Journal of Interdisciplinary Mathematics
ISSN: 0972-0502 (Print), ISSN: 2169-012X (Online)
Vol. 28 (2025), No. 5, pp. 1759–1777
DOI : 10.47974/JIM-1789

A random S-asymptotically δ_1-periodic mild solutions for functional evolution equations with delay and random effect

Amel Benaissa §
Fatima Bouazzaoui †
Laboratory of Mathematics
University of Sidi Bel-Abbès
P. O. Box 89
22000 Sidi Bel-Abbès
Algeria

Abdelkrim Salim *
Laboratory of Mathematics
University of Sidi Bel-Abbès
P. O. Box 89
22000 Sidi Bel-Abbès
Algeria

and

Faculty of Technology
Hassiba Benbouali University
P. O. Box 151
Chlef 02000
Algeria

§ *ORCID-ID:* 0000-0001-7330-1021
† *ORCID-ID:* 0009-0007-0762-6478
* *ORCID-ID:* 0000-0003-2795-6224

§ *E-mail:* amel2222@yahoo.com
† *E-mail:* bouazz_fz@hotmail.fr
* *E-mail:* salim.abdelkrim@yahoo.com; (Corresponding Author)
a.salim@univ-chlef.dz

Mouffak Benchohra ‡
Laboratory of Mathematics
University of Sidi Bel-Abbès
PO Box 89
22000 Sidi Bel-Abbès
Algeria

Abstract

Our aim in this work is to study the existence of S-asymptotically $\boldsymbol{\delta}_1$-periodic mild almost automorphic solutions of a random functional evolution equation with infinite delay and random effects with results based on a random fixed point theorem with stochastic domain. We give an example to show the applicability of our results.

Subject Classification: *34G20, 34K20, 34K30.*

Keywords: *Random fixed point, Functional evolution equation, Mild solution, Infinite delay, Almost automorphic.*

1. Introduction

The almost automorphic functions, which were first proposed in the literature by S. Bochner [10], are a significant generalization of the almost periodic functions. We refer the reader to [19], and [9, 13, 31, 34–36] for fundamental results on the almost periodic functions and their applications to deterministic differential systems. Until now, the almost automorphic functions and the almost automorphic solutions for differential systems have been investigated by many mathematicians [2, 3].

Many authors have expressed interest in the pseudo nearly periodic function, see [33]. The almost periodic solution for stochastic differential equations had been researched progressively [38] using the basic principles and analytical techniques for stochastic differential equations that have been thoroughly explored in the literature. In particular, in [37], the nearly periodic solution for the affine and stochastic evolution equations was researched. Furthermore, in [21], the square-mean almost automorphic process was presented, and the square-mean almost automorphic mild solution for the stochastic differential equations was studied, which extended the almost automorphic theory from the deterministic to the stochastic form.

‡ *ORCID-ID:* `0000-0003-3063-9449`

‡ *E-mail:* `benchohra@yahoo.com`

For several years, functional differential delay equations have been employed to represent scientific phenomena. There are several papers in which authors discuss differential problems with various forms of delays, see the papers [4–8, 11, 25–28] and the references therein.

In [14], Diop *et al.* considered the following random partial integro-differential equations with unbounded delay:

$$\begin{cases} \xi'(\theta,\delta) = \mathrm{A}\xi(\theta,\delta) + \int_0^\theta \beta(\theta-r)\xi(r,\delta)dr + \mathrm{F}(\theta,\xi_\theta(\cdot,\delta),\delta), \ \theta\in[0,\kappa], \\ \xi(\theta,\delta) = \wp(\theta,\delta), \quad \theta\in(-\infty,0], \end{cases}$$

where A is a generator of a C_0 semigroup $\left(\mathrm{S}(\theta)_{\theta\geq 0}\right)$ from a Banach space Ξ into Ξ, $\beta(\theta)$ is a closed linear operator with domain $\mathcal{D}(\mathrm{A}) \subset \mathcal{D}(\beta(\theta))$ and δ is a random variable. The authors used a random fixed point theorem with a stochastic domain combined with Schauder's fixed point theorem and Grimmer's resolvent operator theory in their proofs.

The authors of [5] studied the existence and controllability results for the second order functional differential equation with delay and random effects that follows:

$$\begin{cases} \xi''(\theta,\delta) = \mathcal{F}_1\xi(\theta,\delta) + \psi(\theta,\xi_\theta(\cdot,\delta),\delta) + \mathcal{F}_2 f(\theta,\delta), \text{a.e.}\,\theta\in[0,\kappa], \\ \xi(\theta,\delta) = \varpi_1(\theta,\delta); \theta\in(-\infty,0], \\ \xi'(0,\delta) = \varpi_2(\delta), \end{cases}$$

with results based on a random fixed point theorem with a stochastic domain.

In this paper, we will establish a random version of the theory for the almost automorphic solution. More precisely we will verifies the existence of mild random almost automorphic solution for evolution equations with delay and random effects of the form:

$$\xi'(\theta,\varpi) = \Phi(\theta)\xi(\theta,\varpi) + \Psi(\theta,\xi_\theta(\cdot,\varpi),\varpi), \quad \text{a.e. } \theta\in\Omega := [0,\infty), \tag{1}$$

$$\xi(\theta,\varpi) = \wp(\theta,\varpi), \quad \theta\in(-\infty,0], \tag{2}$$

where $(\Theta,\mathfrak{K},\mathfrak{Q})$ is a complete probability space, $\varpi\in\Theta$, $\Psi:\Omega\times\mathfrak{E}\times\Theta\to\Xi$, $\wp\in\mathfrak{E}\times\Theta$ are given random functions, $\{\Phi(\theta)\}_{0\leq\theta<+\infty}$ is a family of linear closed operators from Ξ into Ξ that generate an evolution system of operators $\{\Im(\theta,\rho)\}_{(\theta,\rho)\in\Omega\times\Omega}$ for $0\leq\rho\leq\theta<+\infty$, $\mathfrak{E}$ is the phase space, and $(\Xi,|\cdot|)$ is a Banach space. For ξ defined on $IR\times\Theta$ and any $\theta\in\Omega$ we denote by $\xi_\theta(\cdot,\varpi)$ the element of $\mathfrak{E}\times\Theta$ given by $\xi_\theta(\kappa,\varpi) = \xi(\theta+\kappa,\varpi)$, $\kappa\in(-\infty,0]$.

The following is how this work is organized. Section 2 presents notations, definitions, and theorems that will be used throughout the work. Section 3 shows the existence of random mild solutions for our differential equations with infinite delay. We will also provide an example to demonstrate the abstract consequence of our effort.

2. Preliminaries

Throughout the manuscript, we will utilize an axiomatic definition of the phase space $\mathfrak{E}$ introduced by Hale and Kato in [24] and follow the terminology used in [29]. Thus, $(\mathfrak{E}, \|\cdot\|_{\mathfrak{E}})$ will be a seminormed linear space of functions mapping $(-\infty, 0]$ into Ξ, meeting the following axioms:

(A1) If $\xi : IR \to \Xi$, is continuous on $\mathbb{R}_+$ and $\xi_0 \in \mathfrak{E}$, then for every $\theta \in \mathbb{R}_+$ the following requirements hold:

(i) $\xi_\theta \in \mathfrak{E}$;

(ii) There exists a positive constant β such that $|\xi(\theta)| \leq \beta \|\xi_\theta\|_{\mathfrak{E}}$;

(iii) There exist $\beta_1(\cdot), \beta_2(\cdot) : R_+ \to R_+$ with β_1 continuous and bounded, and β_2 locally bounded where

$$\|\xi_\theta\|_{\mathfrak{E}} \leq \beta_1(\theta) \sup\{|\xi(\rho)| : 0 \leq \rho \leq \theta\} + \beta_2(\theta) \|\xi_0\|_{\mathfrak{E}}.$$

(A_2) For the function ξ in (A_1), ξ_θ is a $\mathfrak{E}-$valued continuous function on $\mathbb{R}_+$.

