_1, \tau_2, K, L)$ mean an ordered quintuple, with τ_1, τ_2 topologies and K, L ideals considered on Z. In this frame work, we define (k, l) -almost (K, L)-continuous functions and study characterization for this type of continuity. Additionally we give relationships with other types of functions.

2. Preliminaries

In the sequel (Z, τ), (W, σ) (or briefly Z, W) will be topological spaces and $\overline{S}$ (resp. S°) is the closure (resp. interior) of $S \subseteq Z$ in τ, also S is known as:

- regular open [10] iff S is equal to $(\overline{S})^\circ$
- semi-open [11] iff S is contained in $\overline{(S^\circ)}$
- pre-open [12] iff S is contained in $(\overline{S})^\circ$
- semi-preopen [13], if S is contained in $\overline{((\overline{S})^\circ)}$

Taking their complements, we have the following terminology:

- regular closed for the complement of regular open
- semiclosed for the complement of semiopen
- preclosed for the complement of preopen
- semi-preclosed for the complement of semipreopen

A subset $S \subseteq Z$ is K-open [9], if $S \subseteq (S^*)^\circ$ (resp. S is K-closed, if $Z \backslash S$ is an K-open set). The K-closure (resp. K-interior) K-$\overline{A}$ (resp. K-A°) of a subset $A \subseteq Z$ can be defined in similar way (*mutatis mutandis*) as closure (resp. interior) of A. The classes studied above generate the following families of subsets in a structure (Z, τ, K):

- $IO_{(Z,\tau)} = \{S \subseteq Z\colon S \text{ is } K - \text{open}\}$,
- $IC_{(Z,\tau)} = \{S \subseteq Z\colon S \text{ is } K - \text{closed}\}$,
- $RO_{(Z,\tau)} = \{S \subseteq Z\colon S \text{ is regular open set}\}$,
- $RC_{(Z,\tau)} = \{S \subseteq Z\colon S \text{ is regular closed set}\}$,
- $SO_{(Z,\tau)} = \{S \subseteq Z\colon S \text{ is semi open set}\}$,
- $SC_{(Z,\tau)} = \{S \subseteq Z\colon S \text{ is semi closed set}\}$,
- $PO_{(Z,\tau)} = \{S \subseteq Z\colon S \text{ is pre open set}\}$,
- $SPO_{(Z,\tau)} = \{S \subseteq Z\colon S \text{ is semi} - \text{pre open set}\}$,
- $SPC_{(Z,\tau)} = \{S \subseteq Z\colon S \text{ is semi} - \text{pre closed set}\}$.

Observe that for avoid confusions, given a structure (Z, τ_1, τ_2). We denoted $\overline{A}^k = cl_{\tau_k}(A)$ (resp. $A^{\circ k} = int_{\tau_k}(A)$) as the closure (resp. interior) of $A \subseteq Z$ in a space (Z, τ_k) for $k = 1{,}2$.

Definition 2.1: Given (Z, τ_1, τ_2) and $A \subseteq Z$. Then, for $k \neq l$:

(1) $S \subset \overline{(S^{\circ k})}^l$ iff S is (k, l)-semi-open [14]

(2) $S \subset (\overline{S}^l)^{\circ k}$ iff S is (k, l)-preopen [15];

(3) $S \subset \overline{((\overline{S}^l))^{\circ k}}^l$ iff S is (k, l)-semi-preopen [15];

(4) $S = (\overline{S}^l)^{\circ k}$ iff S is (k, l)-regular open [10],

Taking complements in Definition 2.1, we have the following terminology too:

- (k, l)-semi-closed for the complement of (k, l)-semi-open
- (k, l)-preclosed for the complement of (k, l)-preopen
- (k, l)-semi-preclosed for the complement of (k, l)-semi-preopen
- (k, l)-regular closed for the complement of (k, l)-regular open

Also, in a bitological structure (Z, τ_1, τ_2) we distinguish the following families of subsets:

- (k, l)-$SO_{(Z,\tau_1,\tau_2)} = \{A \subseteq Z\colon Ais(k, l) - \text{semi} - \text{open}\}$,
- (k, l)-$PO_{(Z,\tau_1,\tau_2)} = \{A \subseteq Z\colon Ais(k, l) - \text{preopen}\}$,
- (k, l)-$SPO_{(Z,\tau_1,\tau_2)} = \{A \subseteq Z\colon Ais(k, l) - \text{semi} - \text{preopen}\}$,

- (k,l)-$RO_{(Z,\tau_1,\tau_2)} = \{A \subseteq Z: A is (k,l) - \text{regular open}\}$,
- (k,l)-$SC_{(Z,\tau_1,\tau_2)} = \{A \subseteq Z: A is (k,l) - \text{semi} - \text{closed}\}$,
- (k,l)-$PC_{(Z,\tau_1,\tau_2)} = \{A \subseteq Z: A is (k,l) - \text{preclosed}\}$,
- (k,l)-$SPC_{(Z,\tau_1,\tau_2)} = \{A \subseteq Z: A is (k,l) - \text{semi} - \text{preclosed}\}$,
- (k,l)-$RC_{(Z,\tau_1,\tau_2)} = \{A \subseteq Z: A is (k,l) - \text{regular closed}\}$.

