

A proof of cases of de Polignac's conjecture

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Abstract

For $n \geq 1$ let p_n denote the n^{th} prime number. Let

$$S = \{1, 7, 11, 13, 17, 19, 23, 29\},$$

the set of positive integers which are both less than and relatively prime to 30. For $x \geq 0$, let $T_x := \{30x + i \mid i \in S\}$. For each x , T_x contains at most seven primes. Let $[\]$ denote the floor or greatest integer function. For each integer $s \geq 30$ let $\pi_7(s)$ denote the number of integers x , $0 \leq x < \left\lceil \frac{s}{30} \right\rceil$ for which T_x contains seven primes. Let $m \geq 10^{10}$ be an integer and let P_{k_m} denote the largest prime number less than $\sqrt{\prod_{i=1}^m p_i}$. In this paper we show that

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$$\frac{\prod_{i=1}^m p_i}{8(K_m+1)} < \pi_7(\prod_{i=1}^m p_i)$$

and thereby prove that there are infinitely many values of x for which T_x contains seven primes. This, in particular, proves the well known twin prime conjecture as well as several cases of Alphonse de Polignac's conjecture that for every even number k , there are infinitely many pairs of prime numbers p and p' for which $p' - p = k$.

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1. Introduction and main result

An integer $p \geq 2$ is called a prime if its only positive divisors are 1 and p . The prime numbers form a sequence:

$$2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, \dots \quad (1.1)$$

Euclid (300 B.C.) considered prime numbers and proved that there are infinitely many.

In 1849, Polignac [7] conjectured that for every even number k , there are infinitely many pairs of prime numbers p and p' such that $p' - p = k$. The case $k = 2$ is the well known twin prime conjecture, which is proved in [6]. The conjecture has not yet been proven or disproven for any given value of $k \neq 2$. In 2013 an important breakthrough was made by Yitang Zhang who proved the conjecture for some value of $k < 70\,000\,000$ [11]. Later that same year, James Maynard announced a related breakthrough which proved the conjecture for some $k < 600$ (see [4]). In 2014 the D.H.J. Polymath project proved the conjecture for some $k \leq 246$. (see [8])

In this paper we prove cases of de Polignac's conjecture which are implied by the following result. Our arguments are an extension or generalization of the arguments developed in [6].

Let

$$S = \{1, 7, 11, 13, 17, 19, 23, 29\},$$

the set of integers which are both less than and relatively prime to 30. For $x \geq 0$, let $T_x := \{30x + i \mid i \in S\}$. For each x , T_x contains at most seven primes. For $s \geq 30$ let $\pi_7(s)$ denote the number of integers x , $0 \leq x < \lfloor \frac{s}{30} \rfloor$, for which T_x contains seven primes. For example if $x = 0, 1, 2, 49, 62, 79, 89, 188$, then T_x contains seven primes. It is easy to show that $\pi_7(10^8) = 962$.

In this paper we prove the following theorem which is also our main result:

Theorem 1.1: *For $i \geq 1$ let p_i denote the i^{th} prime number. Let $m \geq 10^{10}$ be an integer and let P_{K_m} denote the largest prime number less than $\sqrt{\prod_{i=1}^m p_i}$. Then*

$$\frac{\prod_{i=1}^m p_i}{8(K_m+1)} < \pi_7(\prod_{i=1}^m p_i).$$

Since $\frac{\prod_{i=1}^m p_i}{8(K_m+1)}$ is an unbounded sequence, the theorem shows that there are infinitely many values of x for which T_x contains seven primes. For each x , the elements of T_x differ by

$$2, 4, 6, 8, 10, 12, 16, 18, 22, 28.$$

So we see that Theorem 1.1 implies several cases of Alphonse de Polignac's conjecture. Since two pairs of elements of T_x differ by 2, Theorem 1.1 also establishes the well known twin prime conjecture.

Our work is organized as follows: In Section 2 we recall the definition of the well known sieve of Eratosthenes and record some preliminary results. In Section 3 we record our main observation, which is a comparison of the relative porousness of the sieve of Eratosthenes with that of another sieve that we shall refer to as the "partition sieve." In Section 4, we prove Theorem 1.1. The concepts required are elementary and can be obtained from introductory texts on number theory, discrete mathematics and set theory. Some references are listed in the bibliography [1,2,3,10]

2. Preliminary Results

Eratosthenes (276 – 194 B.C.) was a Greek mathematician whose work in number theory remains significant. Consider the following lemma.