(A_3) The space $\mathfrak{E}$ is complete.

Denote

$$\alpha_1 = \sup\{\beta_1(\theta) : \theta \in \mathbb{R}_+\},$$

and

$$\alpha_2 = \sup\{\beta_2(\theta) : \theta \in \mathbb{R}_+\}.$$

Remark

1. (ii) is equivalent to $|\wp(0)| \leq \beta \|\wp\|_{\mathfrak{E}}$ for every $\wp \in \mathfrak{E}$.
2. Since $\|\cdot\|_{\mathfrak{E}}$ is a seminorm, two elements $\wp, \ell \in \mathfrak{E}$ can verify $\|\wp - \ell\|_{\mathfrak{E}} = 0$ without necessarily $\wp(\kappa) = \ell(\kappa)$ for all $\kappa \leq 0$.
3. From the equivalence of in the first remark, we can see that for all $\wp, \ell \in \mathfrak{E}$ such that $\|\wp - \ell\|_{\mathfrak{E}} = 0$: We necessarily have that $\wp(0) = \ell(0)$.

By BUC we denote the space of bounded uniformly continuous functions defined from $(-\infty, 0]$ to Ξ.

By $BC := BC(\mathrm{IR})$ we denote the Banach space of all bounded and continuous functions from IR into Ξ equipped with the standard norm

$$\|\xi\|_{BC} = \sup_{\theta \in \mathrm{IR}} |\xi(\theta)|.$$

Finally, by $BC' := BC'([0, +\infty))$ we denote the Banach space of all bounded and continuous functions from $(0, +\infty))$ into Ξ equipped with the standard norm

$$\|\xi\|_{BC'} = \sup_{\theta \in [0,+\infty)} |\xi(\theta)|.$$

Suppose that $\{\Phi(\theta), \theta \geq 0\}$ is a family of closed densely defined linear unbounded operators on the Banach space Ξ and with domain $D(\Phi(\theta))$ independent of θ.

Definition 2.2: ([1, 18, 32]) A family of bounded linear operators

$$\{\Im(\theta,\rho)\}_{(\theta,\rho)\in\mathfrak{Z}} : \Im(\theta,\rho) : \Xi \to \Xi \ (\theta,\rho) \in \mathfrak{Z} := \{(\theta,\rho) \in \Omega \times \Omega : 0 \leq \rho \leq \theta < +\infty\}$$

is called en evolution system if the following properties are satisfied:

1. $\Im(\theta,\theta) = I$ where I is the identity operator in Ξ,
2. $\Im(\theta,\rho)\,\Im(\rho,\nu) = \Im(\theta,\nu)$ for $0 \leq \nu \leq \rho \leq \theta < +\infty$,
3. $\Im(\theta,\rho) \in B(\Xi)$ the space of bounded linear operators on Ξ, such that for $(\rho,\theta) \in \mathfrak{Z}$ and $\xi \in \Xi$, $(\theta,\rho) \to \Im(\theta,\rho)\,\xi$ is continuous.

Lemma 2.3: ([12]). *Let $\mathfrak{M} \subset BC(\Omega,\Xi)$ be a set verifying the requirements:*

(i) $\mathfrak{M}$ *is bounded in* $BC(\Omega,\Xi)$;

(ii) *the functions of* $\mathfrak{M}$ *are equicontinuous on any compact interval of* Ω;

(iii) *the set* $\mathfrak{M}(\theta) := \{\xi(\theta) : \xi \in \mathfrak{M}\}$ *is relatively compact on any compact interval of* Ω;

(iv) *the functions from* $\mathfrak{M}$ *are equiconvergent, i.e., given* $\upsilon > 0$, *there corresponds* $\varkappa(\upsilon) > 0$ *such that* $|\xi(\theta) - \xi(+\infty)| < \upsilon$ *for any* $\theta \geq \varkappa(\upsilon)$ *and* $\xi \in \mathfrak{M}$.

Then $\mathfrak{M}$ is relatively compact in $BC(\Omega,\Xi)$.

Let $\mathfrak{X}$ be a separable Banach space with the Borel σ-algebra $B_{\mathfrak{X}}$. $\xi : \Theta \to \mathfrak{X}$ is a random variable with values in $\mathfrak{X}$ if for $B \in B_{\Upsilon}$, $\xi^{-1}(B) \in \mathfrak{K}$. $\varkappa : \Theta \times \mathfrak{X} \to \mathfrak{X}$ is a random operator if $\varkappa(.,\xi)$ is measurable for $\xi \in \mathfrak{X}$ and is expressed as $\varkappa(\varpi,\xi) = \varkappa(\varpi)\xi$.

Definition 2.4: (15). Let $\mathfrak{M}$ be a mapping from Θ into $2^{\mathfrak{X}}$. A mapping $\varkappa : \{(\varpi,\xi) : \varpi \in \Theta \wedge \xi \in \mathfrak{M}(\varpi)\} \to \mathfrak{X}$ is called random operator with stochastic domain $\mathfrak{M}$ iff $\mathfrak{M}$ is measurable (i.e., for all closed $\Phi \subseteq \mathfrak{X}$, $\{\varpi \in \Theta : \mathfrak{M}(\varpi) \cap \Phi \neq \emptyset\} \in \mathfrak{K}$) and for all open $D \subseteq \mathfrak{X}$ and all $\xi \in \mathfrak{X}$, $\{\varpi \in \Theta : \xi \in \mathfrak{M}(\varpi) \wedge \varkappa(\varpi,\xi) \in D\} \in \mathfrak{K}$. $\varkappa$ we be called continuous if every $\theta(\varpi)$ is continuous. For a random operator $\varkappa$, a mapping $\xi : \Theta \to \mathfrak{X}$ is called random (stochastic) fixed point of $\varkappa$ iff for p-almost all $\varpi \in \Theta$, $\xi(\varpi) \in \mathfrak{M}(\varpi)$ and $\varkappa(\varpi)\ \xi(\varpi) = \xi(\varpi)$ and for all open $D \subseteq \mathfrak{X}$, $\{\varpi \in \Theta : \xi(\varpi) \in D\} \in \mathfrak{K}$ (ξ is measurable).

Remark 2.5: If $\mathfrak{M}(\varpi) \equiv \mathfrak{X}$, then the definition of random operator with stochastic domain coincides with the definition of random operator.

