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A proposed improved ratio method for the solution of IVPs in delay differential equations of various characteristics

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Abstract

This paper proposes an improved Ratio Method (IRM) to tackle Delay Differential Equations (DDEs) by employing a rational-type interpolating function. The Lagrange interpolation was used for approximating the delay argument. The consistency and convergence properties of IRM were analyzed. Moreover, the stability property of IRM using a test problem of DDE was investigated, and the corresponding region of stability was also obtained. The efficiency of IRM was demonstrated through illustrative examples of singularly perturbed and stiff DDEs. A comparative study of the results generated via IRM and that of the theoretical solution was presented.

Subject Classification (2010): *34A12, 65L05, 65L20, 65L70.*

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I. Introduction

The popularity of mathematical equations that involve the relationship between a function and its derivatives.in science and technology has been displayed as one of the key major areas in physical phenomena. In other words, DEs have contributed greatly to the real-life situation. In fact, DEs can describe change in populations, how heat moves, how spring vibrates, how radioactive material decays and much more. DEs of different nature such as Delay Differential Equations (DDEs) and Singularly Perturbed Delay Differential Equations (SPDDEs) arise in several areas of science and engineering such as population dynamics [1], chemical kinetics [2], and management systems [3], just to mention a few. There has been a growing focus on numerically investigating solutions to both stiff and non-stiff delay differential equations (DDEs) using various techniques, including the Runge-Kutta method, homotopy perturbation method, Adomian decomposition method, block method, difference method, parameter-uniform fitted mesh method, and parallel two-step row method. just to mention a few [4-10]. There are several models with stiff flavor in DEs that cannot be solved analytically. Therefore, numerical methods are needed to provide approximate solutions to such models. Many researchers have constructed numerical methods by means of interpolating functions of various types for the solution of Initial Value Problems (IVPs) of First Order Differential Equations (FODEs); see [11-18]. In this paper, we introduce an enhanced single-step ratio method aimed at numerically solving DDEs with varying characteristics. The structure of the paper is as follows: Section 2 outlines the development of the proposed method. Section 3 delves into the analysis and investigation of the properties of the improved ratio method (IRM), with the stability region also identified. In Section 4, numerical examples are provided to showcase the effectiveness of the IRM in solving DDEs. Finally, Section 5 offers concluding remarks on the study.

2. The derivation of improved ratio method

Consider the initial value problem (IVP) for first order delay differential equations (DDEs) with a delay parameter τ, which can be expressed in the following form

$$\left.\begin{aligned} y'(t) &= f(t, y(t), y_t), t > t_0, \\ y(t) &= \phi(t), t \leq t_0 \end{aligned}\right\} \tag{1}$$

where $y_t = y\left(t - \tau\big(t, y(t)\big)\right) = \{y(\tau): \tau \leq t\}$ is the trajectory of the past solution and $f(t, y(t), y_t)$ is a functional operator, $f: \mathbb{R} \times \mathbb{R}^n \times \mathbb{C}^1(\mathbb{R}, \mathbb{R}^n) \to \mathbb{R}^n$, $\phi(t)$ is the initial function and τ is the delay term.

Suppose the exact solution $y(t_{n+1}) = y_{n+1}$ to (1) can be locally represented in $[t_n, t_{n+1}]$, $n \geq 0$ as

$$y_{n+1} = y_n + \frac{\alpha_0 t_n}{\sum_{k=0}^{r} \beta_k t_n^k} \tag{2}$$

Expanding (2) with r = 4 further yields

$$y_{n+1} = y_n + \frac{\alpha_0 t_n}{\beta_0+\beta_1 t_n+\beta_2 t_n^2+\beta_3 t_n^3+\beta_4 t_n^4} \tag{3}$$

provided $\beta_0 + \beta_1 t_n + \beta_2 t_n^2 + \beta_3 t_n^3 + \beta_4 t_n^4 \neq 0$
where $\alpha_0, \beta_0, \beta_1, \ldots, \beta_4$ are undetermined coefficients.

