

The convex and strictly convex on graphs

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Abstract

The main goal of this paper introduces convex and strictly convex on graphs when $1 \leq p < \infty$. The convexity was discovered in ancient civilizations and developed in the over time by many scientists and researchers. Here we study the convex with discrete Sobolev seminorm of f and strictly convex on graphs with weighted Sobolev seminorm when $1 \leq p < \infty$.

Subject Classification (2020): 26Bxx, 26B25.

Keywords: Co-regular captive domination number, Co-regular captive dominating set.

1. Introduction and Preliminaries

Convexity is important notion in geometry and it evolved by the time to advanced application in many areas of mathematics. We often found convexity hidden in other areas of mathematics: complex analysis, partial differential equations, functional analysis, graph theory, algebraic geometry, coding theory, probability theory, discrete mathematics, crystallography, calculus of variations many other fields. Convexity consider an important role in areas outside mathematics, such as biology, physics, chemistry and other sciences, but these applications beyond the

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scope our research. Many authors study convexity such as in (sections 7 and 8 in to the book by Hörmander [1]; section 9 in to the book by Abdulaali and Shelash [2], Oda [3] or paper [4]. In the paper of the strong group of Balkan geometers found a more modern, approach to convexity problems, e.g., Dwilewicz [5], Udriște and Oprea [6], Bogaru, Udriște, and Stamin [7], recent papers by Bogaru, Udriște, and Stamin [7] [8], the well-known book by Udriște and Balan [9] and Udriște [10]. We study in [11] and [12, 14, 15-20] the Sobolev spaces on graphs when $0 < p < 1$. Let G be a connected graph and suppose $\mathbb{G}$ a subgraph of a graph G , by V, E we denote to the set of vertices and to the set of edges. In this paper we study the convex on graphs with discrete Sobolev seminorm to E and p

$$\|f\| = \|f\|_{E,p} = (\sum_{(u,v) \in E} |f(u) - f(v)|^p)^{\frac{1}{p}}, 1 \leq p < \infty. [13]$$

And take the strictly convex on graphs with weighted Sobolev seminorm when there is an equation $\mu: E \rightarrow (0, \infty)$

$$\|f\| = \|f\|_{E,p,\mu} = (\sum_{(u,v) \in E} (|f(u) - f(v)|^p \mu(u, v))^{\frac{1}{p}}, 1 \leq p < \infty. [13]$$

2. Main Results

Definition 2.1: Let $\mathbb{G}$ be a subgraph of a graph G is said to be convex if $\mathbb{G}$

Contains edge connect any pair of two vertices. i.e. $\mathbb{G}$ is convex if

$$u, w \in \mathbb{G}, 0 \leq \lambda \leq 1 \Rightarrow \lambda u + (1 - \lambda)w \in \mathbb{G}.$$

Proposition 2.2: Let $\mathbb{G}$ a connected subgraph of a graph G , and let for all $v \in G$ has a unique nearest vertex $u_v \in \mathbb{G}$. So, the nearest vertex map $v \rightarrow u_v$ is continuous.

Theorem 2.3: Let $\mathbb{G}$ be a subgraph of graph G , and let $u \in G$. The set consist of all best approximations to u out of $\mathbb{G}$ denote by $\mathbb{G}_u$, is bounded, convex set.

Proof: From the previous proposition, the set $\mathbb{G}_u$ is a subset of $\{v \in G: \|v\| \leq 2\|u\|\}$ and, so, is bounded.

From the definition of convex we must suppose $v_1, v_2 \in \mathbb{G}_u$ means that

$$\|u - v_1\| = \|u - v_2\| = \min_{v \in \mathbb{G}} \|u - v\|$$

Take $0 \leq \theta \leq 1$, set $v^* = \theta v_1 + (1 - \theta)v_2$

We want to prove $v^* \in \mathbb{G}_u$, but notice that at least have $v^* \in \mathbb{G}$. In the final, we have an estimation

$$\begin{aligned}
 \|u - v^*\| &= (\sum_{u, v^* \in \mathbb{G}} |u - v^*|^p)^{\frac{1}{p}} \\
 &= (\sum_{u, v_1, v_2 \in \mathbb{G}} |u - (\theta v_1 + (1 - \theta)v_2)|^p)^{\frac{1}{p}} \\
 &= (\theta \sum_{u, v_1 \in \mathbb{G}} |u - v_1|^p + (1 - \theta) \sum_{u, v_2 \in \mathbb{G}} |u - v_2|^p)^{\frac{1}{p}} \\
 &\leq \theta (\sum_{u, v_1 \in \mathbb{G}} |u - v_1|^p)^{\frac{1}{p}} + (1 - \theta) (\sum_{u, v_2 \in \mathbb{G}} |u - v_2|^p)^{\frac{1}{p}} \\
 &\leq \theta \|u - v_1\| + (1 - \theta) \|u - v_2\| \\
 &= \min_{v \in \mathbb{G}} \|u - v\|
 \end{aligned}$$

So, $\|u - v^*\| = \min_{v \in \mathcal{G}} \|u - v\|$, that is $v^* \in \mathcal{G}$.

Definition 2.4: We have a norm on a graph G where $1 \leq p \leq \infty$, is called a Strictly convex if, for any $u \neq v \in G$ with $\|u\| = r = \|v\|$, where $\|f\| = \|f\|_{E,p,\mu} = (\sum_{(u,v) \in E} |f(u) - f(v)|^p \mu(u, v))^{\frac{1}{p}}, 1 \leq p < \infty$, we always have

$$\|\theta u + (1 - \theta)v\| < r \text{ for any } 0 < \theta < 1.$$

Proposition 2.5: Let G has a strictly convex norm, so, for any subgraph $\mathbb{G}$ of G and any vertex $u \in G$, there can be at most one best approximation to u out of $\mathbb{G}$. Then, $\mathbb{G}_u$ is either null graph or consist a single vertex.

Theorem 2.6: G has a strictly convex norm $\leftrightarrow$ the triangle inequality is strict on different vertices $\leftrightarrow u \neq \lambda v, v \neq \lambda u, \forall \lambda \in \mathbb{R} \Rightarrow \|u + v\| < \|u\| + \|v\|$ where

$$\|f\| = \|f\|_{E,p,\mu} = (\sum_{(u,v) \in E} |f(u) - f(v)|^p \mu(u, v))^{\frac{1}{p}}, 1 \leq p < \infty.$$

Proof: In the beginning we suppose that G is strictly convex, and let u and v be different vertices in G , So, particularly, the vertices $u/\|u\|$ and $v/\|v\|$ must be different

Thus,

$$\left\| \left(\frac{\|u\|}{\|u\| + \|v\|} \right) \frac{u}{\|u\|} + \left(\frac{\|v\|}{\|u\| + \|v\|} \right) \frac{v}{\|v\|} \right\| < 1 \text{ where } 1 \leq p < \infty,$$

Then, $\|u + v\| < \|u\| + \|v\|$.

Conclusion

Best approximation on convex set in graphs with discrete Sobolev seminorm of f , best approximation on strictly convex set in graphs with weighted Sobolev seminorm, when $1 \leq p < \infty$.

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Received November, 2024