Lemma 2.6: [15] *Let $\mathfrak{M} : \Theta \to 2^{\mathfrak{X}}$ be measurable with $\mathfrak{M}(\varpi)$ closed, convex and solid (i.e., $int\mathfrak{M}(\varpi) \neq \emptyset$) for all $\varpi \in \Theta$. We assume that there exists measurable $\xi_0 : \Theta \to \mathfrak{X}$ with $\xi_0 \in int\mathfrak{M}(\varpi)$ for all $\varpi \in \Theta$. Let $\varkappa$ be a continuous random operator with stochastic domain $\mathfrak{M}$ such that for every $\varpi \in \Theta$, $\{\xi \in \mathfrak{M}(\varpi) : \varkappa(\varpi)\ \xi = \xi\} \neq \emptyset$. Then $\varkappa$ has a stochastic fixed point.*

Let ξ be a mapping of $\Omega \times \Theta$ into X. ξ is said to be a stochastic process if for each $\theta \in \Omega$, $\xi(\theta,\cdot)$ is measurable.

Definition 2.7: [17] A function $\Psi \in BC(\mathbb{R}^+,\Xi)$ is called S-asymptotically δ_1-periodic if there exists $\delta_1 > 0$ such that $\lim_{\theta\to+\infty}(\Psi(\theta+\delta_1) - \Psi(\theta)) = 0.$ In this case we say that δ_1 is an asymptotic period of Ψ and that Ψ is S-asymptotically δ_1-periodic.

We will denote by $SAP_{\delta_1}(\Xi)$, the set of all S-asymptotically δ_1-periodic functions from $\mathbb{R}^+$ to X.

The following result is due to Henriquez-Pierri-Táboas; Proposition 3.5 in [16].

Theorem 2.8: *Endowed with the norm $\|\xi\|_\infty$, $SAP_{\delta_1}(X)$ is a Banach space.*

Sketch of the proof. We consider the translation operator $\nu_{\delta_1} : BC(\mathbb{R}^+,X) \to BC(\mathbb{R}^+,X)$ defined by $\nu_{\delta_1}\Psi(\theta) = \Psi(\theta+\delta_1)$. Clearly the function ν_{δ_1} is linear and continuous. We note that $SAP_{\delta_1}(\Xi) = (\nu_{\delta_1} - I)^{-1}(C_0(\mathbb{R}^+,\Xi))$. And then, since $(\nu_{\delta_1} - I)^{-1}$ is linear continuous and since $C_0(\mathbb{R}^+,\Xi)$ is a closed vector subspace of $BC(\mathbb{R}^+,\Xi)$, $SAP_{\delta_1}(\Xi)$ is a closed vector subspace of the Banach space $BC(\mathbb{R}^+,\Xi)$.

3. Existence of mild solutions

In this section, we study the main existence result for problem (1)-(2).

Definition 3.1: A stochastic process $\xi : \Omega \times \Theta \to \Xi$ is a random mild solution of (1)-(2) if $\xi(\theta,\varpi) = \wp(\theta,\varpi)$, $\theta \in (-\infty, 0)$ and the restriction of $\xi(\cdot,\varpi)$ to the interval $(0,\infty)$ is continuous and verifies

$$\xi(\theta,\varpi) = \Im(\theta,0)\wp(0,\varpi) + \int_0^{\theta} \Im(\theta,\rho)\Psi(\rho,\xi_\rho(\cdot,\varpi),\varpi)d\rho, \theta \in \Omega = (0,\infty). \quad (3)$$

We will need to introduce the following hypotheses which are be assumed there after:

(H1) The evolution family $\{\Im(\theta,\rho)\}_{\theta \geq \rho}$ is compact and there exists a constant $\gamma_1 \geq 1$ and $\zeta > 0$ such that

$$\| \Im(\theta,\rho) \|_{B(\Xi)} \leq \gamma_1 e^{-\zeta(\theta-\rho)} \quad \text{for every } (\rho,\theta) \in \mathfrak{Z}.$$

and

$$\Im(\theta+\delta_1, \rho+\delta_1) = \Im(\theta,\rho).$$

(H2) The function $\Psi : \Omega \times \Xi \times \Theta \to \Xi$ is Carathéodory and satisfies the following:

(a) There exists a constant $\delta_1 > 0$, such that

$$\lim_{\theta \to +\infty} \left| \Psi(\theta+\delta_1,u,\varpi) - \Psi(\theta,u,\varpi) \right| = 0 \text{ for a.e } \theta,\varpi \in \Omega \times \Theta \text{ and each } u \in \Xi.$$

(b) There exist $p \in L^{\infty}(\Omega \times \Theta, R^+)$, and a continuous nondecreasing function $\ell : [0,\infty) \times \Theta \to (0,\infty)$ such that:

$$|\Psi(\theta,u,\varpi)| \leq p(\theta,\varpi)\ell(\| u \|_B, \varpi) \text{ for a.e } \theta \in \Omega \times \Theta \text{ and each } u \in \Xi.$$

(H3) There exists a random function $\chi : \Theta \to \mathbb{R}^+ \setminus \{0\}$ such that:

$$\gamma_1 \| \wp \|_{\mathfrak{E}} + \frac{\gamma_1 \ell(D_1,\varpi)\|p\|_{L^\infty}}{\zeta} \leq \chi(\varpi),$$

where

$$D_1 := \alpha_1 \chi(\varpi) + \alpha_2 \| \wp \|_{\mathfrak{E}} .$$

(H4) For each $\varpi \in \Theta, \wp(\cdot,\varpi)$ is continuous and for each $\theta, \wp(\theta,\cdot)$ is measurable, and $\lim_{\theta \to \infty} (\wp(\theta+\delta_1,\varpi) - \wp(\theta,\varpi)) = 0$.

(H5) For each $(\theta,\rho) \in \mathfrak{Z}$ we have: $\lim_{\theta \to +\infty} \int_0^{\theta} e^{-\zeta(\theta-\rho)} p(\rho,\varpi) d\rho = 0$.

Theorem 3.2: *Assume that* $(H_1)-(H_5)$ *are met. Then, problem (1)-(2) has at least one S-asymptotically* δ_1 *-periodic mild random solution on* $(-\infty,\infty)$.

Proof: Consider the operator $\aleph:\Theta\times SAP_{\delta_1}(X)\to SAP_{\delta_1}(X)$ be the random operator given by

$$(\aleph(\varpi)\xi)(\theta)=\Im(\theta,0)\wp(0,\varpi)+\int_0^\theta \Im(\theta,\rho)\Psi(\rho,\xi_\rho,\varpi)d\rho, \quad \theta\in\Omega, \tag{4}$$

and

$$\xi(\theta,\varpi)=\wp(\theta,\varpi), \theta\in(-\infty,0).$$

Let the operators $\overline{\aleph}=\aleph_1+\aleph_2$, where

$$(\aleph_1(\varpi)\xi)(\theta)=\Im(\theta,0)\wp(0,\varpi),$$

$$(\aleph_2(\varpi)\xi)(\theta)=\int_0^\theta \Im(\theta,\rho)\Psi(\rho,\xi_\rho,\varpi)d\rho.$$

Claim 1: $\aleph_1\in SAP_{\delta_1}(\Xi)$.

For $\theta\geq 0$, one has

$$\begin{aligned}|(\aleph_1(\varpi)\xi)(\theta+\delta_1)-(\aleph_1(\varpi)\xi)(\theta)| &= |\Im(\theta+\delta_1,0)\wp(0,\varpi)-\Im(\theta,0)\wp(0,\varpi)|\\ &\leq \gamma_1\|\wp\|_{\mathfrak{E}}(e^{-\zeta(\theta+\delta_1)}+e^{-\zeta\theta}).\end{aligned} \tag{5}$$

Since $\zeta>0$, we deduce that

$$\lim_{\theta\to+\infty}|(\aleph_1(\varpi)\xi)(\theta+\delta_1)-(\aleph_1(\varpi)\xi)(\theta)|=0.$$

Then $\aleph_1\in SAP_{\delta_1}(X)$.

