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A new regularity criterion for the 3D-Boussinesq equations in homogeneous Besov and Triebel-Lizorkin spaces

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Abstract

This paper aims to establish a new regularity criterion for weak solutions to the three-dimensional (3D) Boussinesq equations in the homogeneous Besov spaces $\dot{B}^{s}_{p,q}(\mathbb{R}^3)$ and the

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homogeneous Triebel–Lizorkin spaces $\dot{F}^{s}_{p,q}(\mathbb{R}^3)$. This regularity is proven if the velocity ω satisfies the following condition

$$\int_{\mathcal{J}} \|\omega\|^2_{\dot{B}^{s}_{p,q}(\mathbb{R}^3)}\, dt < \infty \quad \text{and} \quad \int_{\mathcal{J}} \|\omega\|^2_{\dot{F}^{s}_{p,q}(\mathbb{R}^3)}\, dt < \infty; \quad \mathcal{J} :=]0, T[,$$

with some conditions on exponents s, p and q.

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1. Introduction and the main result

The Boussinesq equations are important in various fields, particularly in fluid dynamics and heat transfer, as they provide a simplified and effective model for describing the behavior of buoyant flows in fluids, such as water and air. These equations are a set of partial differential equations that describe fluid motion with temperature variations, also, Boussinesq equation is one of the important topics for research in nonlinear sciences. For more details, e.g., see references [1], [2], [3], [4] and [5]. To date, a substantial body of literature has focused on the incompressible 3D-Boussinesq equations, with particular attention to the regularity of solutions in various functional spaces. For instance, we can mention: In [6] Ishimura and Morimoto demonstrated that blow-up is controlled by the condition $\nabla\omega \in \mathrm{L}_1(\mathcal{J}; \mathrm{L}_\infty(\mathbb{R}^3))$. Kozono, Ogawa, and Taniuchi [7], they obtained the following regularity condition $\omega \in \mathrm{L}_2(\mathcal{J}; \dot{B}^0_{\infty,\infty}(\mathbb{R}^3))$ or $\mathrm{curl}\omega \in \mathrm{L}_1(\mathcal{J};\ \dot{B}^0_{\infty,\infty}(\mathbb{R}^3))$. Fan and Ozawa [8] and Fan and Zhou [9] established that regularity of the weak solution to the Boussinesq equations is guaranteed under either of the following conditions $\omega \in \mathrm{L}_{2/(1-s)}(\mathcal{J}; \dot{B}^{-s}_{\infty,\infty}(\mathbb{R}^3))$ where $0 < s < 1$, and $\mathrm{curl}\,\omega \in \mathrm{L}_1(\mathcal{J}; \dot{B}^0_{\infty,\infty}(\mathbb{R}^3))$, respectively. In [10] Qiu, Du, and Yao, they presented an improvement and extension of some well-known regularity criteria of smooth solution for the 3D-Boussinesq equations under the incompressibility condition $\nabla \cdot \omega = 0$ in the multiplier space $\dot{\mathcal{X}}_s$, provided that one of the following conditions holds $\omega \in \mathrm{L}_{2/(1-s)}(\mathcal{J}; \dot{\mathcal{X}}_s(\mathbb{R}^3))$ and $\nabla\omega \in \mathrm{L}_{2/(2-\sigma)}(\mathcal{J};\ \dot{\mathcal{X}}_\sigma(\mathbb{R}^3))$ where $0 \le s < 1$ and $0 \le \sigma \le 1$, respectively. In [11] Geng and Fan, used the Serrin-type condition $\omega \in \mathrm{L}_{2/(1+s)}(\mathcal{J}; \dot{B}^{s}_{\infty,\infty}(\mathbb{R}^3))$ where $-1 < s < 1$ and $s \neq 0$. Zhang [12] recently demonstrated that the pair (ω, ϕ) attains regularity as a strong solution under the condition $\nabla\omega, \nabla\phi \in \mathrm{L}_2(\mathcal{J}; \dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3))$. On the other hand, in [13] Barbagallo et al., an extension was given which also improved this result. In [14], Ye improved prior regularity criteria for the 3D-Boussinesq equations by requiring $\omega \in \mathrm{L}_{p_2}(\mathcal{J};\ \dot{B}^{3/p_1+2/p_2-1}_{p_1,\infty}(\mathbb{R}^3))$ for $1 < p_1, p_2 \le \infty$ where $3/p_1 + 2/p_2 \le 2$ and $(p_1, p_2) \neq (\infty, \infty)$. Alghamdi, Gala, and Ragusa [15] established a regularity criterion for the 3D-Boussinesq equations, requiring the velocity field ω to satisfy $\omega \in \mathrm{L}_{2/(1+s)}(\mathcal{J}; \dot{B}^{s}_{\infty,\infty}(\mathbb{R}^3))$ where $0 < s < 1$. Additionally, the work in [16] demonstrated that the weak solution (ω,ϕ) gains regularity provided $\nabla\omega_h,\nabla\phi_h \in \mathrm{L}_{8/3}(\mathcal{J};\dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3))$. In [17], Gala and Ragusa proved a regularity criterion requiring the horizontal gradients $\nabla_h\widetilde{\omega},\nabla_h\phi \in \mathrm{L}_1(\mathcal{J};\dot{B}^{0}_{\infty,\infty}(\mathbb{R}^3))$, where $\nabla_h := (\partial_1,\partial_2)$ and $\widetilde{\omega} := (\omega_1,\omega_2,0)$. Later, Omrane, Gala, and Théra [18] improved this result. From recent studies, in [19] Zou and Li, showed a the smooth solution (ω,ϕ) is regular if the horizonal velocity ω_h holds $\nabla_h\omega_h \in \mathrm{L}_{p_2}(\mathcal{J};\dot{B}^{0}_{p_1,2p_1/3}(\mathbb{R}^3))$ or $\nabla\omega_h \in \mathrm{L}_2(\mathcal{J};\dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3))$, with $\omega_h := (\omega_1,\omega_2)$, and $3/p_1 + 2/p_2 = 2$ for all $3/2 < p_1 \leq \infty$. In the paper [20], the author investigated functions acting on Besov-type spaces and Triebel-Lizorkin-type spaces, offering new insights into their structural and functional properties. Such analyses are particularly valuable for understanding the behavior of solutions to PDEs, such as the regularity of solutions to the Boussinesq equation within homogeneous Besov-type spaces $\dot{B}^{s,\tau}_{p,q}(\mathbb{R}^3)$ and homogeneous Triebel-Lizorkin-type spaces $\dot{F}^{s,\tau}_{p,q}(\mathbb{R}^3)$. Notably, when $\tau = 0$, these spaces simplify to the classical Besov and Triebel-Lizorkin spaces.