Lemma 2.1: *Let $a > 1$ be an integer. If a is not divisible by a prime number $p \leq \sqrt{a}$, then a is a prime.*

Eratosthenes used the above lemma as a basis of a technique called "Sieve of Eratosthenes" for finding all the prime numbers not exceeding a given integer x . The algorithm calls for writing down the integers from 2 to x in their natural order. The composite numbers in the sequence are then sifted out by crossing off from 2, every second number (all multiples of two) in the list, from the next remaining number, 3, every third number, and so on for all the remaining prime numbers less than or equal to $\sqrt{x}$. The integers that are left on the list are primes. We shall refer to the set of integers left as the **residue** of the sieve. The order of the residue set is therefore equal to $\pi(x)$, the number of primes not exceeding the integer x .

We recall that $S = \{1, 7, 11, 13, 17, 19, 23, 29\}$ is the set of integers which are both less than and relatively prime to 30 and that for $x \geq 0$, $T_x = \{30x + i \mid i \in S\}$. For each integer $m \geq 1$ the sieve of Eratosthenes can be extended to the sequence of integers in the interval $0 \leq x \leq m - 1$, or, equivalently, to the sets

$$T_x = \{30x + 1, 30x + 7, 30x + 11, 30x + 13, 30x + 17, 30x + 19, 30x + 23, 30x + 29\},$$

to obtain those values of x for which T_x contains seven primes. More formally define $\phi_3(m)$ to be the number of integers x , $0 \leq x \leq m - 1$, for which $\gcd(m, 30x + i) = 1$ for all $i \in S$. We obtain a formula for evaluating ϕ_3 for certain values of m .

If p is a prime then $\phi_3(p)$ is easy to evaluate. For example $\phi_3(7) = 0$ since for all x , $0 \leq x \leq 6$ the set $\{30x + i \mid i \in S\}$ contains an integer divisible by 7. On the other hand if $p \neq 7$, then $\phi_3(p) \neq 0$. It is easy to check that $\phi_3(p) = p$ if $p = 2, 3$ or 5. Further $\phi_3(11) = 11 - 6$ and $\phi_3(p) = p - 8$ if $p \geq 13$. We note also that $\phi_3(1) = 1$.

In [5] it is shown that ϕ_3 is multiplicative and that we have the following theorem.

Theorem 2.2: *If the integer $m > 1$ has the prime factorization*

$$m = 2^{k_1} 3^{k_2} 5^{k_3} 11^{k_4} p_5^{k_5} \dots p_r^{k_r}$$

with $p_s \neq 7$ for any $s \geq 5$, then

$$\phi_3(m) = 2^{k_1} 3^{k_2} 5^{k_3} (11^{k_4} - 6 \cdot 11^{k_4-1}) (p_5^{k_5} - 8p_5^{k_5-1}) \dots (p_r^{k_r} - 8p_r^{k_r-1}).$$

We have seen that if $p = 7$ then for all x , the set $\{30x + i \mid i \in S\}$ contains an integer divisible by 7. We note further that if $p = 7$, then in S , we have $29 \equiv 1 \pmod{7}$ and hence for all x for which $30x + 1 \equiv 0 \pmod{7}$ we also have $30x + 29 \equiv 0 \pmod{7}$. Thus there exists one integer x between 0 and 6 that simultaneously satisfies the congruence relations $30x + 1 \equiv 0 \pmod{7}$ and $30x + 29 \equiv 0 \pmod{7}$. A procedure for obtaining seven prime subsets must therefore consider the fact that T_x may contain two distinct integers divisible by 7. Taking this into consideration we obtain the following modification of ϕ_3 . Let $p_4 = 7$, $p_5 = 11$, ..., p_n , ... be the ordered sequence of consecutive prime numbers in ascending order and let $m := 7 \cdot 11 \cdot 13 \dots p_n$. If $k \leq n$, then by Theorem 2.2

$$S_7(m, k) = 6 \cdot 5(p_6 - 8) \dots (p_k - 8)p_{k+1} \dots p_n$$

is the number of values of x , $0 \leq x \leq m - 1$, for which T_x contains 7 integers that are relatively prime to $7 \cdot 11 \cdot 13 \dots p_k$. In particular if one such value of x is less than or equal to $\lfloor \frac{p_{k+1}^2 - 1}{30} \rfloor$ then, by Lemma 2.1, T_x contains seven primes. Let

$$R_7(m, n) := \{x \mid 0 \leq x \leq m - 1,$$

$$T_x \text{ contains 7 integers that are relatively prime to } m\}$$

so that $|R_7(m, n)| = S_7(m, n)$. Let

$$T_7(p_{n+1}^2) := \{x \in R_7(m, n) \mid x \leq \lfloor \frac{p_{n+1}^2 - 1}{30} \rfloor\}.$$