From (3), one obtains

$$y_{n+1} = y_n + \alpha_0 t_n[\beta_0 + \beta_1 t_n + \beta_2 t_n^2 + \beta_3 t_n^3 + \beta_4 t_n^4]^{-1} \tag{4}$$

Setting $\beta_0 = 1$, then (4) yields,

$$y_{n+1} = y_n + \alpha_0 t_n[1 + (\beta_1 t_n + \beta_2 t_n^2 + \beta_3 t_n^3 + \beta_4 t_n^4)]^{-1} \tag{5}$$

Let $B = \beta_1 t_n + \beta_2 t_n^2 + \beta_3 t_n^3 + \beta_4 t_n^4$ and by means of the binomial expansion (5) becomes

$$y_{n+1} = y_n + \alpha_0 t_n[1 + B]^{-1} \tag{6}$$

$$= y_n + \alpha_0 t_n(1 - B + B^2 - B^3 + B^4 - \cdots) \tag{7}$$

Using (6) and (7) and ignoring higher powers greater than four in the expansion, one gets

$$y_{n+1} = y_n + \alpha_0 t_n(1 - \beta_1 t_n - \beta_2 t_n^2 - \beta_3 t_n^3 - \beta_4 t_n^4 + \beta_1\beta_2 t_n^2 + {\beta_1}^2 t_n^2 + \\ +\beta_1\beta_3 t_n^4 + +\beta_1\beta_2 t_n^3 + {\beta_2}^2 t_n^4 + \beta_1\beta_3 t_n^4 - {\beta_1}^3 t_n^3 - 3{\beta_1}^2\beta_3 t_n^4 + {\beta_1}^4 t_n^4) \tag{8}$$

Simplifying further (8) yields

$$y_{n+1} = y_n + \alpha_0 t_n\big(1 - \beta_1 t_n + ({\beta_1}^2 - \beta_2){t_n}^2 + (-\beta_3 + 2\beta_1\beta_2 - {\beta_1}^3){t_n}^3 + \\ (-\beta_4 + 2\beta_1\beta_3 + {\beta_2}^2 - 3{\beta_1}^2\beta_2 + {\beta_1}^4){t_n}^4\big) \tag{9}$$

Thus,

$$y_{n+1} = y_n + \alpha_0 t_n - \alpha_0\beta_1 {t_n}^2 + (\alpha_0{\beta_1}^2 - \alpha_0\beta_2){t_n}^3 + (-\alpha_0\beta_3 + 2\alpha_0\beta_1\beta_2 - \\ \alpha_0{\beta_1}^3){t_n}^4 + (-\alpha_0\beta_4 + 2\alpha_0\beta_1\beta_3 + \alpha_0{\beta_2}^2 - 3\alpha_0{\beta_1}^2\beta_2 + \alpha_0{\beta_1}^4){t_n}^5 \tag{10}$$

From the Taylor's series expansion, we have that

$$y_{n+1} = y_n + h{y_n}' + \frac{h^2}{2!}{y_n}'' + \frac{h^3}{3!}{y_n}''' + \frac{h^4}{4!}{y_n}^{iv} + \frac{h^5}{5!}{y_n}^{v} \tag{11}$$

Equating (10) and (11) and comparing term by term for each parameter, one obtains

$$\alpha_0 t_n = h y_n'$$

$$\alpha_0 = \frac{h y_n'}{t_n} \tag{12}$$

$$-\alpha_0\beta_1 {t_n}^2 = \frac{h^2 y_n''}{2} \tag{13}$$

Using (12) in (13), yields

$$\beta_1 = \frac{-h y_n''}{2 y_n' t_n} \tag{14}$$

$${\beta_1}^2 - \beta_2 = \frac{h^2 y_n'''}{6 y_n' t_n^2}$$

$$\beta_2 = \frac{h^2(3(y_n'')^2 - 2y_n' y_n''')}{12(y_n')^2 t_n^2} \tag{15}$$

$$-\beta_3 + 2\beta_1\beta_2 - {\beta_1}^3 = \frac{h^3 y_n^{(iv)}}{24 y_n' t_n^3}$$

$$\beta_3 = \frac{-h^3 y_n^{(iv)}}{24 y_n' t_n} - \frac{3h^3 (y_n'')^3}{24 (y_n')^3 t_n^3} + \frac{4h^3 y_n'' y_n'''}{24 (y_n')^2 t_n^3} \tag{16}$$