Definition 2.2: Given a bitopological structure (Z,τ_1,τ_2,K,L), then:

(1) the elements belong to $IO_{(Z,\tau_1)} \cup JO_{(Z,\tau_2)}$ are called b-(K,L)-open sets.

(2) $S \subseteq Z$ is called b-(K,L)-closed set if $Z\backslash S$ is b-(K,L)-open

$$b-(K,L)O_{(Z,\tau_1,\tau_2)} = \{A \subseteq Z: A \in IO_{(Z,\tau_1)} \cup JO_{(Z,\tau_2)}\};$$
$$b-(K,L)C_{(Z,\tau_1,\tau_2)} = \{A \subseteq Z: X\backslash A \in IO_{(Z,\tau_1)} \cup JO_{(Z,\tau_2)}\}.$$

Remark 2.3: Observe that if in Definition 2.2, $K = L = \{\emptyset\}$, then

$$A \in PO_{(Z,\tau_1)} \cup PO_{(Z,\tau_2)} \Rightarrow A \in b-(K,L)O_{(Z,\tau_1,\tau_2)}.$$

Definition 2.4: In a bitological structure (Z,τ_1,τ_2,K,L);

(1) (τ_1,τ_2)-(K,L) $cl(S) = K - \overline{S} \cap L - \overline{S}$,

(2) (τ_1,τ_2)-(K,L) $int(S) = K - S^{\circ} \cup L - S^{\circ}$,

for all $S \subseteq Z$.

Definition 2.5: Given (Z,τ_1,τ_2) and $A \subseteq Z$, then

$$(k,l) - cl_\delta(A) = \{x \in Z: U \cap A \neq \emptyset, \forall U \in (k,l) - RO_{(Z,\tau_1,\tau_2)}, x \in U\}$$

The elements in $(k,l) - cl_\delta(A)$ are called (k,l)-δ-cluster points of A [16] and A is (k,l)-δ-closed if $A = (k,l) - cl_\delta(A)$.

Complements of (k,l)-δ-closed sets are denominated (k,l)-δ-open sets. Also $\bigcup_{k\in K} \; A_k$ is (k,l)-δ-open if each $A_k \in (k,l)$-$RO_{(Z,\tau_1,\tau_2)}$.

Remark 2.6: In (Z,τ_1,τ_2) given a subset $A \subseteq Z$, $(k,l) - cl_\delta(A)$ (resp. $(k,l) - int_\delta(A)$) is the union (resp. intersection) of all (k,l)-δ-closed super (resp. open sub) sets containing (resp. contained) A.

Lemma 2.7: *Given* (Z,τ_1,τ_2,K,L) *and* $S \subseteq Z$. *Then:*

(1) (τ_1,τ_2)-(K,L) $int(S)$ is b-(K,L)-open;

(2) (τ_1,τ_2)-(K,L) $cl(S)$ is b-(K,L)-closed;

(3) S is b-(K,L)-open iff $S = (\tau_1,\tau_2)$-(K,L) $int(S)$;

(4) S is b-(K,L)-closed iff $S = (\tau_1,\tau_2)$-(K,L) $cl(S)$;

(5) (τ_1,τ_2)-(K,L) $int(Z\backslash S) = Z\backslash(\tau_1,\tau_2)$-$(K,L)$ $cl(S)$;

(6) (τ_1,τ_2)-(K,L) $cl(Z\backslash S) = Z\backslash(\tau_1,\tau_2)$-$(K,L)$ $int(S)$.

Lemma 2.8: *Given* (Z,τ_1,τ_2,K,L), *then*

$(\tau_1,\tau_2)-(K,L)\ cl(A)=\{x\in Z\colon U\cap A\neq\emptyset,\forall U\in b-(K,L)O_{(Z,\tau_1,\tau_2)},x\in U\}$ *for all* $A\subseteq Z$.

3. $(\boldsymbol{k},\boldsymbol{l})$-almost $(\boldsymbol{K},\boldsymbol{L})$-continuity and characterizations

Now we will introduce the main notions of this work, also we find the relationships between them and looking for some characterizations.

Definition 3.1: $(Z,\tau_1,\tau_2,K,L)\xrightarrow{f}(W,\sigma_1,\sigma_2)$ is b-(K,L)-continuous if:

$$\forall x\in Z, g(x)\in V\in\sigma_1\cup\sigma_2,\exists\ U\in b-(K,L)O_{(Z,\tau_1,\tau_2)}\colon x\in U, g(U)\subset V$$

Remark 3.2: Taking $\tau_1=\tau_2$, $K=L=\{\emptyset\}$ and $\sigma_1=\sigma_2$ in Definition 3.1, we obtain the notion of precontinuous function and this means that Definition 3.1, generalize some notions of continuity.