Claim 2: $\aleph_2\in SAP_{\delta_1}(\Xi)$..

For $\theta\geq 0$, one has

$$\begin{aligned}&(\aleph_2(\varpi)\xi)(\theta+\delta_1)-(\aleph_2(\varpi)\xi)(\theta)\\ &=\int_0^{\theta+\delta_1}\Im(\theta+\delta_1,\rho)\Psi(\rho,\xi_\rho,\varpi)d\rho-\int_0^\theta \Im(\theta,\rho)\Psi(\rho,\xi_\rho,\varpi)d\rho\\ &=\int_0^{\delta_1}\Im(\theta+\delta_1,\rho)\Psi(\rho,\xi_\rho,\varpi)d\rho\\ &+\int_{\delta_1}^{\theta+\delta_1}\Im(\theta+\delta_1,\rho)\Psi(\rho,\xi_\rho,\varpi)d\rho-\int_0^\theta \Im(\theta,\rho)\Psi(\rho,\xi_\rho,\varpi)d\rho\\ &=\Upsilon_1(\theta)+\Upsilon_2(\theta),\end{aligned}$$

where

$$\Upsilon_1(\theta)=\int_0^{\delta_1}\Im(\theta+\delta_1,\rho)\Psi(\rho,\xi_\rho,\varpi)d\rho,$$

and

$$\Upsilon_2(\theta)=\int_{\delta_1}^{\theta+\delta_1}\Im(\theta+\delta_1,\rho)\Psi(\rho,\xi_\rho,\varpi)d\rho-\int_0^{\theta}\Im(\theta,\rho)\Psi(\rho,\xi_\rho,\varpi)d\rho.$$

Using the fact that $(\Im(\theta,\rho))_{\theta\geq\rho}$ is exponentially stable, we obtain

$$\begin{aligned}|\Upsilon_1(\theta)| &\leq \gamma_1\int_0^{\delta_1}e^{-\zeta(\theta+\delta_1-\rho)}p(\rho,\varpi)\ell(\|\xi_\rho\|_{\mathfrak{E}},\varpi)d\rho\\ &= \gamma_1 e^{-\zeta\theta}\ell(\|\xi_\rho\|_{\mathfrak{E}},\varpi)\|p\|_{L^\infty}\int_0^{\delta_1}e^{-\zeta(\delta_1-\rho)}d\rho\\ &= \gamma_1 e^{-\zeta\theta}\ell(\|\xi_\rho\|_{\mathfrak{E}},\varpi)\|p\|_{L^\infty}\frac{1-e^{-\zeta\delta_1}}{\zeta},\end{aligned}$$

which shows that

$$\lim_{\theta\to+\infty}\Upsilon_1(\theta)=0.$$

Let us write

$$\Upsilon_2(\theta)=\int_0^{\theta}\Im(\theta+\delta_1,\rho+\delta_1)\Psi(\rho+\delta_1,\xi_{\rho+\delta_1},\varpi)d\rho-\int_0^{\theta}\Im(\theta,\rho)\Psi(\rho,\xi_\rho,\varpi)d\rho$$

and since the evolution family is δ_1-periodic, we obtain

$$\begin{aligned}\Upsilon_2(\theta)&\leq\int_0^{\theta}\Im(\theta,\rho)|\Psi(\rho+\delta_1,\xi_{\rho+\delta_1},\varpi)-\Psi(\rho+\delta_1,\xi_\rho,\varpi)|d\rho\\ &+\int_0^{\theta}\Im(\theta,\rho)|\Psi(\rho+\delta_1,\xi_\rho,\varpi)-\Psi(\rho,\xi_\rho,\varpi)|d\rho\\ &=I_1(\theta)+I_2(\theta),\end{aligned}$$

where

$$\begin{aligned}I_1(\theta)&=\int_0^{T_\varepsilon}\Im(\theta,\rho)|\Psi(\rho+\delta_1,\xi_{\rho+\delta_1},\varpi)-\Psi(\rho+\delta_1,\xi_\rho,\varpi)|d\rho\\ &+\int_{T_\varepsilon}^{\theta}\Im(\theta,\rho)|\Psi(\rho+\delta_1,\xi_\rho,\varpi)-\Psi(\rho,\xi_\rho,\varpi)|d\rho,\end{aligned}$$

$$\begin{aligned}I_2(\theta)&=\int_{T_\varepsilon}^{\theta}\Im(\theta,\rho)|\Psi(\rho+\delta_1,\xi_{\rho+\delta_1},\varpi)-\Psi(\rho+\delta_1,\xi_\rho,\varpi)|d\rho\\ &+\int_0^{T_\varepsilon}\Im(\theta,\rho)|\Psi(\rho+\delta_1,\xi_\rho,\varpi)-\Psi(\rho,\xi_\rho,\varpi)|d\rho,\end{aligned}$$

and $T_\varepsilon >$. Observing that

$$\begin{aligned}|I_1(\theta)| \le & \left|\int_0^{T_\varepsilon} \Im(\theta,\rho)\,|\Psi(\rho+\delta_1,\xi_\rho(\rho+\delta_1),\varpi)-\Psi(\rho+\delta_1,\xi_\rho,\varpi))|\,d\rho\right| \\ & +\left|\int_{T_\varepsilon}^{\theta} \Im(\theta,\rho)(\Psi(\rho+\delta_1,\xi_{\rho+\delta_1},\varpi)-\Psi(\rho+\delta_1,\xi_\rho,\varpi)d\rho\right|.\end{aligned}$$

Since $\xi \in SAP_{\delta_1},$, we know that there is a positive constant $T_\varepsilon > 0$ such that

$$\left|\xi(\theta+\delta_1,\varpi)-\xi(\theta,\varpi)\right| \le \varepsilon, \quad \text{for } \theta \ge T_\varepsilon.$$

By (H_3), we have

$$|\Psi(\theta,\xi_{\rho+\delta_1},\varpi)-\Psi(\theta,\xi_\rho,\varpi)| \le \frac{\zeta\varepsilon}{\gamma_1}, \quad \text{for } \theta \ge T_\varepsilon.$$

Thus, we get

$$\begin{aligned}& \left|\int_{T_\varepsilon}^{\theta} \Im(\theta,\rho)(\Psi(\rho+\delta_1,\xi_\rho(\rho+\delta_1),\varpi)-\Psi(\rho+\delta_1,\xi_\rho,\varpi)d\rho\right| \\ & \le \gamma_1 \int_{T_\varepsilon}^{\theta} e^{-\zeta(\theta-\rho))}\,|\Psi(\rho+\delta_1,\xi_{\rho+\delta_1},\varpi)-\Psi(\rho+\delta_1,\xi_\rho)|\,d\rho \\ & \le \int_{T_\varepsilon}^{\theta} \gamma_1 e^{-\zeta(\theta-\rho)}\frac{\zeta\varepsilon}{\gamma_1}d\rho \\ & \le \varepsilon(e^{-\zeta\theta}-e^{-\zeta(\theta-T_\varepsilon)}) \\ & \le \varepsilon.\end{aligned} \tag{6}$$