This paper establishes a new regularity criterion for weak solutions to the 3D-Boussinesq equations in the homogeneous Besov spaces $\dot{B}^{s}_{p,q}(\mathbb{R}^3)$ and Triebel–Lizorkin spaces $\dot{F}^{s}_{p,q}(\mathbb{R}^3)$. For brevity, we collectively refer to these spaces as $\dot{A}^{s}_{p,q}(\mathbb{R}^3)$, where $\dot{A}^{s}_{p,q}(\mathbb{R}^3) := \dot{B}^{s}_{p,q}(\mathbb{R}^3)$ or $\dot{F}^{s}_{p,q}(\mathbb{R}^3)$. Recall that, as these spaces are defined modulo polynomials, since $\|\omega\|_{\dot{A}^{s}_{p,q}(\mathbb{R}^3)} = 0$ if and only if, ω is a polynomial on $\mathbb{R}^3$. The Boussinesq system is described as follows:

$$\begin{cases} \partial_t\omega + (\omega\cdot\nabla)\omega - \Delta\omega + \nabla\vartheta = \phi\, e_3, \\ \partial_t\phi + (\omega\cdot\nabla)\phi - \Delta\phi = 0, \\ \nabla\cdot\omega = 0, \\ \omega(\cdot\,,0) = \omega_0(\cdot), \phi(\cdot\,,0) = \phi_0(\cdot), \end{cases} \tag{1.1}$$

where $\omega := \omega(\cdot,t): \mathbb{R}^3 \times]0,\infty[\to \mathbb{R}^3$ is the velocity, $\phi := \phi(\cdot,t): \mathbb{R}^3 \times]0,\infty[\to \mathbb{R}$ is the temperature, $\vartheta := \vartheta(\cdot,t) \in \mathbb{R}$ is the scalar function pressure, and $e_3^T := (0,0,1)$. Note that the initial velocity ω_0 and the initial temperature ϕ_0 are given, where $\nabla\cdot\omega_0 = 0$. It is worth noting that when $\phi = 0$, the system (1.1) reduces to the classical 3D-incompressible Navier–Stokes equations.