$$-\beta_4 - 2\beta_1\beta_3 + {\beta_2}^2 - 3{\beta_1}^2\beta_2 + {\beta_1}^4 = \frac{h^4 y_n^v}{120 y_n' t_n^4}$$

$$\beta_4 = \frac{h^4\left[-135(y_n'')^4 + 150 y_n' (y_n'')^2 y_n''' + \left(20(y_n''')^2 - 30 y_n'' y_n^{iv}\right)(y_n')^2 - 6 y_n^v (y_n')^3\right]}{720 (y_n')^4 t_n^4} \tag{17}$$

Substituting (12), (14), (15), (16), and (17) into (3) results in the formulation of the proposed improved ratio method.

$$y_{n+1} = y_n + \frac{A_n}{B_n} \tag{18}$$

where

$$A_n = 720 h (y_n')^5 \tag{19a}$$

$$B_n = 720(y_n')^4 - 360h(y_n')^3 y_n'' + 60h^2 (y_n')^2 (3(y_n'')^2 - 2y_n''' y_n')$$
$$+30h^3 y_n' \left(4 y_n' y_n'' y_n''' - (y_n')^2 y_n^{(iv)} - 3(y_n'')^3\right) + h^4 (-135 (y_n'')^4$$
$$+150 y_n' y_n''' (y_n'')^2 + 20 (y_n')^2 (y_n''')^2 - 30 (y_n')^2 y_n'' y_n^{(iv)} - 6 (y_n')^3 y_n^v) \tag{19b}$$

3. Analysis of the improved ratio method

Rearranging and simplifying further, the ratio method (18) yields by means of Taylor's series,

$$y_{n+1} = y_n + \frac{A_n}{B_n} = y_n + h y_n' + \frac{h^2}{2!} y_n'' + \frac{h^3}{3!} y_n''' + \frac{h^4}{4!} y_n^{iv} + \frac{h^5}{5!} y_n^v$$
$$= y_n + h\left(y_n' + \frac{h}{2!} y_n'' + \frac{h^2}{3!} y_n''' + \frac{h^3}{4!} y_n^{iv} + \frac{h^4}{5!} y_n^v\right) \tag{20}$$

Using the definition of a general one step method and (20), the increment function of the IRM (18) is obtained as

$$\phi(t_n, y_n; h) = y_n' + \frac{h}{2!} y_n'' + \frac{h^2}{3!} y_n''' + \frac{h^3}{4!} y_n^{iv} + \frac{h^4}{5!} y_n^v \tag{21}$$

3.1 *Order of Convergence of the Improved Ratio Method*

From the Taylor's series expansion of $y(t_n)$, we have

$$y(t_n + h) = y(t_n) + h f\big(t_n, y(t_n)\big) + \frac{h^2}{2} f^{(1)}\big(t_n, y(t_n)\big) + + \frac{h^3}{6} f^{(2)}\big(t_n, y(t_n)\big)$$
$$+ \frac{h^4}{24} f^{(3)}\big(t_n, y(t_n)\big) + \frac{h^5}{120} f^{(4)}\big(t_n, y(t_n)\big) + O(h^6) \tag{22}$$

The local truncation error for IRM is defined as

$$T_{n+1} = y(t_n + h) - y_{n+1} \tag{23}$$

Using (18) and (22), (23) becomes

$$T_{n+1} = [y(t_n) + h f\big(t_n, y(t_n)\big) + \frac{h^2}{2} f^{(1)}\big(t_n, y(t_n)\big) + + \frac{h^3}{6} f^{(2)}\big(t_n, y(t_n)\big)$$

$$+\frac{h^4}{24}f^{(3)}\big(t_n, y(t_n)\big) + \frac{h^5}{120}f^{(4)}\big(t_n, y(t_n)\big) + O(h^6)]$$

$$-[y_n + hy_n' + \frac{h^2}{2!}y_n'' + \frac{h^3}{3!}y_n''' + \frac{h^4}{4!}y_n^{iv} + \frac{h^5}{5!}y_n^{v}] \tag{24}$$

Using the fact that $y_n' = f(t_n, y_n)$ and by means of the simplifying assumptions, then (24) gives the local truncation error for IRM

$$T_{n+1} = O(h^6) \tag{25}$$

From (25), we deduce that IRM has accuracy of order five.

