Journal of Interdisciplinary Mathematics
ISSN: 0972-0502 (Print), ISSN: 2169-012X (Online)
Vol. 28 (2025), No. 4, pp. 1355–1365
DOI : 10.47974/JIM-1787

Two classes of conformable fractional evolution problems solved by the U-EM method

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Abstract

In this paper, we apply the U-expansion method to finding new traveling wave solutions for two important classes of nonlinear fractional differential equations. The efficiency of this method is illustrated by some examples.

Subject Classification (2010): *26A33, 24A08, 35C07.*

Keywords: *Conformable derivative, U-expansion method, Fractional differential equation, Traveling wave.*

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1. Introduction

Evolution equations of fractional order have various applications in real word phenomena, in fact, there are several applications of such equations in biology, fluid mechanics, fluid flow, seismology, signal processing, and other areas, see [9, 18, 19, 13, 15, 5].

Numerically, the solutions of such fractional equations are obtained by different methods, for example, the ADM method (Adomian decomposition method) [6, 11], the VIM method (variation iteration method) [11, 24, 16], the HPM (homotopy perturbation method) [3, 8, 22], the tanh method [10, 23], the exp-function method and other numerical ones, see for more information the research papers [25, 14, 7, 4, 21].

In this paper, we apply the U-EM method (U-expansion method) to obtain new traveling wave solutions for any problem of type:

$$T_t^{\alpha} y(t,x) = G(y, T_x^{\beta} y, T_x^{2\beta} y, T_x^{3\beta} y), \tag{1}$$

where $T_x^{\beta}, T_t^{\alpha}$ are the conformable fractional derivative, $0<\alpha,\ \beta\le 1$ and $G:\mathbb{R}^4 \to \mathbb{R}$ is a given function.

On the other hand, the study of the Ulam and Hyers and the generalized Ulam and Hyers stabilities for this equation is also of a great importance. Some research papers dealing with Ulam type stabilities can be found in [1, 12].

The paper is organized as follows: In the second section, some preliminaries related to conformable fractional derivative approach are recalled. In the third section, the U-EM method is discussed for solving some important conformable fractional evolution equations. In the fourth section, numerical examples are presented. Finally, some graphs of the obtained solutions are plotted, and a conclusion follows.

2. Conformable Calculus of Fractional Order

In this section, we present some definitions and properties related to conformable fractional derivative, fore more details, see [17, 2]:

Definition 2.1: The conformable fractional derivative of order α of a function $f:(0,\infty)\to\mathbb{R}$ is defined as

$$T^{\alpha}(f)(x) = \lim_{s\to 0}\left(\frac{f(x+sx^{1-\alpha})-f(x)}{s}\right),\ x>0,\ 0<\alpha\le 1. \tag{2}$$

When $\alpha=1,$ we find the standard derivative of order one.

Definition 2.2: The conformable fractional integral of order α of a continuous function $f:(0,\infty)\to\mathbb{R}$ is defined as

$$I^{\alpha}(f)(x)=\int_{0}^{x}\tau^{\alpha-1}f(\tau)d\tau,\quad 0<\alpha\leq 1. \tag{3}$$

In our work, we need to use these two important formulas:

$$I^{\alpha}T^{\alpha}(f)(x)=f(x)-f(0),$$

and

$$T^{\alpha}(f)(x)=x^{1-\alpha}\frac{df(x)}{dx}.$$

3. U-expansion Method Analysis

In this part, we present the U-expansion method that can be applied to give an approximate solution of nonlinear conformable fractional differential equations. So, we begin by considering the following problem:

$$F(y,T_t^{\alpha}(y),T_x^{\beta}(y),T_t^{\alpha}(T_t^{\alpha}(y)),T_t^{\alpha}(T_x^{\beta}(y)),T_x^{\beta}(T_x^{\beta}(y)),...)=0, \tag{4}$$

where $T_t^{\alpha}(y)$ denotes the conformable fractional derivative of order $\alpha,(0<\alpha\leq 1)$ of y.

Firstly, we introduce a function $y(\xi)$ as follows:

$$y(\xi)=y(t,x),$$

with the new variable:

$$\xi=\frac{c}{\alpha}t^{\alpha}+\frac{k}{\beta}x^{\beta},$$

where k and c are constants.