The main result of the regularity criterion of weak solutions of equations (1.1) is as follows:

Theorem 1.1: *Let $p \in]0,\infty]$ and $q \in]0,1]$ in the $\dot{B}^{s}_{p,q}(\mathbb{R}^3)$-case (with $p \in]0,1]$ in the $\dot{F}^{s}_{p,q}(\mathbb{R}^3)$-case). Suppose $(\omega_0,\phi_0) \in (H^1(\mathbb{R}^3))^2$ with $div\ \omega_0 = 0$ in the sense of distributions, (ω,ϕ) is a weak solution of (1.1) on $\mathbb{R}^3 \times \mathcal{J}$ and satisfies the strong energy inequality. If velocity ω satisfis*

$$\int_{\mathcal{J}} \|\omega\|^2_{\dot{A}^{3/p+1}_{p,q}(\mathbb{R}^3)} \mathrm{d}t < \infty,$$

then the solution (ω,ϕ) is regular.

Corollary 1: *We assume the same conditions as specified in Theorem 1.1. We have if velocity ω satisfis*

$$\sum_{k=0}^{3} \int_J \|\partial_k \omega\|^2_{\dot{A}^{3/p}_{p,q}(\mathbb{R}^3)} \mathrm{d}t < \infty,$$

then the solution (ω, ϕ) *is regular.*

Notation. We denote by $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$, $\|\cdot\|_p$ the quasi-norm of $\mathrm{L}_p(\mathbb{R}^3)$ for all $0 < p \leq \infty$, $\mathcal{S}(\mathbb{R}^3)$ the Schwartz space, $\mathcal{S}'(\mathbb{R}^3)$ its topological dual. For $\omega \in \mathrm{L}_1(\mathbb{R}^3)$ or $\omega \in \mathcal{S}'(\mathbb{R}^3)$, let $\mathcal{F}\omega = \widehat{\omega}$ be the Fourier transform of ω, $\mathcal{F}^{-1}\omega$ the inverse of $\widehat{\omega}$. The symbol $\hookrightarrow$ indicates a continuous embedding. Let $\mathcal{P}_\infty(\mathbb{R}^3)$ denote the space of all polynomial functions on $\mathbb{R}^3$. For every $\omega \in \mathcal{P}_\infty(\mathbb{R}^3)$, we define $\mathcal{S}_\infty(\mathbb{R}^3)$ as the set of all $\psi \in \mathcal{S}(\mathbb{R}^3)$ s.t. $\langle \omega, \psi \rangle = 0$. The topological dual of $\mathcal{S}_\infty(\mathbb{R}^3)$ is denoted by $\mathcal{S}'_\infty(\mathbb{R}^3)$. The quotient space $\mathcal{S}'(\mathbb{R}^3)/\mathcal{P}_\infty(\mathbb{R}^3)$ can identified with $\mathcal{S}'_\infty(\mathbb{R}^3)$ (the tempered distributions modulo $\mathcal{P}_\infty(\mathbb{R}^3)$). Positive constants c, c_1,..., depending only on p, q,..., may vary from line to line.

2. Preliminaries

The Littlewood-Paley decomposition serves as a key tool in the definition of Besov and Triebel-Lizorkin spaces. This analytical framework was introduced in foundational works such as [21] and [22]. Let us first recall: let $\rho \in C^\infty(\mathbb{R}^3)$ (where ρ is the radial function) s.t. $\rho \in [0,1]$, $\rho(\upsilon) = 1$ for $|\upsilon| \leq 1$ and $\rho(\upsilon) = 0$ for $|\upsilon| \geq 3 \cdot 2^{-1}$. For all $\upsilon \in \mathbb{R}^3$, we put $\theta(\upsilon) := \rho(\upsilon) - \rho(2\,\upsilon)$ which is supported by $2^{-1} \leq |\upsilon| \leq 3 \cdot 2^{-1}$ and $\theta(\upsilon) = 1$ in $3 \cdot 2^{-2} \leq |\upsilon| \leq 1$, and we have

$$\forall \upsilon \neq 0, \sum_{j\in\mathbb{Z}} \theta(2^j\,\upsilon) = 1, \text{and } \forall m \in \mathbb{Z},\ \rho(2^{-m}\upsilon) + \sum_{j\geq m+1} \theta(2^{-j}\upsilon) = 1.$$