Then for all $x \in T_7(p_{n+1}^2)$, T_x contains seven primes so that $|T_7(p_{n+1}^2)| \leq \pi_7(p_{n+1}^2)$. We note, by writing

$$S_7(m, n) = m(1 - \frac{1}{7})(1 - \frac{6}{11})(1 - \frac{8}{p_6}) \dots (1 - \frac{8}{p_n}), \quad (2.2)$$

that $S_7(m, n)$ may be computed iteratively as follows: for each j , $4 \leq j \leq n$, let $S_7(m, n, 4) = m(1 - \frac{1}{7})$, $S_7(m, n, 5) = S_7(m, n, 4)(1 - \frac{6}{11})$, and for $j \geq 6$ let $S_7(m, n, j) = S_7(m, n, j-1)(1 - \frac{8}{p_j})$.

Associated with the expression for $S_7(m, n)$ is a sieve on the set of integers

$$0, 1, 2, 3, 4, \dots, m-1$$

which sifts out x if either T_x contains two integers divisible by 7 or an integer y not divisible by 7 but divisible by p_j for some j , $5 \leq j \leq n$. The residue set of the respective sieve is then $R_7(m, n)$. Since the primes p_j are unevenly distributed they sift out the values x in an unevenly distributed manner. However $S_7(m, n)$, when viewed as a sieve in the manner above, is cyclic in the sense that when extended to the set of all integers, then for each $j \leq n$, p_j sifts out the same number of values of x , with the same irregularity, over each interval

$$s \cdot 7 \cdot 11 \cdot \prod_{i=6}^j p_i \leq x < (s+1)7 \cdot 11 \cdot \prod_{i=6}^j p_i.$$

The average density of elements of the residue set is therefore $\frac{30m}{S_7(m, n)}$ or

$$2 \cdot 3 \cdot 5 \cdot \frac{7 \cdot 11}{6 \cdot 5} \left(\prod_{j=6}^n \frac{p_j}{(p_j-8)} \right).$$

We shall require the following results of J.B. Rosser and L. Schoenfeld (see [9] page 69):

Theorem 2.3: [[9] Theorem 1] *Let $n \geq 1$ be an integer. Then:*

- (i) $\frac{n}{\log n - \frac{1}{2}} < \pi(n)$ for $n \geq 67$,
- (ii) $\pi(n) < \frac{n}{\log n - \frac{3}{2}}$ for $n \geq 5$.

Corollary 2.4: *Let $n \geq 1$ be an integer. Then:*

- $\frac{n}{\log n} < \pi(n)$ for $n \geq 17$.

Theorem 2.5: [[9] Theorem 3] *Let $n \geq 1$ be an integer. Then:*

- (i) $n(\log n + \log \log n - \frac{3}{2}) < p_n$ for $n \geq 2$,
- (ii) $p_n < n(\log n + \log \log n - \frac{1}{2})$ for $n \geq 20$.

Corollary 2.6: *Let $n \geq 1$ be an integer. Then:*

- (i) $n(\log n) < p_n$ for $n \geq 1$,
- (ii) $p_n < n(\log n + \log \log n)$ for $n \geq 6$.

As a consequence of the above results we have the following result (see [6]):

Theorem 2.7: [6] *For $n \geq 3$, let p_n denote the n^{th} prime. Then for each integer $b > 0$ there exists an integer $N(b)$ such that*

$$\frac{bp_{n+1}^2}{n+1} < \pi(p_{n+1}^2)$$

for all $n \geq N(b)$.