Example 3.3: Consider $Z=\{1,2,3\}$, $\tau_1=\tau_2=\{\emptyset,Z,\{1\},\{2,2\}\}$, $K=L=\{\emptyset\}$ and (W,σ_1,σ_2) a bitopological space, with $W=\{1,2,3\}$ and $\sigma_1=\sigma_2=\{\emptyset,W\}$. Also $(Z,\tau_1,\tau_2,K,L)\xrightarrow{h}(W,\sigma_1,\sigma_2)$ given as $h(1)=2$, $h(2)=3$, $h(3)=1$. Observe the families of $\{\emptyset\}$-open sets for (Z,τ_1) and (Z,τ_2) are exactly the set $\{\emptyset,Z,\{1\},\{1,2\},\{1,3\}\}$ of preopen sets in Z and b - $(K,L)O(Z,\tau_1,\tau_2)=\{\emptyset,Z,\{1\},\{1,2\},\{1,3\}\}$. Fom here, we concluded the b-(K,L)-continuity of h.

Definition 3.4: $(Z,\tau_1,\tau_2,K,L)\xrightarrow{g}(W,\sigma_1,\sigma_2)$ is (k,l) -almost (K,L) - continuous if:

$$\forall x\in Z|\ g(x)\in V\in\sigma_k,\exists\ U\in b-(K,L)O_{(Z,\tau_1,\tau_2)}\colon x\in U, g(U)\subset((\overline{V}^{l}))^{\circ k}$$

Observe that

$$b-(K,L)-\text{continuity}\Rightarrow(k,l)-\text{almost}(K,L)-\text{continuity}$$

But

$$(k,l)-\text{almost}(K,L)-\text{continuity}\not\Rightarrow b-(K,L)-\text{continuity}$$

As we show in the next example.

Example 3.5: If $Z=\mathbb{R}$, $\tau_1=\tau_2=\{\emptyset,\mathbb{R},\mathbb{R}\backslash\mathbb{Q}\}$, $K=L=\{\emptyset\}$ and (W,σ_1,σ_2) a bitopological space, where $W=\{1,2,3\}$ and $\sigma_1=\sigma_2=\{\emptyset,W,\{1\},\{1,2\}\}$. Also $(Z,\tau_1,\tau_2,K,L)\xrightarrow{g}(W,\sigma_1,\sigma_2)$ given as

$$g(x)=\begin{cases}1 & \text{if } x\in\mathbb{Q},\\ 2 & \text{if } x\notin\mathbb{Q}\end{cases}$$

Then both families $\{\emptyset\}$-open sets for (Z,τ_1) and (Z,τ_2) are just the family of preopen sets. From this, with a simple calculations, we obtain the (k,l)-almost (K,L)-continuity of g. But g not is b-(K,L)-continuous.

Theorem 3.6: *Given* $(Z,\tau_1,\tau_2,K,L)\xrightarrow{g}(W,\sigma_1,\sigma_2)$, *then:*
(1) g is an (k,l)-almost (K,L)-continuous function;
(2) $\forall x\in Z|\ g(x)\in V\in(k,l)-RO_{(W,\sigma_1,\sigma_2)}, \exists\ U\in b-(K,L)O_{(Z,\tau_1,\tau_2)}: x\in U, g(U)\subset V$;
(3) $\forall x\in Z|\ V\ (k,l)-\delta-open, g(x)\in V, \exists\ U\in b-(K,L)O_{(Z,\tau_1,\tau_2)}: x\in U, g(U)\subset V$;
are equivalent.

Proof: $(1)\Rightarrow(2)$: Assume that $g(x)\in V\in(k,l)\text{-}RO_{(W,\sigma_1,\sigma_2)}$. By (1), there exists $U\in b\text{-}(K,L)O_{(Z,\tau_1,\tau_2)}$ containing x and $g(U)\subset(\overline{V}^{l})^{\circ k}$. But $V\in(k,l)\text{-}RO_{(W,\sigma_1,\sigma_2)}$, then $(\overline{V}^{l})^{\circ k}=V$. Thus, $g(U)\subset V$.

$(2)\Rightarrow(3)$: Suppose that $g(x)\in V$ and V is $(k,l)\text{-}\delta$-open. Then there exists $G\in\sigma_k$ such that $g(x)\in G\subset(\overline{G}^{l})^{\circ k}\subset V$. Since $g(x)\in(\overline{G}^{l})^{\circ k}\in(k,l)\text{-}RO_{(W,\sigma_1,\sigma_2)}$. By (2), there exists $U\in b\text{-}(K,L)O_{(Z,\tau_1,\tau_2)}$ such that $x\in U$ and $g(U)\subset(\overline{G}^{l})^{\circ k}\subset V$.