The operators S_j and Q_j are defined by $\widehat{S_j\omega} := \rho(2^{-j}(\cdot))\widehat{\omega}$ and $\widehat{Q_j\omega} := \theta(2^{-j}(\cdot))\widehat{\omega}$, where is defined on $\mathcal{S}'(\mathbb{R}^3)$, and that Q_j is defined on $\mathcal{S}'_\infty(\mathbb{R}^3)$. All of these operators take values in the space of analytic functions of exponential type, as characterized by the Paley-Wiener theorem (see, e.g., [22, Rem. 2.3.1/2, p. 45]).

The convergence of the Littlewood-Paley decomposition of any function is given by:

Proposition 1: [23, Prop. 2]

- For any $\omega \in \mathcal{S}_\infty(\mathbb{R}^3)$ (respectively, $\mathcal{S}'_\infty(\mathbb{R}^3)$) one has the decomposition $\omega = \sum_{j\in\mathbb{Z}} Q_j\omega$ where the series converges in $\mathcal{S}_\infty(\mathbb{R}^3)$ (respectively, $\mathcal{S}'_\infty(\mathbb{R}^3)$).
- For any $\omega \in \mathcal{S}(\mathbb{R}^3)$ (respectively, $\mathcal{S}'(\mathbb{R}^3)$) and any $m \in \mathbb{Z}$, one has the decomposition $\omega = S_m\omega + \sum_{j>m} Q_j\omega$ where the series converges in $\mathcal{S}(\mathbb{R}^3)$ (respectively, $\mathcal{S}'(\mathbb{R}^3)$).

For more details about $\dot{A}^s_{p,q}(\mathbb{R}^3)$ we refer to e.g., [21], [22], [24], however we recall definitions and some properties.

Definition 1: Let $s \in \mathbb{R}$, p, $q \in]0,\infty]$ (with $p \neq \infty$ in the $\dot{F}^s_{p,q}(\mathbb{R}^3)$-case). The homogeneous spaces $\dot{A}^s_{p,q}(\mathbb{R}^3)$ is the set of $\omega \in \mathcal{S}'_\infty(\mathbb{R}^3)$ such that

$$\|\omega\|_{\dot{A}^s_{p,q}(\mathbb{R}^3)} := \begin{cases} (\sum_{j\in\mathbb{Z}} (2^{js}\|Q_j\omega\|_p)^q)^{1/q} < \infty, & \text{in the } \dot{B}^s_{p,q}(\mathbb{R}^3) - case, \\ \|(\sum_{j\in\mathbb{Z}} (2^{js}|Q_j\omega|)^q)^{1/q}\|_p < \infty, & \text{in the } \dot{F}^s_{p,q}(\mathbb{R}^3) - case. \end{cases}$$

On the other hand, it holds that: $\mathcal{S}_\infty(\mathbb{R}^3) \hookrightarrow \dot{A}^s_{p,q}(\mathbb{R}^3) \hookrightarrow \mathcal{S}'_\infty(\mathbb{R}^3)$, $\dot{A}^s_{p,q_1}(\mathbb{R}^3) \hookrightarrow \dot{A}^s_{p,q_2}(\mathbb{R}^3)$ if $q_1 < q_2$, $\dot{B}^s_{p,\min(p,q)}(\mathbb{R}^3) \hookrightarrow \dot{F}^s_{p,q}(\mathbb{R}^3) \hookrightarrow \dot{B}^s_{p,\max(p,q)}(\mathbb{R}^3)$. If $s_1 > s_2$, $0 < p_1 < p_2 < \infty$, $0 < q, r \leq \infty$ and $s_1 - 3/p_1 = s_2 - 3/p_2$ then $\dot{B}^{s_1}_{p_1,q}(\mathbb{R}^3) \hookrightarrow \dot{B}^{s_2}_{p_2,q}(\mathbb{R}^3) \hookrightarrow \dot{B}^{s_2-3/p_2}_{\infty,q}(\mathbb{R}^3)$, $\dot{F}^{s_1}_{p_1,q}(\mathbb{R}^3) \hookrightarrow \dot{B}^{s_2}_{p_2,p_1}(\mathbb{R}^3)$ and $\dot{F}^{s_1}_{p_1,q}(\mathbb{R}^3) \hookrightarrow \dot{F}^{s_2}_{p_2,r}(\mathbb{R}^3)$, see [24, Thm. 2.1].