3. Comparison of the sieve of Eratosthenes and the Partition sieve

We now compare the porousness of the sieve of Eratosthenes with a sieve that may be obtained from the identity

$$1 - \sum_{s=1}^k \frac{1}{s(s+1)} = \frac{1}{k+1}.$$

The idea is to let x be a positive integer. Then we have

$$x - \sum_{s=1}^k \frac{x}{s(s+1)} = \frac{x}{k+1}. \quad (3.3)$$

We then consider $\frac{x}{k+1}$ as a measure of the residue or an estimate of the number of integers that remain after applying $\sum_{s=1}^k \frac{x}{s(s+1)}$ to the sequence

$$1, 2, 3, 4, \dots, x. \quad (3.4)$$

Viewed in this way we shall refer to the sieve 3.4 as the **partition sieve**. Written in this manner, the partition sieve could then be applied inductively to the sequence 3.4. We have seen, from definition, that the sieve of Eratosthenes may also be applied inductively for each given value of x .

For each integer $s \geq 1$, let p_s denote the s^{th} prime number. Let n, k with $n \geq k$ be a pair of integers. For each integer $x \geq p_{n+1}^2 - 1$, let $S(x, k)$ denote the sum

$$S(x, k) := x + \sum_{j=1}^k (-1)^j \left\{ \sum_{1 \leq s_1 < \dots < s_j \leq k} \left\lfloor \frac{x}{\prod_{i=1}^j p_{s_i}} \right\rfloor \right\}. \quad (3.5)$$

The sum in Equation (3.5) is based on the inclusion-exclusion principle and can be considered as a sieve on the sequence of integers;

$$1, 2, 3, 4, 5, \dots, x \quad (3.6)$$

which sifts out all integers y for which $\text{g.c.d.}(y, p_s) \neq 1$ for some s , $1 \leq s \leq k$. We note that the expression for the value $S(x, k)$ sifts out the primes p_j , $1 \leq j \leq k$, from the sequence (3.6). This is the only difference between the sieve represented by this expression and the sieve of Eratosthenes. Let $x = p_{n+1}^2 - 1$ for some $n \geq k$. We now compare the values $S(p_{n+1}^2 - 1, n)$ with $\frac{p_{n+1}^2 - 1}{n+1}$, the order of the residue of the sieve $p_{n+1}^2 - 1 - \sum_{s=1}^n \frac{p_{n+1}^2 - 1}{s(s+1)}$. The comparison can be achieved inductively. Let $S(x, k)$, denote the set of all positive integers not exceeding x which are relatively prime to the primes p_j , $1 \leq j \leq k$. Then $S(x, k) = |S(x, k)|$. Note that for $x = p_{n+1}^2 - 1$, the effect of $S(x, n)$ on the sequence 3.6 coincides with that of the sieve of Eratosthenes apart from the fact that $S(x, n)$ also sifts out the primes $2, 3, 5, \dots, p_n$. Thus $\pi(p_{n+1}^2) = S(p_{n+1}^2 - 1, n) + n - 1$.

In the following result we show, in particular, that the number of primes between p_n and p_{n+1}^2 is unbounded as n increases. The result is an immediate consequence of Theorem 2.7 (see [6]). In fact the rest of the results in this section and their proofs are mere extensions of our work [6].

Corollary 3.1: *For $n \geq 12$, let p_n denote the n^{th} prime. Then for each integer $d \geq 1$ there exists an integer $N(d)$ such that $\frac{dp_{n+1}^2}{n+1} < S(p_{n+1}^2 - 1, n)$, for all $n \geq N(d)$.*

Proof: If $n \geq 12$, then $\frac{bp_{n+1}^2}{n+1} < \pi(p_{n+1}^2)$ for some integer $b \geq 2$. Since $S(p_{n+1}^2 - 1, n) = \pi(p_{n+1}^2) - n + 1$ and $n - 1 < \frac{(n+1)^2 \log^2(n+1)}{n+1} < \frac{p_{n+1}^2}{n+1}$, we have

$$\frac{(b-1)p_{n+1}^2}{n+1} < \frac{bp_{n+1}^2}{n+1} - n + 1 < \pi(p_{n+1}^2) - n + 1 = S(p_{n+1}^2 - 1, n)$$

for all $n \geq 12$. The result of the corollary follows if we put $d = b - 1$.