Remark 1: (a) It is clearly seen from (20), IRM is of order five by virtues of Taylor's series.

3.2. *Consistency Property of IRM*

By the definition of consistency, we see that IRM is found to be consistent, that is

$$\phi(t_n, y_n; 0) = y_n' + 0 = y_n' = f(t_n, y_n). \tag{26}$$

Alternatively, the consistency of IRM can be checked as follows.

$$\lim_{h\to 0}\left(\frac{T_{n+1}}{h}\right) = \lim_{h\to 0}\left(\frac{O(h^6)}{h}\right) = 0 \tag{27}$$

Remark 2

(a) It is evident that IRM is consistent since it is of order five.
(b) The following result summarizes the consistency property of IRM.
(c) The consistency property of IRM is given by the following result.

Theorem 1: *The consistency of the improved ratio method, incorporating an increment function $\phi(t_n, y_n; h)$ is established when the condition*

$$\phi(t_n, y_n; 0) = f(t_n, y_n) \tag{28}$$

holds true.

Proof: From (18),

$$y_{n+1} = y_n + \frac{A_n}{B_n}$$

$$y_{n+1} = y_n + h\left[\frac{720(y_n')^5}{\begin{array}{c}720(y_n')^4 - 360h(y_n')^3y_n'' + 60h^2(y_n')^2(3(y_n'')^2 - 2y_n'''y_n') \\ +30h^3y_n'\left(4y_n'y_n''y_n''' - (y_n')^2y_n^{(iv)} - 3(y_n'')^3\right) \\ +h^4(-135(y_n'')^4 + 150y_n'y_n'''(y_n'')^2 + 20(y_n')^2(y_n''')^2 - 30(y_n')^2y_n''y_n^{(iv)} - 6(y_n')^3y_n^{v})\end{array}}\right] \tag{29}$$

Equation (29) reduces to

$$y_{n+1} = y_n + h\,\phi(t_n, y_n; h)$$

So, at h = 0,

$$\phi(t_n, y_n; 0) = \frac{720h(y_n')^5}{720(y_n')^4} = y_n' = f(t_n, y_n) \tag{30}$$

Hence

$$\phi(t_n, y_n; 0) = f(t_n, y_n)$$

This establishes (28) and this completes the proof.

3.3 *Stability Polynomial and Region of Absolute Stability of IRM using DDE*

In this subsection, we examine a widely utilized linear test equation characterized by a constant delay, represented as follows $\tau = mh$, where m is a positive integer.

$$y'(t) = \lambda y(t) + \mu y(t-\tau), \quad t > t_0$$
$$y(t) = \phi(t), \qquad t \le t_0$$

where $\lambda, \mu \in C, \tau > 0$ and $\phi(t)$ is continuous.

Simplifying (19), we obtain

$$y_{n+1} = y_n + hy_n' + \frac{h^2}{2}y_n'' + \frac{h^3}{6}y_n''' + \frac{h^4}{24}y_n^{iv} + \frac{h^5}{120}y_n^{v} \tag{31}$$

Applying (31) on the test equation, one obtains

$$y_{n+1} = y_n + h\big(\lambda y_n + \mu y(t_n-\tau)\big) + \frac{h^2}{2}\Big(\lambda y'_n + \mu y'(t_n-\tau)\Big) + \frac{h^3}{6}\Big(\lambda y''_n + \mu y''(t_n-\tau)\Big) + \frac{h^4}{24}\big(\lambda y'''_n + \mu y'''(t_n-\tau)\big) + \frac{h^5}{120}\Big(\lambda y_n{}^{iv} + \mu y^{iv}(t_n-\tau)\Big) \tag{32}$$

Using the Lagrange interpolation, the delay term is approximated as

$$y(t_n - mh) = y(t_{n-m}) = \sum_{l=-r_1}^{s_1} L_l(c_i) y_{n-m+l}$$

with

$$L_l(c_i) = \prod_{j_1=-r_1}^{s_1} \frac{c_i - j_1}{l - j_1}, \qquad j_1 \ne l \quad \text{and } r_1, s_1 > 0$$