Proposition 2: Let $s \in \mathbb{R}$, p, $q \in]0,\infty]$ (with $p \neq \infty$ in the $\dot{F}^s_{p,q}(\mathbb{R}^3)$-case). A distribution ω of $\mathcal{S}'_\infty(\mathbb{R}^3)$ belongs to $\dot{A}^s_{p,q}(\mathbb{R}^3)$ iff $\partial_k\omega$ $(k = 1,2,3)$, belong to $\dot{A}^{s-1}_{p,q}(\mathbb{R}^3)$. Moreover, $\sum_{k=1}^3 \|\partial_k\omega\|_{\dot{A}^{s-1}_{p,q}(\mathbb{R}^3)}$ is an equivalent quasi-seminorm in $\dot{A}^s_{p,q}(\mathbb{R}^3)$. (see, e.g., [25, Prop. 2.1.1] for the case of $\dot{B}^s_{p,q}(\mathbb{R}^3)$, and [26, Prop. 5] for the case of $\dot{F}^s_{p,q}(\mathbb{R}^3)$).

Remark 1: From in Proposition 2, we have case particular, we note that there exists a universal constant c_1 and $c_2 := 1/c_1$ such that

$$c_2\|\omega\|_{\dot{A}^s_{p,q}(\mathbb{R}^3)} \leq \|\nabla\omega\|_{\dot{A}^{s-1}_{p,q}(\mathbb{R}^3)} \leq c_1\|\omega\|_{\dot{A}^s_{p,q}(\mathbb{R}^3)}.$$

We now make use of the following classical result, known as the Bernstein inequality:

Proposition 3: [22, Rem. 1.3.2/1] Let $p, q \in]0,\infty]$, $\alpha \in \mathbb{N}_0^3$ and $R > 0$. If $p \leq q$, then

$$\|\omega^{(\alpha)}\|_q \leq c\, R^{|\alpha|+3/p-3/q}\|\omega\|_p,$$

for all $\omega \in \mathrm{L}_p(\mathbb{R}^3)$ satisfying $supp\ \widehat{\omega} \subseteq \{\xi \in \mathbb{R}^3 : |\xi| \leq R\}$.

Definition 2: [15] Let $(\omega_0, \phi_0) \in (\mathrm{L}_2(\mathbb{R}^3))^2$ where $\operatorname{div}\omega_0 = 0$ in the sense of distributions. Let $\mathrm{T} > 0$, a measurable pair (ω, ϕ) is said to be a weak solution of (1.1) on $\mathcal{J}$, provided that

- ω, $\phi \in \mathrm{L}_\infty(\mathcal{J}; L_2(\mathbb{R}^3)) \cap L_2(\mathcal{J}; \mathrm{H}^1(\mathbb{R}^3))$.
- The system (1.1) holds in the sense of distributions.
- For $t \in [\varepsilon, \mathrm{T}]$ with $\varepsilon \geq 0$, the following strong energy inequality is satisfied:

$$\|\omega(\cdot,t)\|_2^2 + \|\phi(\cdot,t)\|_2^2 + 2\int_\varepsilon^t (\|\nabla\omega(\cdot,s)\|_2^2 + \|\nabla\phi(\cdot,s)\|_2^2)\mathrm{d}s \leq \|\omega(\cdot,\varepsilon)\|_2^2 + \|\phi(\cdot,\varepsilon)\|_2^2.$$

Remark 2: If a weak solution (ω,ϕ) satisfies

$$\omega,\phi \in L_\infty(\mathcal{J}\,;H^1(\mathbb{R}^3)) \cap L_2(\mathcal{J}\,;H^2(\mathbb{R}^3)),$$

then (ω,ϕ) is a strong solution (classical). It is well established that strong solutions are both regular and unique.