Since $\frac{d(p_{n+1}^2-1)}{n+1} < \frac{dp_{n+1}^2}{n+1}$ we see that if for each integer $n \geq 12$, we put $d_n = \frac{(n+1)S(p_{n+1}^2-1, n)}{p_{n+1}^2-1}$, then, from the result of Corollary 3.1, we have an unbounded sequence of rational numbers $\{d_n\}_{n \geq 12}$ such that

$$S(p_{n+1}^2 - 1, n) = \frac{d_n(p_{n+1}^2 - 1)}{n+1}.$$

But for each $n \geq 4$, $S(p_{n+1}^2 - 1, n)$ may be computed inductively from $S(p_{n+1}^2 - 1, 3)$, forming a finite sequence of values $S(p_{n+1}^2 - 1, k)$, $3 \leq k \leq n$. For each $n \geq 4$ and k , $3 \leq k \leq n$, we have

$$S(p_{n+1}^2 - 1, k+1) = S(p_{n+1}^2 - 1, k) - T(p_{n+1}^2 - 1, k+1), \text{ where}$$

$$T(p_{n+1}^2 - 1, k+1) := \left\lfloor \frac{p_{n+1}^2 - 1}{p_{k+1}} \right\rfloor + \sum_{j=1}^k (-1)^j \left\{ \sum_{1 \leq s_1 < \dots < s_j \leq k} \left\lfloor \frac{p_{n+1}^2 - 1}{p_{k+1} \prod_{i=1}^j p_{s_i}} \right\rfloor \right\}. \quad (3.7)$$

In the same vain, d_n is the last term of a sequence of numbers $\{a_k(n)\}$, $3 \leq k \leq n$, defined, for each fixed integer $n \geq k$, by $a_k(n) := \frac{(k+1)S(p_{n+1}^2-1, k)}{p_{n+1}^2-1}$.

The following is our main observation in this section (see [6]):

Lemma 3.2: *Let n, k with, $k \leq n$, be a pair of positive integers. Then $8T(p_{n+1}^2 - 1, k+1) < \frac{a_k(n)(p_{n+1}^2-1)}{(k+1)(k+2)}$ for all $k \geq 10^{10}$.*

Proof: We first show that for each fixed n and all $k \leq n$, $\{a_k(n)\}$ is a nondecreasing sequence. To get a more explicit estimate for $a_k(n)$ for a given value of $k \geq 30$, we note that if $k = n$, then

$$\begin{aligned}
a_n(n) &= \frac{(n+1)S(p_{n+1}^2-1, n)}{p_{n+1}^2-1} \\
&= \frac{(n+1)(\pi(p_{n+1}^2)-(n-1))}{p_{n+1}^2-1} \\
&> \frac{(n+1)(\frac{p_{n+1}^2}{(2\log(p_{n+1}))}-(n-1))}{p_{n+1}^2-1} \\
&> \frac{(n+1)}{(2\log(p_{n+1}))} - \frac{(n+1)(n-1)}{(n+1)^2(\log(n+1))^2} > \frac{(n+1)}{(2\log(p_{n+1}))} - \frac{1}{10}.
\end{aligned}$$

By Theorem 2.3, $\frac{m}{(\log(\frac{m}{1.64}))} < \frac{m}{(\log(m-\frac{1}{2}))} < \pi(m)$ for $m \geq 67$. We see therefore that we must have

$$a_n(n) > \frac{(n+1)}{(2\log(p_{n+1}))} \text{ for all } n \geq 30.$$

Recall that $S(p_{n+1}^2-1, k)$ represents the residue set of the sieve of Equation 3.5, that is, is the set of all positive integers not exceeding p_{n+1}^2-1 which are relatively prime to the primes p_s , $1 \leq s \leq k$. Let $\{q_r^k\}$, $r \geq 1$ be the ordered sequence of elements of $S(p_{n+1}^2-1, k)$, so that $q_1^k = 1, q_2^k = p_{k+1}, q_3^k = p_{k+2}, \dots$. Then q_r^k is a prime whenever $q_r^k < p_{k+1}^2$. For $n > k$, $S(p_{n+1}^2-1, k+1)$ is obtained from $S(p_{n+1}^2-1, k)$ by sifting out all products $q_r^k p_{k+1}$ less than p_{n+1}^2-1 , where, for each r , q_r^k is an element of $S(p_{n+1}^2-1, k)$ or, equivalently,

$$S(p_{n+1}^2-1, k+1) = S(p_{n+1}^2-1, k) - T(p_{n+1}^2-1, k+1).$$

Now

$$\frac{a_k(n)(p_{n+1}^2-1)}{k+1} = S(p_{n+1}^2-1, k)$$

and

$$\begin{aligned}
\frac{a_k(n)(p_{n+1}^2-1)}{k+2} &= \frac{a_k(n)(p_{n+1}^2-1)}{k+1} - \frac{a_k(n)(p_{n+1}^2-1)}{(k+1)(k+2)} \\
&= \frac{p_{n+1}^2-1}{\frac{1}{a_k(n)}(k+1)} - \frac{p_{n+1}^2-1}{\frac{1}{a_k(n)}(k+1)(k+2)}
\end{aligned}$$