Thus,

$$y(t_n - \tau) = \sum_{l=-r_1}^{s_1} L_l(c_i) y_{n-m+l}\,, \tag{33}$$

Substituting (33) and its first- fourth derivatives into (32) and rearranging terms, we get

$$\begin{aligned} y_{n+1} &= y_n + \lambda h y_n + \frac{\lambda^2h^2}{2}y_n + \frac{\lambda^3h^3}{6}y_n + \frac{\lambda^4h^4}{24}y_n + \frac{\lambda^5h^5}{120}y_n \\ &+ \sum_{l=-r_1}^{s_1} L_l(c_i)y_{n-m+l}\left(\mu h + \mu\lambda h^2 + \frac{\lambda^2\mu h^3}{2} + \frac{\lambda^3\mu h^4}{6} + \frac{\lambda^4\mu h^5}{24}\right) \\ &+ \sum_{l=-r_1}^{s_1} L_l(c_i)y_{n-2m+l}\left(\frac{\mu^2h^2}{2} + \frac{\lambda\mu^2h^3}{2} + \frac{\lambda^2\mu^2h^4}{4} + \frac{\lambda^3\mu^2h^5}{12}\right) \\ &+ \sum_{l=-r_1}^{s_1} L_l(c_i)y_{n-3m+l}\left(\frac{\mu^3h^3}{6} + \frac{\lambda\mu^3h^4}{6} + \frac{\lambda^2\mu^3h^5}{12}\right) \\ &+ \sum_{l=-r_1}^{s_1} L_l(c_i)y_{n-4m+l}\left(\frac{\mu^4h^4}{24} + \frac{\lambda\mu^4h^5}{24}\right) + \sum_{l=-r_1}^{s_1} L_l(c_i)y_{n-5m+l}\left(\frac{\mu^5h^5}{120}\right) \end{aligned} \tag{34}$$

Setting $\alpha = \lambda h$ and $\beta = \mu h$. Thus, (34) yields,

$$y_{n+1} = y_n\left(1+\alpha+\frac{\alpha^2}{2!}+\frac{\alpha^3}{3!}+\frac{\alpha^4}{4!}+\frac{\alpha^5}{5!}\right)+\left(\beta\left(1+\alpha+\frac{\alpha^2}{2}+\frac{\alpha^3}{6}+\frac{\alpha^4}{24}\right)\right)\sum_{l=-r_1}^{s_1} L_l(c_i)y_{n-m+l}+\left(\beta^2\left(\frac{1}{2}+\frac{\alpha}{2}+\frac{\alpha^2}{4}+\frac{\alpha^3}{12}\right)\right)\sum_{l=-r_1}^{s_1} L_l(c_i)y_{n-2m+l}+\left(\beta^3\left(\frac{1}{6}+\frac{\alpha}{6}+\frac{\alpha^2}{12}\right)\right)\sum_{l=-r_1}^{s_1} L_l(c_i)y_{n-3m+l}+\beta^4\left(\frac{1}{24}+\frac{\alpha}{24}\right)\sum_{l=-r_1}^{s_1} L_l(c_i)y_{n-4m+l}+\left(\frac{\beta^5}{120}\right)\sum_{l=-r_1}^{s_1} L_l(c_i)y_{n-5m+l} \tag{35}$$

The stability characteristic polynomial for (18) is obtained as

$$S(\alpha,\beta:\zeta) = \zeta^{n+1}-\left(1+1+\alpha+\frac{\alpha^2}{2!}+\frac{\alpha^3}{3!}+\frac{\alpha^4}{4!}+\frac{\alpha^5}{5!}\right)\zeta^n \tag{36}$$
$$-\left(\beta+\frac{\beta^2}{2}+\frac{\beta^3}{6}+\frac{\beta^4}{24}+\frac{\beta^5}{120}+\frac{\alpha\beta^2}{2}+\frac{\alpha^2\beta}{2}+\frac{\alpha\beta^3}{6}+\frac{\alpha^3\beta}{6}+\frac{\alpha\beta^4}{24}+\frac{\alpha^4\beta}{24}+\frac{\alpha^2\beta^2}{4}+\frac{\alpha^2\beta^3}{12}+\frac{\alpha^3\beta^2}{12}+\alpha\beta\right)\zeta$$

Equation (19) is said to be stable if the roots ζ_i of (36) satisfy the root condition $|\zeta_i| \leq 1$.