Also, we have

$$\begin{aligned}& \left|\int_0^{T_\varepsilon} \Im(\theta,\rho)(\Psi(\rho+\delta_1,\xi_{\rho+\delta_1},\varpi)-\Psi(\rho+\delta_1,\xi_\rho)d\rho\right| \\ & \le \gamma_1 \int_0^{T_\varepsilon} e^{-\zeta(\theta-\rho)}(|\Psi(\rho+\delta_1,\xi_{\rho+\delta_1},\varpi)-\Psi(\rho+\delta_1,\xi_\rho)|)d\rho \\ & \le \gamma_1 \int_0^{T_\varepsilon} e^{-\zeta(\theta-\rho)}(\,|\Psi(\rho+\delta_1,\xi_{\rho+\delta_1},\varpi)|+|\Psi(\rho+\delta_1,\xi_\rho,\varpi)|\,)d\rho \\ & \le 2\gamma_1 \int_0^{T_\varepsilon} e^{-\zeta(\theta-\rho)} p(\rho,\varpi)\ell(\,|\xi(\rho)|\,,\varpi)d\rho \\ & \le 2\gamma_1 \ell(\|\xi\|_{BC},\varpi)\int_0^{T_\varepsilon} e^{-\zeta(\theta-\rho)} p(\rho,\varpi)d\rho.\end{aligned}$$

For $\theta \ge 0$, it follows from the Hölder inequality that

$$\begin{aligned}
&\left|\int_0^{T_\varepsilon}\Im(\theta,\rho)(\Psi(\rho+\delta_1,\xi_{\rho+\delta_1},\varpi)-\Psi(\rho+\delta_1,\xi_\rho,\varpi)d\rho\right| \\
&\leq 2\gamma_1\|p\|_{L_\infty}\ \ell(\|\xi\|_{BC},\varpi)\left[\int_0^{T_\varepsilon}e^{-\zeta(\theta-\nu)}d\nu\right] \\
&\leq \frac{2\gamma_1\|p\|_{L_\infty}\ \ell(\|\xi\|_{BC},\varpi)}{\zeta}(e^{-\zeta\theta}-e^{-\zeta(\theta-T_\varepsilon)}) \\
&\leq \varepsilon.
\end{aligned} \tag{7}$$

By (6), (7) which implies that $\lim_{\theta\to\infty} I_1(\theta)=0.$

By $(H_2)(a),$ we can see that, for every $\varepsilon>0,$ there is a positive constant $T_\varepsilon>0$ such that

$$\left|\Psi(\theta+\delta_1,\xi_.(\theta,\varpi),\varpi)-\Psi(\theta,\xi_.(\theta,\varpi),\varpi)\right|\leq\varepsilon \quad \text{for } \theta\geq T_\varepsilon.$$

Similarly, we find

$$\begin{aligned}
I_2(\theta) \leq\ & \gamma_1\int_{T_\varepsilon}^{\theta}e^{-\zeta(\theta-\rho))}\,|\Psi(\rho+\delta_1,\xi_{\rho+\delta_1},\varpi)-\Psi(\rho,\xi_\rho,\varpi)|\,d\rho \\
+\ & \gamma_1\int_0^{T_\varepsilon}e^{-\zeta(\theta-\rho))}\,|\Psi(\rho+\delta_1,\xi_\rho)-\Psi(\rho,\xi_\rho,\varpi)|\,d\rho \\
\leq\ & \int_{T_\varepsilon}^{\theta}\gamma_1 e^{-\zeta(\theta-\rho)}\frac{\zeta\varepsilon}{\gamma_1}d\rho \\
+\ & \gamma_1\int_0^{T_\varepsilon}e^{-\zeta(\theta-\rho)}(\,|\Psi(\rho+\delta_1,\xi_\rho,\varpi)|+|\Psi(\rho,\xi_\rho,\varpi)|\,)d\rho \\
\leq\ & \varepsilon(1-e^{-\zeta(\theta-T_\varepsilon)}) \\
+\ & 2\gamma_1\int_0^{T_\varepsilon}e^{-\zeta(\theta-\rho)}p(\rho,\varpi)\ell(\,|\xi(\rho)|\,,\varpi)d\rho \\
\leq\ & \varepsilon(1-e^{-\zeta(\theta-T_\varepsilon)}) \\
+\ & 2\gamma_1\ell(\|\xi\|_\infty,\varpi)\int_0^{T_\varepsilon}e^{-\zeta(\theta-\rho)}p(\rho,\varpi)d\rho.
\end{aligned}$$

For $\theta\geq 0,$ it follows from the Hölder inequality that

$$\begin{aligned}
I_2(\theta) \leq\ & \varepsilon(1-e^{-\zeta(\theta-T_\varepsilon)}) \\
+\ & 2\gamma_1\|p\|_{L_\infty}\ \ell(\|\xi\|_\infty,\varpi)\left[\int_0^{T_\varepsilon}e^{-\zeta(\theta-\nu)}d\nu\right] \\
\leq\ & \varepsilon(1-e^{-\zeta(\theta-T_\varepsilon)}) \\
+\ & 2\varepsilon\frac{\gamma_1\|p\|_{L_\infty}\ \ell(\|\xi\|_\infty,\varpi)}{\zeta}(e^{-\zeta(\theta-T_\varepsilon)}-e^{-\zeta\theta}) \\
\leq\ & \varepsilon.
\end{aligned}$$

Consequently, $\lim_{\theta\to\infty} I_2(\theta)=0$, we conclude that $\aleph_2 \in SAP_{\delta_1}$. From **Claim 1** and **Claim 2** we deduce $\aleph \in SAP_{\delta_1}(\Xi)$, and $\wp(\cdot,\varpi)\in SAP_{\delta_1}(\Xi)$.

Next, we will prove that the operator $\aleph$ satisfies all the assumptions of Lemma 2.6.

Let $\mathfrak{X} = SAP_{\delta_1}(\Xi)$. Then we show that the mapping defined by (4) is a random operator. To do this, we need to prove that for any $\xi\in\mathfrak{X}$, $\aleph(\cdot)(\xi)$ $:\Theta\to\mathfrak{X}$ is a random variable. Then we prove that $\aleph(\cdot)(\xi):\Theta\to\mathfrak{X}$ is measurable since the mapping $\Psi(\theta,\xi,\cdot)$, $\theta\in\Omega$, $\xi\in\mathfrak{X}$ is measurable by assumption (H_2) and (H_4).

Let $\sigma:\Theta\to 2^{\mathfrak{X}}$ be defined by:

$$\sigma(\varpi)=\{\xi\in\mathfrak{X}:\|\xi\|\le\chi(\varpi)\}.$$

$\sigma(\varpi)$ is bounded, closed, convex and solid for all $\varpi\in\Theta$. Then σ is measurable by Lemma 17 (see [30]).