Thus $a_{k+1}(n) = a_k(n)$ if

$$T(p_{n+1}^2-1, k+1) = \frac{p_{n+1}^2-1}{\frac{1}{a_k(n)}(k+1)(k+2)}.$$

It follows that $a_{k+1}(n) \geq a_k(n)$ only if

$$T(p_{n+1}^2-1, k+1) \leq \frac{p_{n+1}^2-1}{\frac{1}{a_k(n)}(k+1)(k+2)}.$$

From our remarks above, it suffices to show that

$$r\left(\frac{1}{a_k(n)}(k+1)(k+2)\right) < p_{k+1}q_r^k \quad (3.8)$$

for each $r \geq 1$ for which both products are less than p_{n+1}^2-1 . Since $(k+1)(\log(k+1)) < p_{k+1}$, it suffices to show that $r\left(\frac{1}{a_k(n)}(k+2)\right) < (\log(k+1))$

1)) q_r^k or, equivalently, $\frac{r}{\log(k+1)}(\frac{1}{a_k(n)}(k+2)) < q_r^k$. If $1 < q_r^k < p_{k+1}^2$, then q_r^k is equal to a prime number p_s with $s > k$. We know that $\text{slogs} < p_s$. Treating slogs as a function of s we get its derivative to be $1 + \text{slogs}$. Treating $\frac{r}{\log(k+1)}(\frac{1}{a_k(n)}(k+2))$ as a function of r we get its derivative to be equal to $\frac{1}{\log(k+1)}(\frac{1}{a_k(n)}(k+2))$. By virtue of our estimation of $a_n(n)$ above we assume that $a_k(n) \geq \frac{(k+1)}{(2\log(p_{k+1}))}$ or that $\frac{(k+1)}{(2\log(p_{k+1}))}$ is a close estimate for $a_k(n)$. Then we would have that $\frac{1}{\log(k+1)}(\frac{1}{a_k(n)}(k+2))$ is less than or approximately equal to $\frac{(k+2)(2\log(p_{k+1}))}{(k+1)(\log(k+1))}$. Now

$$\begin{aligned} \frac{(k+2)(2\log(p_{k+1}))}{(k+1)(\log(k+1))} &= \frac{2\log(p_{k+1})}{\log(k+1)} + \frac{2\log(p_{k+1})}{(k+1)\log(k+1)} \\ &< \frac{2\log((k+1)(\log(k+1)+\log\log(k+1)))}{\log(k+1)} + 0.1 \\ &= 2 + \frac{2\log(\log(k+1)+\log\log(k+1))}{\log(k+1)} + 0.1 < 3 \end{aligned}$$

But $3 < 1 + \text{slogs}$, $s \geq k+1$ and $k \geq 30$. This establishes the Inequality (3.8) for $1 \leq q_r^k < p_{k+1}^2$. For $k \geq 30$, the set $S(p_{n+1}^2 - 1, k)$ is more dense over the interval $1 \leq q_r^k < p_{k+1}^2$, than over the interval $q_r^k \geq p_{k+1}^2$. The above argument therefore suffices for the cases $q_r^k \geq p_{k+1}^2$. For $s \geq 10^{10}$, $24 = 8 \cdot 3 < 1 + \text{slogs}$ and this completes the proof of the lemma.

Thus when $k \geq 10^{10}$, then at each inductive stage, the partition sieve is more porous (leaves a smaller residue set) by at least eight times the porousness of the sieve of Eratosthenes or that of Equation 3.5. In particular, the result of Lemma 3.2 is not dependent on x being equal to $p_{n+1}^2 - 1$, but could be extended to any value of $x \geq p_{n+1}^2$.

4. Proof of Theorem 1.1

The result of Theorem 1.1 shall be seen to be a consequence of the following observation.