The plot of the corresponding region of stability for (18) is displayed in Figure 1.

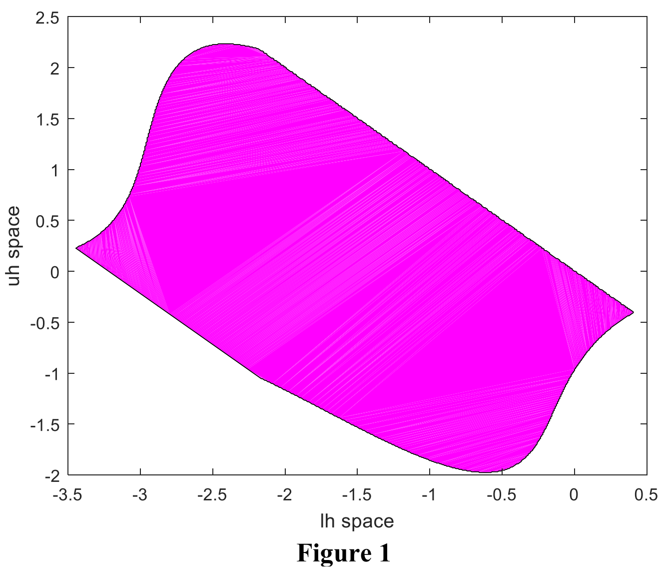

Figure 1
The corresponding region of stability for IRM

3.4. *Convergence Property of IRM*

From the above analyses and investigation, IRM is found to be consistent and stable. Hence, we conclude that IRM is convergent.

4. Illustrative examples

Example 1: Consider the DDE of the form

$$y_1'(t) = -\frac{1}{2}y_1(t) - \frac{1}{2}y_2(t-1) + \frac{1}{2}e^{-(t-1)},$$

$$y_2'(t) = -y_2(t) - \frac{1}{2}y_1\left(t - \frac{1}{2}\right) + \frac{1}{2}e^{-(t-1/2)/2}, \quad 0 \le t \le 1$$

subject to,

$$y_1(t) = e^{-t/2}, \qquad \frac{-1}{2} \le t \le 0,$$

$$y_2(t) = e^{-t}, \qquad -1 \le t \le 0$$

whose exact solution is given by

$$y_1(t) = e^{-t/2}, \qquad y_2(t) = e^{-t}$$

The exact, numerical and error results are given in Tables 1 and 2.

Table 1
Exact solution versus numerical solution y_1

t	Exact	y_1		Absolute error	
		m = 100	m = 1000	m = 100	m = 1000
0.2	0.9048374180	0.9048374399	0.9048374181	2.186013e-08	2.191303e-11
0.4	0.8187307531	0.8187307950	0.8187307531	4.189721e-08	4.200029e-11
0.6	0.7408182207	0.7408182810	0.7408182207	6.034777e-08	6.049816e-11
0.8	0.6703200460	0.6703201235	0.6703200461	7.742898e-08	7.762457e-11
1.0	0.6065306597	0.6065307531	0.6065306598	9.334127e-08	9.358125e-11

Table 2
Exact solution versus numerical solution y_2

t	Exact	y_2		Absolute error	
		m = 100	m = 1000	m = 100	m = 1000
0.2	0.8187307531	0.8187299241	0.8187294799	8.289571e-07	1.273217e-06
0.4	0.6703200460	0.6703186204	0.6703180067	1.425630e-06	2.039301e-06
0.6	0.5488116361	0.5488097972	0.5488091284	1.838939e-06	2.507728e-06
0.8	0.4493289641	0.4493268559	0.4493261251	2.108191e-06	2.839062e-06
1.0	0.3679074412	0.3790807906	0.3689818113	1.120135e-02	1.102370e-03

Example 2: Consider the state dependent DDE of the form

$$y'(t) = \cos(t)y(\mathrm{y(t)} - 2), \qquad t \ge 0$$

subject to

$$y(t) = 1, t \le 0$$

whose exact solution is given by

$$y(t) = \sin(t) + 1,\ 0 \le t \le 1$$

The exact, numerical and error results are displayed in Table 3.

