*Journal of Interdisciplinary Mathematics*
ISSN: 0972-0502 (Print), ISSN: 2169-012X (Online)
Vol. 28 (2025), No. 5, pp. 1973–1980
DOI : 10.47974/JIM-2184

# A mathematical model of water pollution measurement in a stream using a collocation method with a higher order Legendre polynomial

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**Abstract**

In environmental research, challenges with water contamination assessment are generally prevalent. Through data collection, pollution levels in a system may be determined. This is quite challenging and involved; the measurements of what was measured vary from one point to another in every location. The governing equations for a uniform flow pollution dispersion model are used in water quality modeling. The advection-diffusion-reaction equation used in water quality model-ing for a uniform flow stream is a stable pollution dispersion model. This study presents a one-dimensional mathematical model for measuring stream water quality by collocation higher order Legendre polynomial functions. A water pol-lutant concentration can be approximated using the collocation method. A related water quality quantification method may also be employed with the suggested mathematical simulation to approximate the solution.

***Subject Classification:*** *65N06.*

***Keywords:*** *Water quality, Mathematical model, Collocaton method, Stream, Legendre polynomial.*

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## 1. Introduction

The collection of data supports the measurement of stream contamination levels. This is quite challenging and detailed, and even when the stream's water flow remains steady, the measurements of the findings vary from one location to another. A steady pollutant dispersion model is the governing equation in water quality modeling for a stream with uniform flow. There are several numerical methods for solving such problems. In order to solve a stable water pollution management problem at the lowest possible cost, the finite element approach is utilized in [1]. The one-dimensional advection-dispersion-reaction equation and other numerical methods for solving the uniform flow of a stream water quality model are described in [2-5] and [6].

The current velocity must be known at any location and at any time within the scope of the non-uniform flow model. The tidal elevation and water velocity field are provided by a hydrodynamic model. [7-9] and [10] employ the hydrodynamic model and the advection-dispersion equation, respectively, to calculate the current velocity in a bay, uniform reservoir, and sea. For one-dimensional domains like longitudinal stream systems, finite difference methods, including both explicit and implicit schemes, are most frequently utilized [11, 12].

In systems with non-uniform water flow, the water quality is measured using two mathematical models. The first is the hydrodynamic one. This gives the water's height and velocity field. The second model, which provides dispersion, identifies the zone where contaminants are concentrated. The traditional Crank-Nicolson technique is employed in hydrodynamic models. The first model's estimated flow velocity fields are inputs into the second model at each step [9, 10, 13]. The one-dimensional advection-dispersion-reaction equation that describes the non-uniform flow of stream water is solved numerically in [10]. The hydrodynamic equation and the backward time-centric space (BTCS) of the dispersion model are solved by a completely implicit scheme (the Crank-Nicolson technique). The hydrodynamic model in [13] is similarly solved using the Crank-Nicolson technique, while the dispersion model is solved using the explicit Saulyev scheme.

By combining the collocation method with higher-order Legendre polynomial functions, we suggest straightforward updates to a one-dimensional mathematical model of water-quality monitoring in a stream. It will be explained how to formulate the collocation approach using higher-order Legendre polynomial functions. The suggested numerical method is discussed in terms of the accuracy aspect.

## 2. Water Pollution Measurement Model

### 2.1 *Water Pollution Measurement Model in a Stream*

The dispersion of water pollutant is described by the advection-diffusion equation [1] in the domain, $[a, b]$,

$$-D_x \frac{d^2C}{dx^2} + u\frac{dC}{dx} + RC - Q = 0, \tag{1}$$

where $C(x)$ is the concentration of pollutant at the point $x \in [a, b]$ (kg/m$^3$), $u$ is the flow velocity in the x-direction (m/s), $D_x$ is the diffusion coefficient (m$^2$/s), R is the substance decay rate (s$^{-1}$) and $Q$ is the rate of change of substance concentration due to a source (kg/m$^3$s). The boundary conditions are $C = C_a$ at $x = a$ and $C = C_b$ at $x = b$.