So, the problem (4) will be converted to the following ordinary differential equation:

$$g(y,y',y'',y''',...)=0. \tag{5}$$

In view of U-EM method, the travelling wave solutions can be presented as the form:

$$y(\xi)=\sum_{i=0}^{n}a_iY^i,$$

$$Y_{\xi}=\alpha_1Y+\alpha_2Y^2 \tag{6}$$

where α_1 and α_2 are real constants, n is a positive integer to be determined.

The equation (6) will have solutions in the form

$$Y(\xi)=\frac{\alpha_1}{-\alpha_2+\alpha_1 A\exp(-\xi\alpha_1)}.$$

To determine the positive integer n, we balance the highest-order derivatives and the nonlinear terms in equation Eq.(5). Then we solve algebraic system using Mapple. Consequently, this allows us to obtain solutions of the above introduced evolution problem.

4. Applications

We consider the following problem:

$$T_t^{\alpha}(y(t,x))=f(y,T_x^{\beta}(y),T_x^{2\beta}(y),T_x^{3\beta}(y)), \tag{7}$$

where $0<\alpha,\beta\leq 1$ and $f:\mathbb{R}^4\rightarrow\mathbb{R}$ is a given function.

We need the following transformations:

$$y(t,x)=y(\xi), \tag{8}$$

$$\xi=\frac{c}{\alpha}t^{\alpha}+\frac{k}{\beta}x^{\beta}, \tag{9}$$

where k and c are constants.

Substituting (9) into (7), can reduce (7) into the following ODE

$$cy_{\xi}=f(y,ky_{\xi},k^2y_{\xi\xi},k^3y_{\xi\xi\xi})$$

By the U-EM method, the solutions of *Eq.*(7) can be expressed in the following form:

$$y(\xi)=a_0+\sum_{i=1}^{n}a_iY^i, \tag{10}$$

where a_i are constants which are unknown to be further determined.

4.1 *Example 1*

We begin this section by considering the following problem:

$$T_t^{\alpha}y(t,x)=6y^2T_x^{\beta}y-T_x^{3\beta}y. \tag{11}$$

Note that Eq.(11) (when $\alpha=\beta=1$ and $f(u,v,w,r)=6u^2v-r)$ can be transformed into the KdV equation [20].

$$y_t - 6y^2 y_x + y_{xxx} = 0,$$

Substituting (9) into (11), reduces (11) into the following equation

$$cy_\xi = 6ky^2 y_\xi - k^3 y_{\xi\xi\xi}. \tag{12}$$

Then, integrating (11) yields the following expression

$$cy - 2ky^3 + k^3 y_{\xi\xi} = 0. \tag{13}$$

Balancing the y^3 and $y_{\xi\xi}$ by using homogenous principle, we have, $n = 1$. Then, Eq.(10) can be reduced to

$$y(\xi) = a_0 + a_1 Y. \tag{14}$$

The system of algebraic equations obtained from equations (14), (13) and (6) is given as follows:

$$\begin{cases} a_0(c - 2ka_0^2) = 0 \\ a_1(c + k^3\alpha_1^2 - 6ka_0^2) = 0 \\ ka_1(k^2\alpha_1\alpha_2 - 2a_0a_1) = 0 \\ ka_1(k^2\alpha_2^2 - a_1^2) = 0. \end{cases}$$

By Maple, we find the traveling wave solutions of Eq(13) as follows:

Family 1

$$a_0 = \frac{1}{2}k\alpha_1, a_1 = k\alpha_2, c = \frac{1}{2}k^3\alpha_1^2$$

$$Y(\xi) = \frac{1}{2}k\alpha_1 + \frac{k\alpha_1\alpha_2}{-\alpha_2 + \alpha_1 A \exp\left(-k\alpha_1\left(\frac{k^2\alpha_1^2}{2\alpha}t^\alpha + \frac{1}{\beta}x^\beta\right)\right)} \tag{15}$$