$(3)\Rightarrow(1)$: If $g(x)\in V\in\sigma_k$, $(\overline{V}^{l})^{\circ k}$ is $(k,l)\text{-}\delta$-open set of W containing $g(x)$. By (3), there exists $U\in b\text{-}(K,L)O_{(Z,\tau_1,\tau_2)}$ with $x\in U$ and $g(U)\subset(\overline{V}^{l})^{\circ k}$. Hence g is an (k,l)-almost (K,L)-continuous function.

Theorem 3.7: *Given* $(Z,\tau_1,\tau_2,K,L)\xrightarrow{g}(W,\sigma_1,\sigma_2)$, *then:*
(1) g is an (k,l)-almost (K,L)-continuous function.
(2) $g^{-1}((\overline{V}^{l})^{\circ k})\in b\text{-}(K,L)O_{(Z,\tau_1,\tau_2)}$ for all $V\in\sigma_k$.
(3) $g^{-1}(\overline{(F)^{\circ l}}^{k})\in b\text{-}(K,L)C_{(Z,\tau_1,\tau_2)}$ for every σ_k-closed set $F\subseteq W$.
(4) $g^{-1}(\overline{(F)^{\circ l}}^{k})\in b\text{-}(K,L)C_{(Z,\tau_1,\tau_2)}$ for all $F\in(k,l)\text{-}RC_{(W,\sigma_1,\sigma_2)}$.
(5) $g^{-1}((\overline{V}^{l})^{\circ k})\in b\text{-}(K,L)O_{(Z,\tau_1,\tau_2)}$ for every $V\in(k,l)\text{-}RO_{(W,\sigma_1,\sigma_2)}$.
are equivalent.

Proof: $(1)\Rightarrow(2)$: Suppose $V\in\sigma_k$ and $x\in g^{-1}((\overline{V}^{l})^{\circ k})$, hence $g(x)\in(\overline{V}^{l})^{\circ k}\in(k,l)\text{ - }RO_{(W,\sigma_1,\sigma_2)}$. Being g (k,l)-almost (K,L)-continuous, by Theorem 3.6, $x\in U$ and $g(U)\subset(\overline{V}^{l})^{\circ k}$ for some $U\in b\text{-}(K,L)O_{(Z,\tau_1,\tau_2)}$. Thus $x\in U\subset g^{-1}((\overline{V}^{l})^{\circ k})$, in consequence $g^{-1}((\overline{V}^{l})^{\circ k})\in b\text{-}(K,L)O_{(Z,\tau_1,\tau_2)}$.

$(2)\Rightarrow(3)$: If $F\subseteq W$ is any σ_k-closed, then $W\setminus F\in\sigma_k$. Using (2), $g^{-1}((\overline{W\setminus F}^{l})^{\circ k})\in b\text{ - }(K,L)O_{(Z,\tau_1,\tau_2)}$. But $Z\setminus g^{-1}((\overline{F}^{l})^{\circ k})=g^{-1}(W\setminus\overline{(F)^{\circ l}}^{k})=g^{-1}(\overline{W(F)^{\circ l}}^{k})=g^{-1}((\overline{W\setminus F}^{l})^{\circ k})$. In consequence $g^{-1}(\overline{(F)^{\circ l}}^{k})\in b\text{ - }(K,L)$

$C_{(Z,\tau_1,\tau_2)}$.

(3) $\Rightarrow$ (4): $F \in (k,l)\text{-}RC_{(W,\sigma_1,\sigma_2)}$, implies that $F \subset W$ is a σ_k-closed set. Now using (3), $g^{-1}(\overline{(F)^{\circ_l}}^{k}) \in b\text{-}(K,L)C_{(Z,\tau_1,\tau_2)}$, since $F \in (k,l)\text{-}RC_{(W,\sigma_1,\sigma_2)}$. Then $g^{-1}(F) = g^{-1}(\overline{(F)^{\circ_l}}^{k}) \in b\text{-}(K,L)C_{(Z,\tau_1,\tau_2)}$.

(4) $\Rightarrow$ (5): If $V \in (k,l)\text{-}RO_{(W,\sigma_1,\sigma_2)}$, then $W\backslash V \in (k,l)\text{-}RC_{(W,\sigma_1,\sigma_2)}$. Using (4), $Z\backslash g^{-1}(V) = g^{-1}(W\backslash V) \in b\text{-}(K,L)C_{(Z,\tau_1,\tau_2)}$ and then $g^{-1}(V) \in b\text{-}(K,L)$ $O_{(Z,\tau_1,\tau_2)}$.

(5) $\Rightarrow$ (1): If $g(x) \in V$ and $V \in (k,l)\text{-}RO_{(W,\sigma_1,\sigma_2)}$, by (5), $g^{-1}(V) \in$ b - $(K,L)O_{(Z,\tau_1,\tau_2)}$ and $x \in g^{-1}(V)$. Since $g\left(g^{-1}(V)\right) \subset V$. Then, using Theorem 3.6, g is (k,l)-almost (K,L)-continuous.

Theorem 3.8: *Given* $(Z,\tau_1,\tau_2,K,L) \xrightarrow{g} (W,\sigma_1,\sigma_2)$, *then:*

(1) g is (k,l)-almost (K,L)-continuous.