Let $\varpi\in\Theta$ be fixed, then for $\xi\in\sigma(\varpi)$, and by $(A1)$, we have

$$\begin{aligned}\|\xi_\rho\|_{\mathfrak{E}} &\le \beta_1(\rho)|\xi(\rho)|+\beta_2(\rho)\|\xi_0\|_{\mathfrak{E}}\\ &\le \alpha_1|\xi(\rho)|+\alpha_2\|\wp\|_{\mathfrak{E}}.\end{aligned}$$

By (H_3) and (H_2), we have

$$\begin{aligned}|(\aleph(\varpi)\xi)(\theta)| &\le \|\Im(\theta,0)\|_{B(\Xi)}|\wp(0,\varpi)|+\int_0^\theta\|\Im(\theta,\rho)\|_{B(\Xi)}|\Psi(\rho,\xi_\rho,\varpi)|\,d\rho\\ &\le \gamma_1 e^{-\zeta\theta}\|\wp\|_{\mathfrak{E}}+\gamma_1\int_0^\theta e^{-\zeta(\theta-\rho)}p(\rho,\varpi)\,\ell(\|\xi_\rho\|_{\mathfrak{E}},\varpi)\,d\rho.\end{aligned}$$

Set

$$D_1:=\alpha_1\chi(\varpi)+\alpha_2\|\wp\|_{\mathfrak{E}}.$$

Then, we have

$$|(\aleph(\varpi)\xi)(\theta)|\le\gamma_1\|\wp\|_{\mathfrak{E}}+\gamma_1\ell(D_1,\varpi)\|p\|_{L_\infty}(1-e^{-\zeta\theta})d\rho.$$

Thus

$$\|(\aleph(\varpi)\xi)\|\le\gamma_1\|\wp\|_{\mathfrak{E}}+\frac{\gamma_1\ell(D_1,\varpi)\|p\|_{L_\infty}}{\zeta}\le\chi(\varpi).$$

Thus, $\aleph$ is a random operator with stochastic domain σ and $\mathfrak{K}(\varpi):$ $\sigma(\varpi)\to\sigma(\varpi)$ for each $\varpi\in\Theta$.

Step 1: $\aleph$ is continuous.

Let ξ^n be a sequence where $\xi^n \to \xi$ in $\mathfrak{X}$. Since the function Ψ is Carathéodory, we can see that

$$\left|\Psi(\rho,\xi_\rho^n,\varpi)-\Psi(\rho,\xi_\rho,\varpi)\right| \le \frac{\zeta\varepsilon}{\gamma_1}. \tag{8}$$

If $\theta \in \Omega,$ then (8) and our hypotheses gives us that

$$\begin{aligned}
&\left|\aleph(\varpi)\xi_\rho^n(\theta)-\aleph(\varpi)\xi_\rho(\theta)\right| \\
&\le \int_0^\theta \|\Im(\theta,\rho)\|_{B(\Xi)} |\Psi(\rho,\xi_n(\rho),\varpi)-\Psi(\rho,\xi(\rho),\varpi)|\, d\rho \\
&\le \gamma_1 \frac{\zeta\varepsilon}{\gamma_1}\int_0^\theta e^{-\zeta(\theta-\rho)} d\rho \\
&\le \frac{\gamma_1}{\zeta}\frac{\zeta\varepsilon}{\gamma_1}(1-e^{-\zeta\theta}) \\
&\le \varepsilon.
\end{aligned} \tag{9}$$

Then the inequality (9) reduces to

$$\|\aleph(\varpi)\xi^n-\aleph(\varpi)\xi\|_\infty \to 0 \quad \text{as } n\to\infty.$$

Now, we conclude that $\aleph$ is continuous.

Step 2: We demonstrate that for $\varpi \in \Theta$, $\{\xi \in \sigma(\varpi) : \aleph(\varpi)\xi = \xi\} \neq \emptyset.$ For prove this we apply Schauder's theorem [23].

$\aleph(\sigma(\varpi))$ **relatively compact:** To prove the compactness, we will use Corduneanu's lemma.

(a) Firstly, it is clear that the assumption (i) is holds. Then we will demonstrate that $\aleph(\sigma(\varpi))$ is equicontinuous set for each closed bounded interval $[0,\mathfrak{A}]$ in Ω. Let ν_1, $\nu_2 \in [0,\mathfrak{A}]$ with $\nu_2 > \nu_1$, $\sigma(\varpi)$ be a bounded set as in Step 2, and $\xi \in \sigma(\varpi)$. Thus

$$\begin{aligned}
|(\aleph(\varpi)\xi)(\nu_2)-(\aleph(\varpi)\xi)(\nu_1)| &\le \|\Im(\nu_2,0)-\Im(\nu_1,0)\|_{B(\Xi)}\|\wp\|_B \\
&\quad + \left|\int_0^{\nu_1}[\Im(\nu_2,\rho)-\Im(\nu_1,\rho)]\Psi(\rho,\xi_\rho,\varpi)d\rho\right| \\
&\quad + \left|\int_{\nu_1}^{\nu_2}\Im(\nu_2,\rho)\Psi(\rho,\xi_\rho,\varpi)d\rho\right| \\
&\le \|\Im(\nu_2,0)-\Im(\nu_1,0)\|_{B(\Xi)}\|\wp\|_B \\
&\quad + \int_0^{\nu_1}|\Im(\nu_2,\rho)-\Im(\nu_1,\rho)|\,|\Psi(\rho,\xi_\rho,\varpi)|d\rho
\end{aligned}$$

$$+\int_{v_1}^{v_2} |\Im(v_2,\rho)|\ |\Psi(\rho,\xi_\rho,\varpi)|d\rho$$
$$\leq \|\Im(v_2,0)-\Im(v_1,0)\|_{B(\Xi)}\|\wp\|_B$$
$$+\ \ell(D_1,\varpi)\int_0^{v_1}\|\Im(v_2,\rho)-\Im(v_1,\rho)\|_{B(\Xi)}\ p(\rho,\varpi)d\rho$$
$$+\frac{\gamma_1\|p\|_{L_\infty}\ \ell(D_1,\varpi)}{\zeta}(e^{-\zeta(\theta-v_2)}-e^{-\zeta(\theta-v_1)}).$$

The right-hand of the above inequality tends to zero as $v_2-v_1\to 0$, as $\aleph$ is bounded and equicontinuous .

(b) Now we demonstrate that $Q(\theta,\varpi) = \{(\aleph(\varpi)\xi)(\theta) : \xi\in\sigma(\varpi)\}$ is precompact in Ξ. Let $\theta\in[0,\mathfrak{A}]$ be fixed and let ε be verifying $0<\varepsilon<\theta$. For $\xi\in\sigma(\varpi)$ we define

$$(\aleph_\varepsilon(\varpi)\xi)(\theta)=\Im(\theta,0)\wp(0,\varpi)+\Im(\theta,\theta-\varepsilon)\int_0^{\theta-\varepsilon}\Im(\theta-\varepsilon,\rho)\Psi(\rho,\xi_\rho,\varpi)d\rho.$$

Since $\Im(\theta,\rho)$ is a compact operator and the set $Q_\varepsilon(\theta,\varpi) = \{(\aleph_\varepsilon(\varpi)\xi)(\theta) : \xi\in\sigma(\varpi)\}$ is the image of bounded set of Ξ then $Q_\varepsilon(\theta,\varpi)$ is precompact in Ξ for every ε, $0<\varepsilon<\theta$. Moreover

$$|(\aleph(\varpi)\xi)(\theta)-(\aleph_\varepsilon(\varpi)\xi)(\theta)|\leq\int_{\theta-\varepsilon}^{\theta}\|\Im(\theta,\rho)\|_{B(\Xi)}|\Psi(\rho,\xi_\rho,\varpi)|\,d\rho$$
$$\leq\gamma_1\ell(D_1,\varpi)\int_{\theta-\varepsilon}^{\theta}e^{-\zeta(\theta-\rho)}p(\rho,\varpi)d\rho.$$

Thus, $Q(\theta,\varpi)=\{(\aleph(\varpi)\xi)(\theta):\xi\in\sigma(\varpi)\}$ is precompact in Ξ.