Family 2

$$a_0 = -\frac{1}{2}k\alpha_1, a_1 = -k\alpha_2, c = \frac{1}{2}k^3\alpha_1^2$$

$$Y(\xi) = -\frac{1}{2}k\alpha_1 - \frac{k\alpha_1\alpha_2}{-\alpha_2 + \alpha_1 A \exp\left(-k\alpha_1\left(\frac{k^2\alpha_1^2}{2\alpha}t^\alpha + \frac{1}{\beta}x^\beta\right)\right)} \tag{16}$$

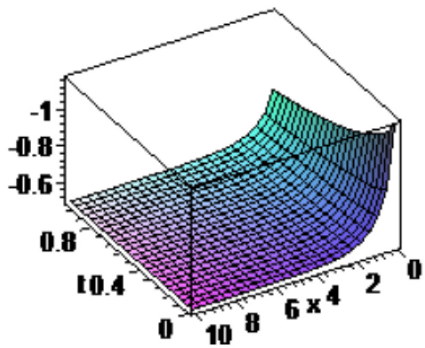

Figure 1

Figure 1 presents the graph of solution (15) for Eq.(11) with $\alpha_1 = 1$, $\alpha_2 = \frac{1}{2}$, $k = 1$, $A = \frac{1}{5}$, $c = 3$, $\alpha = \frac{3}{4}$ **and** $\beta = \frac{9}{10}$.

4.2 *Example 2*

We consider the following problem:

$$T_t^{\alpha} y(t,x) = 2y T_x^{\beta} y + T_x^{2\beta} y. \tag{17}$$

Substituting (9) into (17) can reduce (17) into:

$$c y_{\xi} - 2k y y_{\xi} - k^2 y_{\xi\xi} = 0,$$

Hence, we have

$$c y - k y^2 - k^2 y_{\xi} = 0, \tag{18}$$

For simplicity, we set $n = 1$, so (10) is reduces to the following equation

$$y(\xi) = a_0 + a_1 Y, \tag{19}$$

substituting (19) into (18) yields a set of algebraic equations for a_0, a_1, α_1, α_2, k and c.

The resulting algebraic system can be written as follows:

$$\begin{cases} a_0 (c - 2k a_0) = 0 \\ c a_1 - 2k a_0 a_1 - k^2 \alpha_1 = 0 \\ k^2 \alpha_2 - k a_1^2 = 0 \end{cases}$$

Hence Maple allows us to obtain the wave solutions of (17):

Family 1

$$a_0 = 0, a_1 = a_1, c = \frac{a_1^3 \alpha_1}{\alpha_2^2}, k = -\frac{a_1^2}{\alpha_2}$$

$$Y(\xi) = \frac{a_1 \alpha_1}{-\alpha_2 + \alpha_1 A \exp\left(-\alpha_1\left(\frac{a_1^3 \alpha_1}{\alpha \alpha_2^2} t^\alpha - \frac{a_1^2}{\beta \alpha_2} x^\beta\right)\right)} \tag{20}$$

Family 2

$$a_0 = a_0, a_1 = \frac{\alpha_2 a_0}{\alpha_1}, c = -\frac{a_0^3 \alpha_2}{\alpha_1^2}, k = -\frac{a_0^2 \alpha_2}{\alpha_1^2}$$

$$Y(\xi) = a_0 + \frac{\alpha_2 a_0}{-\alpha_2 + \alpha_1 A \exp\left(-\alpha_1\left(-\frac{a_0^3 \alpha_2}{\alpha \alpha_1^2} t^\alpha - \frac{a_0^2 \alpha_2}{\beta \alpha_1^2} x^\beta\right)\right)} \tag{21}$$

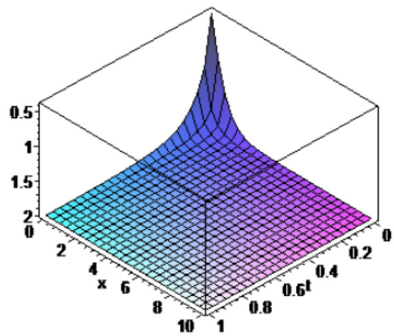

Figure 2

Figure 2 presents the graph of solution (20) for Eq.(17) with $\alpha_1 = 1$, $\alpha_2 = -\frac{1}{2}$, $k = 1$, $A = 2$, $\alpha = \frac{1}{2}$ and $\beta = \frac{3}{4}$.