3. Proofs of the main results

Proof of Theorem 1.1: As [15], we apply ∇ to the first and second equations of system (1.1), take their inner products with $\nabla\omega$ and $\nabla\phi$, respectively, and employ integration by parts to derive the equality

$$\frac{1}{2}\frac{d}{dt}(\|\nabla\omega\|_2^2+\|\nabla\phi\|_2^2)+\|\Delta\omega\|_2^2+\|\Delta\phi\|_2^2$$
$$=-\int_{\mathbb{R}^3}\nabla(\omega\cdot\nabla\omega)\cdot\nabla\omega\,dx+\int_{\mathbb{R}^3}\nabla(\phi\,e_3)\cdot\nabla\omega\,dx-\int_{\mathbb{R}^3}\nabla(\omega\cdot\nabla\phi)\cdot\nabla\phi\,dx$$
$$:=J_1+J_2+J_3. \tag{3.2}$$

By Proposition 1, we first have

$$\nabla\omega=\sum_{j\in\mathbb{Z}}\nabla(Q_j\omega).$$

Hence, for clarity, we divide the proof into the following steps.

Step 1: *Estimate of* J_1. We will do the proof in two steps.

Substep 1.1: *Estimate of* J_1 *in the* $\dot{B}^s_{p,q}(\mathbb{R}^3)$*-case*. By Proposition 3, we have

$$\forall j\in\mathbb{Z}, \left\|\nabla(Q_j\omega)\right\|_\infty\le c\sum_{k=1}^3\left\|\partial_k(Q_j\omega)\right\|_\infty\le c_1 2^j\left\|Q_j\omega\right\|_\infty.$$

Now, applying Hölder's inequality and Proposition 3, we get

$$\begin{aligned}J_1:=-\int_{\mathbb{R}^3}\nabla(\omega\cdot\nabla\omega)\cdot\nabla\omega\,dx &=-\int_{\mathbb{R}^3}\sum_{j\in\mathbb{Z}}\nabla(Q_j\omega)\nabla\omega\cdot\nabla\omega\,dx\\ &\le\sum_{j\in\mathbb{Z}}\int_{\mathbb{R}^3}|\nabla(Q_j\omega)|\,|\Delta\omega|\,|\nabla\omega|\,dx\\ &\le\sum_{j\in\mathbb{Z}}\left\|\nabla(Q_j\omega)\right\|_\infty\|\Delta\omega\|_2\|\nabla\omega\|_2\\ &\le c_2\sum_{j\in\mathbb{Z}}2^{j(3/p+1)}\left\|Q_j\omega\right\|_p\|\Delta\omega\|_2\|\nabla\omega\|_2\\ &\le c_2\|\omega\|_{\dot{B}^{3/p+1}_{p,1}(\mathbb{R}^3)}\|\Delta\omega\|_2\|\nabla\omega\|_2.\end{aligned}$$

Then, by taking into account of the embedding $\dot{B}^{3/p+1}_{p,q}(\mathbb{R}^3)\hookrightarrow\dot{B}^{3/p+1}_{p,1}(\mathbb{R}^3)$, and applying Young inequality, we immediately obtain

$$J_1\le\frac{1}{2}\|\Delta\omega\|_2^2+c\,\|\omega\|^2_{\dot{B}^{3/p+1}_{p,q}(\mathbb{R}^3)}\|\nabla\omega\|_2^2.$$

Substep 1.2: *Estimate of* J_1 *in the* $\dot{F}^s_{p,q}(\mathbb{R}^3)$-case. We use the inequality:

$$\left(\sum_{j\in\mathbb{Z}}w_j\right)^d\le\sum_{j\in\mathbb{Z}}w_j^d,\quad\forall w_j\ge 0,\quad\forall d\in]0,1],$$

then

$$J_1\le\sum_{j\in\mathbb{Z}}\left\|\nabla(Q_j\omega)\right\|_\infty\|\Delta\omega\|_2\|\nabla\omega\|_2$$

$$\leq c_1 (\sum_{j\in\mathbb{Z}} 2^{jp} \|Q_j\omega\|_\infty^p)^{1/p} \|\Delta\omega\|_2 \|\nabla\omega\|_2$$
$$\leq c_1 \|\omega\|_{\dot{B}^1_{\infty,p}(\mathbb{R}^3)} \|\Delta\omega\|_2 \|\nabla\omega\|_2,$$

then we use the embeddings $\dot{F}^{3/p+1}_{p,q}(\mathbb{R}^3) \hookrightarrow \dot{B}^1_{\infty,p}(\mathbb{R}^3)$, we obtain

$$J_1 \leq c_1 \|\omega\|_{\dot{F}^{3/p+1}_{p,q}(\mathbb{R}^3)} \|\Delta\omega\|_2 \|\nabla\omega\|_2.$$