(c) We prove that $\aleph$ is equiconvergent.
Let $\xi\in\sigma(\varpi)$, then from (H_1), (H_2) we have

$$|(\aleph(\varpi)\xi)(\theta)|\leq\gamma_1 e^{-\zeta\theta}\|\wp\|_{\mathfrak{C}}+\gamma_1\int_0^{\theta}e^{-\zeta(\theta-\rho)}p(\rho,\varpi)\,\ell(D_1,\varpi)\,d\rho,$$

it follows immediately by (H_5) that $|(\aleph(\varpi)\xi)(\theta)|\to 0$ as $\theta\to+\infty$.
Then

$$\lim_{\theta\to+\infty}|(\aleph(\varpi)\xi)(\theta)-(\aleph(\varpi)\xi)(+\infty)|=0.$$

Therefore, $\aleph$ is equiconvergent.

Consequently, $\aleph(\varpi):\sigma(\varpi)\to\sigma(\varpi)$ is continuous and compact. By Schauder's theorem [23], $\aleph(\varpi)$ has a fixed point $\xi(\varpi)$ in $\sigma(\varpi)$. Since $\bigcap_{\varpi\in\Theta}\sigma(\varpi)\neq\emptyset$, the hypothesis that a measurable selector of $\operatorname{int}\sigma$ exists holds. By Lemma 2.6, $\aleph$ has a stochastic fixed point $\xi^*(\varpi)$, which is a random mild solution of (1)-(2).

4. An example

Consider the following example:

$$\frac{\partial}{\partial\theta}\vartheta(\theta,\alpha,\varpi)=j(\theta,\alpha)\frac{\partial^2}{\partial\alpha^2}\vartheta(\theta,\alpha,\varpi)+C_0(\varpi)K(\varpi)\frac{\theta^{\frac{1}{r}}}{2(1+\theta^2)^{\frac{2}{r}}}$$

$$\times\ln\left(1+\frac{2\theta^{\frac{r-1}{r}}}{(1+\theta^2)^{\frac{2r-2}{r}}}|\vartheta(\theta,\alpha,\varpi)|\right), \alpha\in(0,\pi), \theta\in(0,+\infty), \tag{10}$$

$$\vartheta(\theta,0,\varpi)=\vartheta(\theta,\pi,\varpi)=0, \theta\in(0,+\infty), \tag{11}$$

$$\vartheta(\rho,\alpha,\varpi)=\vartheta_0(\rho,\alpha,\varpi), \rho\in(-\infty,0), \alpha\in[0,\pi], \tag{12}$$

where $j(\theta,\xi)$ is continuous and uniformly Hölder continuous in θ, K and C_0 are a real-valued random variable.

Let $\Xi=L^2(0,\pi)$, $(\Theta,\mathfrak{K},\mathfrak{Q})$ be a complete probability space, and define $\Phi(\theta)$ by

$$\Phi(\theta)\eta=j(\theta,\xi)\eta''$$

with domain

$$\sigma(\Phi)=\{\eta\in\Xi,\eta,\eta' \text{ are absolutely continuous}, \eta''\in\Xi, \eta(0)=\eta(\pi)=0\}.$$

Then $\Phi(\theta)$ generates an evolution system $\Im(\theta,\rho)$ verifying (H_1) (see [20, 22]).

Let $\mathfrak{E}=BUC(\mathrm{IR}^-;\Xi)$ be the space of bounded uniformly continuous functions endowed with

$$\|\xi\|=\sup_{\rho\le 0}|\xi(\rho)|, \quad \text{for } \xi\in\mathfrak{E}$$

If $\wp\in BCU(\mathrm{IR}^-;\Xi),\alpha\in(0,\pi)$ and $\varpi\in\Theta$, then

$$\xi(\theta,\alpha,\varpi)=\vartheta(\theta,\alpha,\varpi),\theta\in(0,\mathfrak{A})$$

$$\wp(\rho,\alpha,\varpi)=\vartheta_0(\rho,\alpha,\varpi), \rho\in(-\infty,0), \alpha\in[0,\pi],\varpi\in\Theta.$$

Set

$$\Psi(\theta,\vartheta,\varpi)=C_0(\varpi)K(\varpi)\frac{\theta^{\frac{1}{r}}}{2(1+\theta^2)^{\frac{2}{r}}}\ln\left(1+\frac{2\theta^{\frac{r-1}{r}}}{(1+\theta^2)^{\frac{2r-2}{r}}}|\vartheta(\theta,\alpha,\varpi)|\right),$$

and we have $\Psi(\theta,\vartheta,\varpi)=C_0(\varpi)K(\varpi)g(\theta,\vartheta,\varpi)$

$$\begin{aligned}
&|g(\theta+1,\vartheta,\varpi)(v)-g(\theta,\vartheta,\varpi)(v)|\\
&\le\frac{\theta^{\frac{1}{r}}}{2(1+\theta^2)^{\frac{2}{r}}}\ln\left(1+\frac{2\theta^{\frac{r-1}{r}}}{(1+\theta^2)^{\frac{2r-2}{r}}}|\vartheta(\theta+1,v,\varpi)|\right)\\
&-\frac{\theta^{\frac{1}{r}}}{2(1+\theta^2)^{\frac{2}{r}}}\ln\left(1+\frac{2\theta^{\frac{r-1}{r}}}{(1+\theta^2)^{\frac{2r-2}{r}}}|\vartheta(\theta,v,\varpi)|\right)\\
&\le\frac{\theta^{\frac{1}{r}}}{2(1+\theta^2)^{\frac{2}{r}}}\ln\left(\frac{1+2(\theta+1)^{\frac{r-1}{r}}|\vartheta(\theta+1,v,\varpi)|}{1+2\theta^{\frac{r-1}{r}}|\vartheta(\theta,v,\varpi)|}\right)\\
&\le\frac{\theta^{\frac{1}{r}}}{2(1+\theta^2)^{\frac{2}{r}}}\ln\left(\left(\frac{\theta+1}{\theta}\right)^{\frac{r-1}{r}}+\frac{1-\left(\frac{\theta+1}{\theta}\right)^{\frac{r-1}{r}}}{1+2\theta^{\frac{r-1}{r}}|\vartheta(\theta,v,\varpi)|}\right)\\
&\le\frac{\theta^{\frac{1}{r}}}{2(1+\theta^2)^{\frac{2}{r}}}\ln\left(\left(\frac{\theta+1}{\theta}\right)^{\frac{r-1}{r}}+1\right).
\end{aligned}\tag{13}$$

Now from (13), we can see $|g(\theta+1,\ \vartheta,\ \varpi)\ (v)\ -\ g(\theta,\ \vartheta,\ \varpi)\ (v)|$ tends to zero as $\theta\to\infty,$ then $|\Psi(\theta+1,\ \vartheta)\ (v,\varpi)\ -\ \Psi(\theta,z)\ (v,\varpi)|$ tends to zero as $\theta\to\infty.$ Hence condition $(H3)(a)$ is satisfied.