Table 3
Exact solution versus numerical solution

t	Exact	IRM		Absolute error	
		m = 100	m = 1000	m = 100	m = 1000
0.2	1.1986693308	1.1986693460	1.1986693308	1.523283e-08	1.503198e-11
0.4	1.3894183423	1.3894183869	1.3894183424	4.459951e-08	4.391465e-11
0.6	1.5646424734	1.5646425625	1.5646424735	8.915054e-08	8.763479e-11
0.8	1.7173560909	1.7173562402	1.7173560910	1.492533e-07	1.482003e-10
1.0	1.8414709848	1.8248883197	1.8414709850	1.658267e-02	2.301086e-10

Example 3: Consider the singularly perturbed DDE of the form

$$\varepsilon u'(t) + u(t) = \frac{1}{2}u(t-1), \quad t \in (0, \infty),$$

subject to

$$u(t) = 2, \quad -1 \le t \le 0.$$

whose exact solution is given by

$$u(t) = 1 + e^{-t/\varepsilon}, \quad t \in [0,1]$$

The exact, numerical and error results for $\varepsilon = 2^{-1}$, $\varepsilon = 2^{-2}$ and $\varepsilon = 2^{-3}$ are given in Table 4, 5 and 6, respectively.

Table 4
Exact solution versus numerical solution for $\varepsilon = 2^{-1}$

t	Exact	IRM		Absolute error	
		m = 100	m = 1000	m = 100	m = 1000
0.2	1.6703200460	1.6703234747	1.6703200465	4.286914e-07	4.294760e-10
0.4	1.4493289641	1.4493295706	1.4493289647	6.064367e-07	6.079031e-10
0.6	1.3011942119	1.3011948708	1.3011942126	6.588835e-07	6.610323e-10
0.8	1.2018965180	1.2018971774	1.2018965187	6.594406e-07	6.623815e-10
1.0	1.1353352832	1.1353359349	1.1353352839	6.517057e-07	6.556768e-10

Table 5
Exact solution versus numerical solution for $\varepsilon = 2^{-2}$

t	Exact	IRM		Absolute error	
		m = 100	m = 1000	m = 100	m = 1000
0.2	1.4493289641	1.4493329244	1.4493289681	3.960247e-06	3.962731e-09
0.4	1.2018965180	1.2019001896	1.2018965217	3.671638e-06	3.677044e-09
0.6	1.0907179533	1.0907206433	1.0907179560	2.690039e-06	2.700477e-09
0.8	1.0407622040	1.0407642424	1.0407622060	2.038398e-06	2.059112e-09
1.0	1.0183156389	1.0183176730	1.0183156410	2.034139e-06	2.077291e-09

Table 6
Exact solution versus numerical solution for $\varepsilon = 2^{-3}$

t	Exact	IRM		Absolute error	
		m = 100	m = 1000	m = 100	m = 1000
0.2	1.2018965180	1.2019243170	1.2018965458	2.779896e-05	2.779643e-08
0.4	1.0407622040	1.0407399655	1.0407622158	1.176152e-05	1.179627e-08
0.6	1.0082297470	1.0082356123	1.0082297531	5.865216e-06	6.046668e-09
0.8	1.0016615573	1.0016470721	1.0016615708	1.251486e-05	1.351144e-08
1.0	1.0003354626	1.0003832467	1.0003355246	4.778406e-05	6.198608e-08

5. Conclusion

In this paper, IRM was developed using a rational interpolating function for the solution of DDEs. From the analysis of the properties of IRM, it is noteworthy to mention that IRM was found to be of order five, consistent, stable, and convergent. Moreover, it is observed from Tables 1-6 that IRM is suitable for solving DDEs of different characteristics. Hence, IRM is computationally efficient, robust, easy to implement, and a good numerical method to include in the class of linear explicit single-step methods.

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