(2) (τ_1,τ_2)-(K,L) $cl(g^{-1}(V)) \subset g^{-1}(\overline{V}^{k})$ for each $V \in (k,l)\text{-}SPO_{(W,\sigma_1,\sigma_2)}$.

(3) $g^{-1}((F)^{\circ_k}) \subset (\tau_1,\tau_2)$-$(K,L)$- $int(g^{-1}(F))$ for each $F \in (k,l)\text{-}SPC_{(W,\sigma_1,\sigma_2)}$.

(4) $g^{-1}((F)^{\circ_k}) \subset (\tau_1,\tau_2)$-$(K,L)$-$int(g^{-1}(F))$ for each $F \in (k,l)\text{-}SC_{(W,\sigma_1,\sigma_2)}$.

(5) (τ_1,τ_2)-(K,L) $cl(g^{-1}(V)) \subset g^{-1}(\overline{V}^{k})$ for each $V \in (k,l)\text{-}SO_{(W,\sigma_1,\sigma_2)}$.

are equivalent.

Proof: (1) $\Rightarrow$ (2): If we take $V \in (k,l)\text{-}SPO_{(W,\sigma_1,\sigma_2)}$. Since $\overline{V}^{k} \in (k,l)$ - $RC_{(W,\sigma_1,\sigma_2)}$ and g is an (k,l)-almost (K,L)-continuous function. Using Theorem 3.7, $g^{-1}(V) \in b$ - $(K,L)C_{(Z,\tau_1,\tau_2)}$. Thus, we obtain the inclusion (τ_1,τ_2)-(K,L) $cl(g^{-1}(V)) \subset g^{-1}(\overline{V}^{l})$.

(2) $\Rightarrow$ (3): If $F \subseteq W$ is a (k,l)-semi-preclosed subset, then $W\backslash F$ is (k,l)-semi-preopen subset of w and then using (2), we obtain $(\tau_1,\tau_2) - (K,L)cl\ \left(g^{-1}(W\backslash F)\right) \subset g^{-1}\left(\overline{W\backslash F}^{k}\right) \subset g^{-1}(W\backslash (F)^{\circ_k} \subset Z\backslash g^{-1}((F)^{\circ_k})$. Therefore $g^{-1}((F)^{\circ_k}) \subset (\tau_1,\tau_2)$-$(K,L)$ $int(g^{-1}(F))$.

(3) $\Rightarrow$ (4): Observe that $(k,l)\text{-}SC_{(W,\sigma_1,\sigma_2)}$ (k,l)- $SPC_{(W,\sigma_1,\sigma_2)}$.

(4) $\Rightarrow$ (5): Suppose that $V \in (k,l)\text{-}SO_{(W,\sigma_1,\sigma_2)}$. Then $W\backslash V \in (k,l)$ - $SC_{(W,\sigma_1,\sigma_2)}$, using (4), $g^{-1}((W\backslash V)^{\circ_k}) \subset (\tau_1,\tau_2) - (K,L)int\left(g^{-1}(W\backslash V)\right) \subset (\tau_1,\tau_2) - (K,L)int\left(Z\backslash g^{-1}(V)\right) \subset Z\backslash(\tau_1,\tau_2) - (K,L)cl(g^{-1}(V))$. Therefore, (τ_1,τ_2)-(K,L) $cl(g^{-1}(V)) \subset g^{-1}(\overline{V}^{k})$.

(5) $\Rightarrow$ (1): If $F \in (k,l)\text{-}RC_{(W,\sigma_1,\sigma_2)}$, then $F \in (k,l)\text{-}SO_{(W,\sigma_1,\sigma_2)}$. Using (5), (τ_1,τ_2) - $(K,L)cl\left(g^{-1}(F)\right) \subset g^{-1}(\overline{F}^{k}) = g^{-1}(F)$. Thus $g^{-1}(F) \in b$ - (K,L) $C_{(Z,\tau_1,\tau_2)}$. In consequence, applying Theorem 3.7, g is (k,l)-almost (K,L)-continuous.

Theorem 3.9: *$(Z,\tau_1,\tau_2,K,L) \xrightarrow{g} (W,\sigma_1,\sigma_2)$ is (k,l)-almost (K,L)-continuous iff $g^{-1}(V) \in (\tau_1,\tau_2)$-$(K,L)$ $int(g^{-1}(\overline{V}^l)^{\circ_k})$, for all $V \in \sigma_k$.*