## 3. Numerical Techniques

3.1 *Collocation Method*

We now provide a collocation technique that will provide a function-based approximation of the solution to a boundary value problem. Assume we are given a general linear two-point boundary value problem of the form,

$$Ly(t) = f(t), t \in [a, b], \tag{2}$$

$$y(a) = \alpha, y(b) = \beta. \tag{3}$$

Letting that

$$V = span\{v_1, \dots, v_n\}. \tag{4}$$

denote an approximation space we wish to represent the approximate solution in. We will express the approximate solution in the form,

$$y(t) = \sum_{j=1}^{n} c_j v_j(t), t \in [a, b], \tag{5}$$

with unknown coefficients $c_1, \dots, c_n$. Since $L$ is assumed to be linear we have,

$$Ly(t) = \sum_{j=1}^{n} c_j L v_j(t), t \in [a, b]. \tag{6}$$

Then Eqs.(2-3) become,

$$\sum_{j=1}^{n} c_j L v_j(t) = f(t), t \in [a, b], \tag{7}$$

$$\sum_{j=1}^{n} c_j v_j(a) = \alpha, \tag{8}$$

$$\sum_{j=1}^{n} c_j v_j(b) = \beta. \tag{9}$$

In order to determine the $n$ unknown coefficients $c_1, \dots, c_n$ in this formulation we impose $n$ collocation conditions to obtain an $n \times n$ system of linear equations for the $c_j$. The Eqs.(7-9) ensure that the boundary conditions are satisfied, and give us the first two collocation equations. To obtain the other $n - 2$ equations we choose $n - 2$ collocation points $t_2, \dots, t_{n-1}$, at which we enforce the differential equation.

If we let $t_1 = a$ t and $t_n = b$, then Eqs.(7-9) becomes,

$$\sum_{j=1}^{n} c_j v_j(t_1) = \alpha, \tag{10}$$

$$\sum_{j=1}^{n} c_j v_j(t_i) = f(t_i), i = 2, \dots, n - 1, \tag{11}$$

$$\sum_{j=1}^{n} c_j v_j(t_n) = \beta. \tag{12}$$

In matrix form we have the linear system [15], [16] and [17],

$$\begin{bmatrix} v_1(t_1) & \dots & v_n(t_1) \\ Lv_1(t_2) & \dots & Lv_n(t_2) \\ \vdots & \ddots & \vdots \\ Lv_1(t_{n-1}) & \dots & Lv_n(t_{n-1}) \\ v_1(t_n) & \dots & v_n(t_n) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_{n-1} \\ c_n \end{bmatrix} = \begin{bmatrix} \alpha \\ f(t_2) \\ \vdots \\ f(t_{n-1}) \\ \beta \end{bmatrix} \tag{13}$$

If the space $V$ and the collocation points $t_i$, $i = 1, \dots, n$, are chosen such that the collocation matrix in Eq.(13) is nonsingular then we can represent an approximate solution of Eqs.(2-3) from the space $V$ uniquely as

$$y(t) = \sum_{j=1}^{n} c_j v_j(t), t \in [a, b], \tag{14}$$

Note that this provides the solution in the form of a function that can be evaluated anywhere in $[a, b]$.

3.2 *Legendre Polynomial*

In this research, the Legendre polynomial function is chosen to be the approximation space.

**Definition 1:** For each nonnegative integer $n$, the Legendre polynomial $P_n(x)$ satisfies the recurrence relation [14],

$$P_n(x) = \frac{2n-1}{n} x P_{n-1}(x) - \frac{n-1}{n} P_{n-2}(x), \tag{15}$$

with $P_0(x) = 1$ and $P_1(x) = x$.

## 4. Numerical Simulation

4.1 *The third-ordered method*

Allow us to assume a plant releases waste water with diffusion coefficient $1.64\times10^{-5}$ $cm^2/s$ and a concentration of 0.75 mg/l into the canal specific area. 0.8000 mg/l while measuring the pollution level at a place one kilometer away. It was discovered that the stream's 0.15 m/s average velocity in the canal. The quantity of contaminants released from the environment may be detected at a rate of 1% of the pollution produced from this industrial unit, and this specific pollution may decay in the environment by roughly 5%. These occurrences make it possible to define parameters, and the following are the boundary conditions for the water quality measuring model, as shown in Table 1.

**Table 1**
**Parameter setting of the scenario**

| Parameter | Value | Parameter | Value |
|---|---|---|---|
| $D_x$ | $1.64\times10^{-5}$ $cm^2/s$ | $Q$ | 0.01 $kg/m^3s$ |
| $u$ | 0.15 m/s | $C_a$ | 0.7500 mg/l |
| $R$ | 0.05 $s^{-1}$ | $C_b$ | 0.8000 mg/l |

The collocation method of Eq.(13) is employed with the third-order Legendre polynomial approximation space of Eq.(15). It turns out that the approximated solution is obtained by

$$C(x) \approx 2.7782 - 5.1958x + 4.0564\left(\frac{3x^2}{2} - \frac{1}{2}\right) + 1.6308\left(\frac{5x^3}{2} - \frac{3x}{2}\right). \quad (16)$$

The approximated solution is shown in Table 2 and Fig. 1.