Applying Young inequality, we get

$$J_1 \leq \frac{1}{2}\|\Delta\omega\|_2^2 + c\,\|\omega\|^2_{\dot{F}^{3/p+1}_{p,q}(\mathbb{R}^3)} \|\nabla\omega\|_2^2.$$

And in the end we have

$$J_1 \leq \frac{1}{2}\|\Delta\omega\|_2^2 + c\,\|\omega\|^2_{\dot{A}^{3/p+1}_{p,q}(\mathbb{R}^3)} \|\nabla\omega\|_2^2. \tag{3.3}$$

Step 2: *Estimate of J_2*. By inequalities of Hölder and Young, we find

$$J_2 := \int_{\mathbb{R}^3} \nabla(\phi\, e_3)\cdot\nabla\omega\,\mathrm{d}x \leq \|\nabla\omega\|_2 \|\nabla\phi\|_2 \leq \frac{1}{2}\|\nabla\omega\|_2^2 + \frac{1}{2}\|\nabla\phi\|_2^2. \tag{3.4}$$

Step 3: *Estimate of J_3*. This can be done as the estimate of J_1, we have

$$\begin{aligned} J_3 := -\int_{\mathbb{R}^3} \nabla(\omega\cdot\nabla\phi)\cdot\nabla\phi\,dx &= -\int_{\mathbb{R}^3} \nabla\omega\cdot\nabla\phi\cdot\nabla\phi\,\mathrm{d}x \\ &\leq \sum_{j\in\mathbb{Z}} \int_{\mathbb{R}^3} |\nabla(Q_j\omega)|\,|\Delta\phi|\,|\nabla\phi|\,\mathrm{d}x \\ &\leq c\sum_{j\in\mathbb{Z}} 2^j \|Q_j\omega\|_\infty \|\Delta\phi\|_2 \|\nabla\phi\|_2. \end{aligned}$$

Again, from Proposition 3, by Hölder and Young inequalities and by taking into account of the embeddings $\dot{B}^{3/p+1}_{p,q}(\mathbb{R}^3) \hookrightarrow \dot{B}^{3/p+1}_{p,1}(\mathbb{R}^3)$ and $\dot{F}^{3/p+1}_{p,q}(\mathbb{R}^3) \hookrightarrow \dot{B}^1_{\infty,p}(\mathbb{R}^3)$, we immediately obtain

$$J_3 \leq \frac{1}{2}\|\Delta\phi\|_2^2 + c\,\|\omega\|^2_{\dot{A}^{3/p+1}_{p,q}(\mathbb{R}^3)} \|\nabla\phi\|_2^2. \tag{3.5}$$

Inserting the estimates (3.3), (3.4) and (3.5) into (3.2), we get

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|\nabla\omega\|_2^2 + \|\nabla\phi\|_2^2) + \|\Delta\omega\|_2^2 + \|\Delta\phi\|_2^2 \\ &\leq \frac{1}{2}(\|\Delta\omega\|_2^2 + \|\Delta\phi\|_2^2) + c(1 + \|\omega\|^2_{\dot{A}^{3/p+1}_{p,q}(\mathbb{R}^3)})(\|\nabla\omega\|_2^2 + \|\nabla\phi\|_2^2), \end{aligned}$$

this implies that

$$\frac{d}{dt}(\|\nabla\omega\|_2^2 + \|\nabla\phi\|_2^2) + \|\Delta\omega\|_2^2 + \|\Delta\phi\|_2^2 \leq c(1 + \|\omega\|^2_{\dot{A}^{3/p+1}_{p,q}(\mathbb{R}^3)})(\|\nabla\omega\|_2^2 + \|\nabla\phi\|_2^2).$$

Using Gronwall's inequality, we find

$$\omega, \phi \in \mathrm{L}_\infty(\mathcal{J}\,;\ \mathrm{H}^1(\mathbb{R}^3)) \cap \mathrm{L}_2(\mathcal{J}\,;\ \mathrm{H}^2(\mathbb{R}^3)).$$

Thus, means that

$$\omega, \phi \in C^2(\mathbb{R}^3 \times \mathcal{J}).$$

The proof of Theorem 1.1 is completed.

Proof of Corollary 1: A consequence of Theorem 1.1 and Proposition 2.

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