Proof: If $V \in \sigma_k$, then $V \subset (\overline{V}^l)^{\circ_k} \in (k,l)$-$RO_{(W,\sigma_1,\sigma_2)}$. Being g (k,l)-almost (K,L)-continuous, Theorem 3.7 ensures that $g^{-1}((\overline{V}^l)^{\circ_k}) \in b$-$(K,L)O_{(Z,\tau_1,\tau_2)}$ and then $g^{-1}(V) \subset g^{-1}((\overline{V}^l)^{\circ_k}) = (\tau_1,\tau_2)$ - (K,L) $int(g^{-1}(\overline{V}^l)^{\circ_k})$. Reciprocally, if $V \in (k,l)$-$RO_{(W,\sigma_1,\sigma_2)}$ then $V \in \sigma_k$. By hypothesis, $g^{-1}(V) \subset (\tau_1,\tau_2)$-$(K,L)$ $int(g^{-1}(\overline{V}^l)^{\circ_k}) = (\tau_1,\tau_2)$-$(K,L)$ $int(g^{-1}(V))$. Thus, $g^{-1}(V) \in b$-$(K,L)O_{(Z,\tau_1,\tau_2)}$ and then applying Theorem 3.7, g is an (k,l)-almost (K,L)-continuous function.

Corollary 3.10: *$(Z,\tau_1,\tau_2,K,L) \xrightarrow{g} (W,\sigma_1,\sigma_2)$ is (k,l)-almost (K,L) -continuous iff $(\tau_1,\tau_2) - (K,L)$ $cl(g^{-1}\overline{(F)^{\circ_k}}^l) \subseteq g^{-1}(F)$, for all σ_k-closed $F \subseteq W$.*

Theorem 3.11: *If $(Z,\tau_1,\tau_2,K,L) \xrightarrow{g} (W,\sigma_1,\sigma_2)$ is (k,l)-almost (K,L)-continuous, then:*

$$\forall x \in Z, \forall V \in \sigma_k \mid x \in (\tau_1,\tau_2) - (K,L)\ cl(g^{-1}(V)) \backslash g^{-1}(V);\ g(x) \in (\tau_1,\tau_2) - (K,L)\ cl(V)$$

Proof: Assume that $x \in (\tau_1,\tau_2) - (K,L)$ $cl(g^{-1}(V)) \backslash g^{-1}(V)$ and $g(x) \notin (\tau_1,\tau_2)$-$(K,L)$ $cl(V)$. Then for some $H \in b$-$(K,L)O_{(Z,\tau_1,\tau_2)}$, $g(x) \in H$ and $H \cap V = \emptyset$. Thus $(\overline{H}^l)^{\circ_k} \cap V \subseteq \overline{H}^l \cap V = \emptyset$ and $(\overline{H}^l)^{\circ_k} \in (k,l)$-$RO_{(W,\sigma_1,\sigma_2)}$. Now g is (k,l)-almost (K,L)-continuous and by Theorem 3.7, there exists $U \in b$-$(K,L)O_{(Z,\tau_1,\tau_2)}$ containing x and $g(U) \subset (\overline{H}^l)^{\circ_k}$. Hence $g(U) \cap V = \emptyset$. Being $x \in (\tau_1,\tau_2)$ - $(K,L)cl\big(g^{-1}(V)\big)$, $U \cap g^{-1}(V) \neq \emptyset$ for each $U \in b$ - $(K,L)O_{(Z,\tau_1,\tau_2)}$ with $x \in U$. So $x \in U$ and then $g(U) \cap V \neq \emptyset$, a contradiction. Therefore $g(x) \in (\tau_1,\tau_2)$-(K,L) $cl(V)$.

Theorem 3.12: *$(Z,\tau_1,\tau_2,K,L) \xrightarrow{g} (W,\sigma_1,\sigma_2)$ is not (k,l)-almost (K,L)-continuous at $x \in Z$ iff there exists $V \in (k,l)$-$RO_{(W,\sigma_1,\sigma_2)}$ and x belong to the (τ_1,τ_2)-(K,L)-frontier of $g^{-1}(V)$.*

Proof: g isn't (k,l)-almost (K,L)-continuous at $x \in Z$, implies that $g(x)$ belong to some $V \in (k,l)$-$RO_{(W,\sigma_1,\sigma_2)}$ such that for every $U \in b$-$(K,L)O_{(Z,\tau_1,\tau_2)}$ containing x, $g(U) \subset (W \backslash V) \neq \emptyset$. In consequence, for every $U \in b$ - $(K,L)O_{(Z,\tau_1,\tau_2)}$ containing x, $U \cap \big(Z \backslash g^{-1}(V)\big) \neq \emptyset$. From this fact $x \in (\tau_1,\tau_2)$ - $(K,L)cl\big(Z \backslash g^{-1}(V)\big)$. Hence $x \in g^{-1}(V)$, so $x \in (\tau_1,\tau_2)$ -(K,L) $cl(g^{-1}(V))$. In consequence, x belongs to the (τ_1,τ_2)-(K,L)-frontier of $g^{-1}(V)$. Reciprocally, if we claim that x belong to the (τ_1,τ_2)-(K,L)-frontier of

$g^{-1}(V)$ for some $V \in (k,l)$ - $RO_{(W,\sigma_1,\sigma_2)}$ such that $g(x) \in V$ and g is (k,l)-almost (K,L)-continuous at the point x. Applying Theorem 3.6, for some $U \in b$-$(K,L)O_{(Z,\tau_1,\tau_2)}$ with $x \in U$ and $g(U) \subset V$. Then $U \subset g^{-1}(V)$. This shows that $x \in (\tau_1,\tau_2)$-(K,L) $int(g^{-1}(V))$. Hence $x \in (\tau_1,\tau_2)$-(K,L) $cl(Z\setminus g^{-1}(V))$ and $x \notin (\tau_1,\tau_2)$-$(K,L)Fr(g^{-1}(V))$, a contradiction. Consequently g is not almost b-(K,L)-continuous at the point x.