Let p_i denote the i^{th} prime number and let $m \geq T = 10^{10}$. Let P_{K_m} denote the largest prime number less than

$$\sqrt{\prod_{i=1}^m p_i}.$$

By the result of Lemma 3.2, it suffices to show that

$$\frac{\prod_{i=1}^m p_i}{8(T+1)} < S_7(\prod_{i=4}^m p_i, T) = 6 \cdot 5 \prod_{i=6}^T (p_i - 8) \prod_{i=T+1}^m p_i \quad (4.9)$$

or, equivalently, that

$$\frac{8 \cdot (T+1)}{2 \cdot 3 \cdot 5} \cdot \frac{6}{7} \cdot \frac{5}{11} \left(\prod_{j=6}^T \frac{(p_j - 8)}{p_j} \right) > 1.$$

This can be shown to be the case by direct computation when $m = T$. This, in turn, implies that the inequality 4.9 holds for all $m \geq 10^{10}$.

Now for each m apply the sieve 3.3 inductively to $\frac{\prod_{i=1}^m p_i}{8(T+1)}$ starting with $k = T$. In practice, for each fixed m we apply the sieve

$$\frac{\prod_{i=1}^m p_i}{8(T+1)} - \sum_{r=T+1}^k \frac{\prod_{i=1}^m p_i}{8r(r+1)} = \frac{\prod_{i=1}^m p_i}{8(k+1)},$$

inductively, replacing $\frac{\prod_{i=1}^m p_i}{8(T+1)}$ by its respective residue at each stage and letting $T < k \leq K_m$.

On the other hand there is no simply expression for a sieve that can be applied to $\prod_{i=4}^m p_i$ to yield a residue set of order $\pi_7(\prod_{i=1}^m p_i)$. Instead, for each k , $T < k \leq K_m$, and, as explained in Chapter 2, consider the sieve associated with the expression

$$S_7(\prod_{i=4}^{K_m} p_i, k) = 6 \cdot 5 \prod_{i=6}^k (p_i - 8) \prod_{i=k+1}^{K_m} p_i.$$

Then the residue set for the sieve will be

$$R_7(\prod_{i=4}^{K_m} p_i, k) = \{x \mid 0 \leq x < \prod_{i=4}^{K_m} p_i,$$

T_x contains 7 integers that are relatively prime to $\prod_{i=4}^k p_i\}$.

Now let $k = K_m$ and let

$$T_7(\prod_{i=1}^m p_i) = \{x \in R_7(\prod_{i=4}^{K_m} p_i, k) \mid x < \prod_{i=4}^m p_i\}.$$

By Lemma 2.1 if $x \in T_7(\prod_{i=1}^m p_i)$, then T_x contains seven primes so that

$$|T_7(\prod_{i=1}^m p_i)| < \pi_7(\prod_{i=1}^m p_i).$$

Now let $k, T < k \leq K_m$ be given. By Lemma 3.2, for each value of x for which T_x contains a multiple of a prime p_j ($10^{10} < j \leq K_m$ and with p_j as its smallest divisor), there corresponds at least eight integers r_i (unique to k) for which each satisfies

$$r_i \left(\frac{1}{a_k(n)} (k+1)(k+2) \right) < \prod_{i=1}^m p_i$$

and bears the relation 3.8 with p_j . There therefore exists an integer r , determined by the integers r_i , (hence unique to k) such that

$$r \left(\frac{8}{a_k(n)} (k+1)(k+2) \right) < \prod_{i=1}^m p_i.$$

Thus in proceeding from $k = T$ to $k = K_m$ the Sieve 3.3 sifts out more elements from the initial residue set of order $\frac{\prod_{i=1}^m p_i}{8(T+1)}$ than the extension of the Sieve 3.5 from its initial residue set of order

$$6 \cdot 5 \prod_{i=6}^T (p_i - 8) \prod_{i=T+1}^m p_i.$$

Note that $|T_x| = 8$ and dividing by $\frac{8}{a_k(n)} (k+1)(k+2)$ sifts out eight more integers than dividing by $\frac{1}{a_k(n)} (k+1)(k+2)$. But

$$\frac{\prod_{i=1}^m p_i}{8(T+1)} < 6 \cdot 5 \prod_{i=6}^T (p_i - 8) \prod_{i=T+1}^m p_i$$

so, by the foregoing, the orders of the final residue sets must bear the relation:

$$\frac{\prod_{i=1}^m p_i}{8 \cdot (K_m + 1)} < |T_7(\prod_{i=1}^m p_i)| < \pi_7(\prod_{i=1}^m p_i).$$

This completes the proof of the theorem.

Competing Interest

The authors declare that they do not have any competing interest.

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An earlier version of this paper is presented as a preprint on arxiv.org. and may be viewed at the following link: <https://arxiv.org/abs/1912.09290>"

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