4.3 *Example 3*

We consider the following problem:

$$T_t^\alpha y(t,x) = -T_x^\beta y + a y^2 T_x^\beta y - T_x^{3\beta} y, \tag{22}$$

where a is a nonzero positive constant.

Note that Eq. (22) (when $\alpha = \beta = 1$ and $f(u,v,w,r) = -u + au^2 v - r$) is transformed into the nonlinear Benjamin-Bona Mahony equation, see [26].

Substituting (9) into (22), reduces (22) to the ODE

$$-cy_{\xi}+ky_{\xi}-aky^2y_{\xi}+k^3y_{\xi\xi\xi}=0. \tag{23}$$

By integraton of (23) yields the equation

$$(c-k)y+\frac{ak}{3}y^3-k^3y_{\xi\xi}=0, \tag{24}$$

Setting $n=1$, so (10) reduces to

$$y(\xi)=a_0+a_1Y, \tag{25}$$

substituting (25) into (24) yields a set of algebraic equations for a_0, a_1, α_1, α_2, k and c.

The obtained algebraic system can be written by the following four expressions

$$\begin{cases} a_0(3c-3k+aka_0^2)=0 \\ a_1(c-k+aka_0^2-k^3\alpha_1^2)=0 \\ ka_1(aa_0a_1-3k^2\alpha_1\alpha_2)=0 \\ ka_1(aa_1^2-6k^2\alpha_2^2)=0 \end{cases}$$

Hence Maple allows us to obtain the wave solutions of (22):

Family 1

$$a_0=\frac{a_1\alpha_1}{2\alpha_2}, a_1=a_1, c=-\frac{a_1\sqrt{6a}}{72\alpha_2^3}(aa_1^2\alpha_1^2-12\alpha_2^2), k=\frac{a_1\sqrt{6a}}{6\alpha_2}$$

$$Y(\xi)=\frac{a_1\alpha_1}{2\alpha_2}+\frac{a_1a_1}{-\alpha_2+\alpha_1A\exp\left(\alpha_1a_1\sqrt{6a}\left(\frac{aa_1^2\alpha_1^2-12\alpha_2^2}{72\alpha\alpha_2^3}t^{\alpha}-\frac{1}{\beta6\alpha_2}x^{\beta}\right)\right)} \tag{26}$$

Family 2

$$a_0=\frac{a_1\alpha_1}{2\alpha_2}, a_1=a_1, c=\frac{a_1\sqrt{6a}}{72\alpha_2^3}(aa_1^2\alpha_1^2-12\alpha_2^2), k=-\frac{a_1\sqrt{6a}}{6\alpha_2}$$

$$Y(\xi)=\frac{a_1\alpha_1}{2\alpha_2}+\frac{a_1a_1}{-\alpha_2+\alpha_1A\exp\left(\alpha_1a_1\sqrt{6a}\left(-\frac{aa_1^2\alpha_1^2-12\alpha_2^2}{72\alpha\alpha_2^3}t^{\alpha}+\frac{1}{\beta6\alpha_2}x^{\beta}\right)\right)}. \tag{27}$$

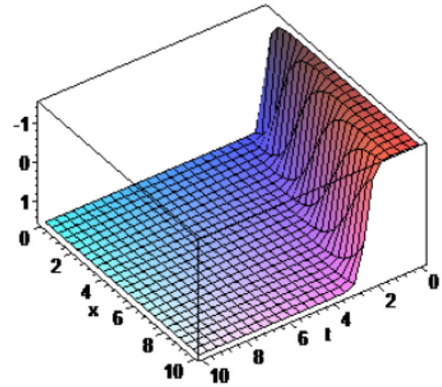

Figure 3

Figure 3 presents the graph of solution (26) for Eq.(22) with $\alpha_1 = 3,\ \alpha_2 = -1,\ k = 1,\ a = 5,\ a = \frac{3}{4}$ **and** $\beta = \frac{7}{10}$.

5. Conclusion

In this study, we have found new travelling wave solutions of two important classes of conformable fractional nonlinear evolution equations by the U-expansion method. The two fractional order nonlinear problems of KdV and Benjamin-Bona Mahony type are solved. The results and the graphs of their solutions show that the U-EM method is effective for solving nonlinear fractional differential equations.

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Received July, 2023