Theorem 3.13: *Given* $(Z,\tau_1,\tau_2,K,L) \overset{g}{\rightarrow} (W,\sigma_1,\sigma_2)$, *then:*

(1) g *is* (k,l)*-almost* (K,L)*-continuous;*

(2) $g^{-1}(C) \in b$-$(K,L)O_{(Z,\tau_1,\tau_2)}$ *for all* (k,l)-δ*-open subset* C *of* W*;*

(3) $g^{-1}(C) \in b$-$(K,L)C_{(Z,\tau_1,\tau_2)}$ *for every* (k,l)-δ*-closed subset* C *of* W*;*

(4) $g((\tau_1,\tau_2)$-(K,L) $clS) \subset (k,l)$- $cl_\delta(g(S))$ *for all subset* S *of* Z*;*

(5) (τ_1,τ_2)-(K,L) $cl(g^{-1}(C)) \subset g^{-1}((k,l)$-$cl_\delta(C)$ *for every subset* C *of* W*;*

(6) $g^{-1}((k,l)$-$int_\delta(C)) \subset (\tau_1,\tau_2)$-$(K,L)$ $int(g^{-1}(C))$ *for all* $C \subseteq W$*; are equivalent.*

Proof: (1)⇒ (2): Suppose $C \subseteq W$ a (k,l)-δ-open set. Then C can be written as $C = \bigcup_{\alpha\in K} C_\alpha$, where $C_\alpha \in (k,l)$ - $RO_{(W,\sigma_1,\sigma_2)}$, for each $\alpha \in K$. Therefore $g^{-1}(C) = \bigcup_{\alpha\in K} g^{-1}(C_\alpha) \in b$-$(K,L)O_{(Z,\tau_1,\tau_2)}$.

(2)⇒(3): Using the complement, we can easily obtain the result.

(3)⇒(4): For every $S \subseteq Z$, (k,l)-$cl_\delta(g(S)) \subseteq W$ is (k,l)-δ closed. Now, being $g^{-1}((k,l)$ - $cl_\delta(g(S))) \in b$ - $(K,L)C_{(Z,\tau_1,\tau_2)}$, we obtain the inclusions (τ_1,τ_2) - (K,L) $cl(S) \subset (\tau_1,\tau_2)$ - (K,L) $cl(g^{-1}(g(S))) \subset (\tau_1,\tau_2)$ - (K,L) $cl(g^{-1}((k,l)$ - $cl_\delta(g(S))) = g^{-1}((k,l)$ - $cl_\delta(g(S)))g((\tau_1,\tau_2)$ - (K,L) $cl(S)) \subset (k,l)$- $cl_\delta(g(S))$.

(4) ⇒ (5): For any $C \subseteq W$. By hypothesis, it follows that (τ_1,τ_2) - (K,L) $cl(g^{-1}(C)) \subset g^{-1}(g(\tau_1,\tau_2)$-$(K,L)$ $cl(g^{-1}(C))) \subset g^{-1}((k,l)$-$cl_\delta(C))$.

(5)⇒(6): Again, using the complement we get the result.

(6)⇒(1): If $C \in (k,l)$ - $RO_{(W,\sigma_1,\sigma_2)}$, then $C = (k,l)$ - $int_\delta(C)$. From the assumption $g^{-1}(C) = g^{-1}((k,l)$ - $int_\delta(C)) \subset (\tau_1,\tau_2)$ - (K,L) $int(g^{-1}(C)) \subset g^{-1}(C)$. Thus $g^{-1}(C) = (\tau_1,\tau_2) - (K,L)$ $int(g^{-1}(C)) \in$ -(K, L)O (Z,τ_1,τ_2).

Hence we conclude (1).

Theorem 3.14: *A bijection* $(Z,\tau_1,\tau_2,K,L) \overset{g}{\rightarrow} (W,\sigma_1,\sigma_2)$ *is* (k,l) *-almost* (K,L)*-continuous iff* (k,l)-$int_\delta(g(S)) \subset g((k,l)$-$int(S))$, *for every subset* S *of* Z.

Proof: Assuming the hypothesis,

$$\begin{aligned}(g^{-1}((k,l) - \ int_\delta(g(S)) &= (\tau_1,\tau_2)\text{-}(K,L)\ int(g^{-1}((k,l)\text{-}\ int_\delta(g(S))) \\ &\subseteq (\tau_1,\tau_2)\text{-}(K,L)\ int(g^{-1}(g(S))) \\ &= (\tau_1,\tau_2)\text{-}(K,L)\ int(S).\end{aligned}$$

Being g bijective, $(k,l)-\ int_\delta(g(S)) = g(g^{-1}((k,l)-\ int_\delta(g(S)))$, and then $g((\tau_1,\tau_2)-(K,L)\ int(S))$.