**Table 2**
**The approximated pollutant concentration of the 3rd-ordered method.**

| Distance (km) | Pollutant concentration | Distance (km) | Pollutant concentration |
|---|---|---|---|
| 0.1 | 0.7500 | 0.6 | 0.4101 |
| 0.2 | 0.5318 | 0.7 | 0.4083 |
| 0.3 | 0.4109 | 0.8 | 0.3570 |
| 0.4 | 0.3627 | 0.9 | 0.2318 |
| 0.5 | 0.3628 | 1.0 | 0.0080 |

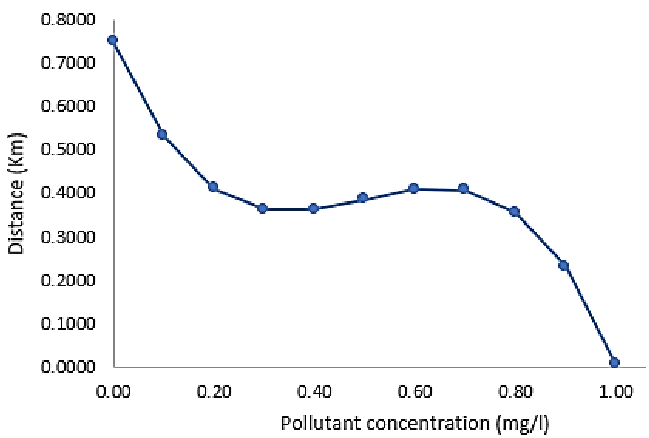


**Figure 1**
**The approximated pollutant concentration of the 3rd-ordered method.**

4.2 *The fourth-ordered method*

The collocation method of Eq.(13) is employed with the fourth- and fifth-order Legendre polynomial approximation space of Eq.(15). These turns out that the approximated solution is obtained by

$$C_4(x) \approx -0.6575 + 3.0109x - 3.4156\left(\frac{3x^2}{2} - \frac{1}{2}\right)$$
$$+1.871\left(\frac{5x^3}{2} - \frac{3x}{2}\right) - 0.8008\left(\frac{35x^4}{8} - \frac{15x^2}{4} + \frac{3}{8}\right), \quad (17)$$
$$C_5(x) \approx 0.7410 + 0.3301x - 0.0492\left(\frac{3x^2}{2} - \frac{1}{2}\right)$$

$$-0.3896\left(\frac{5x^3}{2}-\frac{3x}{2}\right)-0.0415\left(\frac{35x^4}{8}-\frac{15x^2}{4}-\frac{3}{8}\right)$$
$$-0.5828\left(\frac{65x^5}{8}-\frac{35x^3}{4}-\frac{15x}{8}\right) \tag{18}$$

The approximated solution is shown in Tables 3-4 and Fig. 2.

**Table 3**
**The approximated pollutant concentration of the 4th-ordered method.**

| Distance (km) | Pollutant concentration | Distance (km) | Pollutant concentration |
|---|---|---|---|
| 0.1 | 0.7500 | 0.6 | 0.6656 |
| 0.2 | 0.7536 | 0.7 | 0.6173 |
| 0.3 | 0.7379 | 0.8 | 0.5163 |
| 0.4 | 0.7184 | 0.9 | 0.3277 |
| 0.5 | 0.7022 | 1.0 | 0.0080 |

**Table 4**
**The approximated pollutant concentration of the 5th-ordered method.**

| Distance (km) | Pollutant concentration | Distance (km) | Pollutant concentration |
|---|---|---|---|
| 0.1 | 0.7500 | 0.6 | 1.1832 |
| 0.2 | 0.7371 | 0.7 | 1.2654 |
| 0.3 | 0.7489 | 0.8 | 1.1938 |
| 0.4 | 0.8027 | 0.9 | 0.8342 |
| 0.5 | 0.9042 | 1.0 | 0.0080 |