The function $\Psi(\theta,\varphi(\alpha),\varpi)$ is Carathéodory, and verifies (H_2) with

$$p(\theta,\varpi)=K(\varpi)\frac{\theta^{\frac{1}{r}}}{2(1+\theta^2)^{\frac{2}{r}}}\ \text{and}\ \ell(\alpha,\varpi)=|C_0(\varpi)|\frac{1}{4}\ln(1+2t).$$

Then the problem (10)-(11) in an abstract formulation of the problem (1)-(2), and conditions $(H_1)-(H_5)$ are satisfied. Theorem 3.2 implies that the random problem (10)-(11) has at least one random mild solutions.

References

[1] N. U. Ahmed, Semigroup Theory with Applications to Systems and Control, Harlow, John Wiley & Sons, Inc., New York (1991).

[2] D. Araya and C. Lizama, "Almost automorphic mild solutions to fractional differential equations," *Nonlinear Analysis,* vol. 69, pp. 3692-3705 (2008).

[3] L. Arnold and C. Tudor, "Stationary and almost periodic solutions of almost periodic affine stochastic differential equations," *Stochastic and Stochastic Reports*, vol. 64, pp. 177-193 (1998).

[4] M. Benchohra and S. Abbas, Advanced Functional Evolution Equations and Inclusions, Springer, Cham (2015).

[5] M. Benchohra, F. Bouazzaoui, E. Karapinar, and A. Salim, "Controllability of second order functional random differential equations with delay," *Mathematics*, vol. 10, p. 16 (2022), doi: 10.3390/math10071120.

[6] N. Benkhettou, K. Aissani, A. Salim, M. Benchohra, and C. Tunc, "Controllability of fractional integro-differential equations with infinite delay and non-instantaneous impulses," *Applied Analysis and Optimization,* vol. 6, pp. 79-94 (2022).

[7] N. Benkhettou, A. Salim, K. Aissani, M. Benchohra, and E. Karapinar, "Non-instantaneous impulsive fractional integro-differential equations with state-dependent delay," *Sahand Communications in Mathematical Analysis*, vol. 19, pp. 93-109 (2022), doi: 10.22130/scma.2022.542200.1014.

[8] A. Bensalem, A. Salim, M. Benchohra, and G. N'Guérékata, "Functional integro-differential equations with state-dependent delay and non-instantaneous impulses: existence and qualitative results," *Fractal and Fraction*, vol. 6, pp. 1-27 (2022), doi: 10.3390/fractalfract6100615.

[9] A. T. Bharucha-Reid, Random Integral Equations, Academic Press, New York (1972).

[10] S. Bochner, "Uniform convergence of monotone sequences of functions," Proceedings of the National Academy of Sciences of the United States of America, vol. 47, pp. 582-585 (1961).

[11] S. Bouriah, A. Salim, and M. Benchohra, "On nonlinear implicit neutral generalized Hilfer fractional differential equations with terminal conditions and delay," *Topology and Algebra with Applications*, vol. 10, pp. 77-93 (2022), doi: 10.1515/taa-2022-0115.

[12] C. Corduneanu, Integral Equations and Stability of Feedback Systems, Academic Press, New York (1973).

[13] W. Lu and T. Chen, "Global exponential stability of almost periodic solution for a large class of delayed dynamical systems," *Science in China Series A: Mathematics*, vol. 48, no. 8, pp. 1015-1026 (2005).

[14] A. Diop, M. A. Diop, and K. Ezzinbi, "Existence results for a class of random delay integrodifferential equations," *Random Operators and Stochastic Equations*, vol. 29, pp. 79-86 (2021).

[15] H. W. Engl, "A general stochastic fixed-point theorem for continuous random operators on stochastic domains," *Analysis and Applications*, vol. 66, pp. 220-231 (1978).

[16] H. R. Henríquez, M. Pierri, and P. Taboas, "Existence of S-asymptotically ω-periodic solutions for abstract neutral equations," *Bulletin of the Australian Mathematical Society*, vol. 78, pp. 365-382 (2008).

[17] H. R. Henríquez, M. Pierri, and P. Taboas, "On S-asymptotically ω-periodic functions on Banach spaces and applications," *Journal of Mathematical Analysis and Applications*, vol. 343, pp. 1119-1130 (2008).

[18] K. J. Engel and R. Nagel, One-Parameter Semigroups for Linear Evolution Equations, Springer-Verlag, New York (2000).

[19] A. M. Fink, Almost Periodic Differential Equations, Springer-Verlag, Berlin (1974).

[20] A. Freidman, Partial Differential Equations, Holt, Rinehart and Winston, New York (1969).

[21] M. M. Fu and Z. X. Liu, "Square-mean almost automorphic solutions for some stochastic differential equations," *Proceedings of the American Mathematical Society*, vol. 138, pp. 3689-3701 (2010).

[22] S. G. Krein, Linear Differential Equations in Banach Spaces, American Mathematical Society, Providence (1971).

[23] A. Granas and J. Dugundji, Fixed Point Theory, Springer-Verlag, New York (2003).

[24] J. Hale and J. Kato, "Phase space for retarded equations with infinite delay," *Functional Ekvacient*, vol. 21, pp. 11-41 (1978).

[25] A. Heris, A. Salim, M. Benchohra, and E. Karapinar, "Fractional partial random differential equations with infinite delay," *Results in Physics* (2022), doi: 10.1016/j.rinp.2022.105557.

[26] E. Hernandez, "On abstract differential equations with state-dependent non-local conditions," *Journal of Mathematical Analysis and Applications,* vol. 466, pp. 408-425 (2018).

[27] E. Hernandez and D. O'Regan, "On state-dependent non-local conditions," *Applied Mathematics Letters*, vol. 83, pp. 103-109 (2018).

[28] E. Hernandez, K. Azevedo, and V. Rolnik, "Well-posedness of abstract differential equations with state-dependent delay," *Mathematica Nachr*. vol. 291, pp. 2045-2056 (2018).

[29] Y. Hino, S. Murakami, and T. Naito, Functional Differential Equations with Unbounded Delay, Springer-Verlag, Berlin (1991).

[30] S. Itoh, "Random fixed point theorems with an application to random differential equations in Banach space," *Analysis and Applications*, vol. 67, pp. 261-273 (1979).

[31] S. H. Malih and S. S. Abed, "Approximating random fixed points under a new iterative sequence," *Journal of Interdisciplinary Mathematics*, vol. 22, no. 8, pp. 1407-1414 (2019), doi: 10.1080/09720502.2019.1700927.

[32] A. Pazy, Semigroups of Linear Operators and Applications to Partial Differential Equations, Springer-Verlag, New York (1983).

[33] D. X. Piao, "Pseudo almost periodic solutions for the systems of differential equations with piecewise constant argument," *Science in China Series A: Mathematics,* vol. 1, pp. 31-38 (2004).

[34] R. Ortega and M. Tarallo, "Almost periodic linear differential equations with non-separated solutions," *Journal of Functional Analysis,* vol. 237, pp. 402-426 (2006).

[35] T. T. Soong, Random Differential Equations in Science and Engineering, Academic Press, New York (1973).

[36] C. P. Tsokos and W. J. Padgett, Random Integral Equations with Applications to Life Sciences and Engineering, Academic Press, New York (1974).

[37] C. Tudor, "Almost periodic solutions of affine stochastic evolution equations," *Stochastic and Stochastic Reports*, vol. 38, pp. 251-266 (1992).

[38] C. A. Tudor, M. Tudor, "Pseudo almost periodic solutions of some stochastic differential equations," *Mathematica Reports (Bucur),* vol. 1, no. 51, pp. 305-314 (1999).

Received November, 2022