Reciprocally. If C is a (k,l)-δ-open subset of W, (k,l)- $int_\delta(C) = C$ and from hypothesis

$$\begin{aligned} g((\tau_1,\tau_2)\ \text{-}(K,L)\ int(g^{-1}(C))) &\supset (k,l)\text{-}\ int_\delta(g(g^{-1}(C))) \\ &= (k,l)\text{-}\ int_\delta(C) \\ &= C. \end{aligned}$$

In consequence, $g^{-1}(g((\tau_1,\tau_2)\text{-}(K,L)\ int(g^{-1}(C)))) \supset g^{-1}(C)$. Again, being g bijective

$$(\tau_1,\tau_2)-(K,L)\ int(g^{-1}(C)) = g^{-1}(g((k,l)-\ int(g^{-1}(C)))) \supset g^{-1}(C).$$

Therefore $g^{-1}(C) = (\tau_1,\tau_2)$ - $(K,L)\ int(g^{-1}(C)) \in b$ - $(K,L)O_{(Z,\tau_1,\tau_2)}$. Then g is (k,l)-almost (K,L)-continuous.

Definition 3.15: Given $(Z,\tau_1,\tau_2,K,L) \xrightarrow{g} (W,\sigma_1,\sigma_2)$, the set $D_{a(K,L)}g$ is defined as

$$D_{a(K,L)}g = \{x \in Z | g \text{ is not } (k,l) - \text{almost } (K,L) - \text{continuous at } x\}$$

Example 3.16: According with Example 3.5, $D_{a(K,L)}g = \{\emptyset\}$.

Example 3.17: Consider $Z = \{1,2,3\}$, $\tau_1 = \{\emptyset, Z, \{1\}, \{1,2\}$, $\tau_2 = \{\emptyset, Z, \{2,3\}$, $K = \{\emptyset\}$, $L = \{\emptyset, \{2\}\}$ and (W,σ_1,σ_2) where $W = \{1,2,3\}$, $\sigma_1 = \{\emptyset, W, \{1\}\}$ and $\sigma_2 = \{\emptyset, W, \{2\}\}$. Observe that:

$$\begin{aligned} IO_{(Z,\tau_1)} &= \{\emptyset, Z, \{1\}, \{1,2\}, \{1,3\}\}; \\ JO_{(Z,\tau_2)} &= \{\emptyset, Z, \{3\}, \{1,3\}, \{2,3\}\}; \\ IO_{(Z,\tau_1)} \cup JO_{(Z,\tau_2)} &= \{\emptyset, Z, \{1\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}\}; \\ IC_{(Z,\tau_1)} \cap JC_{(Z,\tau_2)} &= \{\emptyset, Z, \{2\}\}. \end{aligned}$$

Now, for $(Z,\tau_1,\tau_2,K,L) \xrightarrow{g} (W,\sigma_1,\sigma_2)$ given as: $g(1) = 2$, $g(2) = 1$, $g(3) = 3$. $D_{a(K,L)}g = \{2\}$ and g is not (k,l)-almost (K,L)-continuous.

Theorem 3.18: *Given* $(Z,\tau_1,\tau_2,K,L) \xrightarrow{g} (W,\sigma_1,\sigma_2)$, *then:*

$$\begin{aligned} D_{a(K,L)}g &= \cup_{V\in(l,k)-SPO_{(W,\sigma_1,\sigma_2)}} \{(\tau_1,\tau_2)\text{-}(K,L)\ cl(f^{-1}(V)) \backslash g^{-1}(\overline{V}^k)\} \\ &= \cup_{F\in(l,k)-SPC_{(W,\sigma_1,\sigma_2)}} \{g^{-1}((F)^{\circ k}) \backslash (\tau_1,\tau_2)\text{-}(K,L)\ int(g^{-1}(F))\} \\ &= \cup_{F\in(l,k)-SC_{(W,\sigma_1,\sigma_2)}} \{g^{-1}((F)^{\circ k}) \backslash (\tau_1,\tau_2)\text{-}(K,L)\ int(g^{-1}(F))\} \\ &= \cup_{V\in(l,k)-SO_{(W,\sigma_1,\sigma_2)}} \{(\tau_1,\tau_2)\text{-}(K,L)\ cl(g^{-1}(V)) \backslash g^{-1}(\overline{V}^k)\} \\ &= \cup_{V\in\sigma_k} \{g^{-1}(V) \backslash (\tau_1,\tau_2)\text{-}(K,L)\ int(g^{-1}(\overline{V}^l)^{\circ k})\}. \end{aligned}$$

Proof: The proof follows from Theorems 3.8-3.9.

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