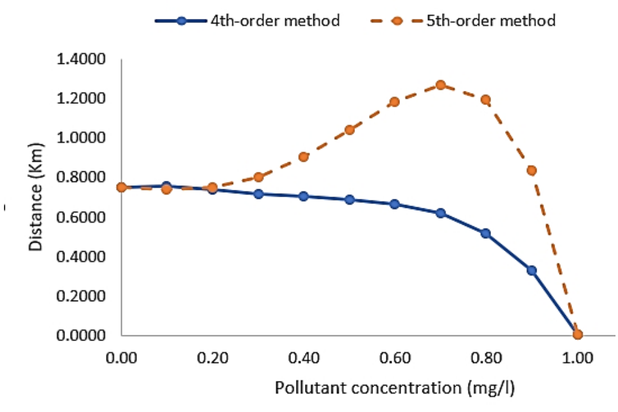


**Figure 2**
**The approximated pollutant concentration of the 4th- and 5th-ordered methods.**

## 5. Conclusion

While it is accurate to assess pollution in the canal using field data collection techniques, it is not always possible to detect contaminants in the water. However, if it could be measured, it would be a waste of time and money. In order to determine the concentration of contaminants in the water, a mathematical method is described. The benefit is that one can compute the concentration of contaminants in the water at any place using mathematics. Both money and time may be saved. The calculation's precision was determined to be satisfactory. The old approach should be improved numerically by include some inner collocated points. The better numerical technique is to incorporate some interior collocated points into the traditional algorithm.

## References

[1] N. Pochai, S. Tangmanee, L. J. Crane, and J. J. H. Miller, "A mathematical model of water pollution control using the finite element method," *Proc. Appl. Math. Mech.* (PAMM), vol. 6, no. 1, pp. 755–756 (2006).

[2] N. Pochai, S. Tangmanee, L. J. Crane, and J. J. H. Miller, "A water quality computation in the uniform channel," *J. Interdiscip. Math.,* vol. 11, no. 6, pp. 803–814 (2008).

[3] N. Pochai, "A numerical computation of non-dimensional form of stream water quality model with hydrodynamic advection-dispersion-reaction equations," *J. Nonlinear Anal.: Hybrid Syst.,* vol. 3, pp. 666–673 (2009).

[4] N. Pochai and R. Depana, "A numerical computation of water quality measurement in a uniform channel using a finite difference method," *Procedia Eng.,* vol. 8, pp. 85–88 (2011).

[5] N. Pochai, "A numerical treatment of non-dimensional form of water quality model in a non-uniform flow stream using Saulyev scheme," *Math. Probl. Eng.,* vol. 2011, Art. ID 491317 (2011).

[6] N. Pochai and R. Depana, "An optimal control of water pollution in a stream using a finite difference method," *World Acad. Sci. Eng. Technol.,* vol. 6, no. 56, pp. 1186–1188 (2011).

[7] N. Pochai, "A numerical computation of non-dimensional form of a nonlinear hydrodynamic model in a uniform reservoir," *J. Nonlinear Anal.: Hybrid Syst.,* vol. 3, pp. 463–466 (2009).

[8] N. Pochai, S. Tangmanee, L. J. Crane, and J. J. H. Miller, "A water quality computation in the uniform reservoir," *J. Interdiscip. Math.,* vol. 12, no. 1, pp. 19–28 (2009).

[9] N. Pochai, S. Tangmanee, L. J. Crane, and J. J. H. Miller, "A finite element simulation of water quality measurement in the open reservoir," *Thai J. Math.,* vol. 7, no. 2, pp. 77–93 (2009).

[10] N. Pochai and J. Sornsri, "A non-dimensional form of hydrodynamic model with variable coefficients in a uniform reservoir using Lax-Wendroff method," *Procedia Eng.,* vol. 8, pp. 89–93 (2011).

[11] F. Mabood and N. Pochai, "Asymptotic solution for a water quality model in a uniform stream," *Int. J. Eng. Math.,* vol. 2013, Art. ID 135140, 4 pp (2013).

[12] N. Pochai, "Numerical treatment of a modified MacCormack scheme in a nondimensional form of the water quality models in a nonuniform flow stream," *J. Appl. Math.,* vol. 2014, Art. ID 274263, 8 pp (2014).

[13] W. Klaychang and N. Pochai, "A numerical treatment of a non-dimensional form of a water quality model in the Rama-nine reservoir," *J. Interdiscip. Math.,* vol. 18, no. 4, pp. 375–394 (2015).

[14] B. Bradie, A Friendly Introduction to Numerical Analysis. Upper Saddle River, NJ, USA: Pearson Prentice Hall (2006).

[15] G. E. Fasshauer, Numerical Methods for Differential Equations Handouts. Chicago, IL, USA (2007).

[16] G. E. Fasshauer, Meshfree Approximation Methods with MATLAB, Interdisciplinary Mathematical Sciences – Vol. 6. Singapore: World Scientific Publishers (2007).

[17] T. D. Rao and S. Chakraverty, "Solving uncertain differential equations using interval Legendre polynomials based collocation method," *J. Interdiscip. Math.,* vol. 22, no. 4, pp. 473–491 (2019).

*Received August, 2024*