

Solving n^{th} order non-homogenous complex partial differential equation using two dimensional differential transform method

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Abstract

This paper utilizes two-dimensional differential transformation for some complex two-variables functions. A recurrence formula related to a complex partial differential equation of order n is presented. In addition, 2-dimensional differential transformation method is used to introduce analytical solutions to complex partial differential equations by the obtained recurrence relation.

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1. Introduction

Many important real-world applications can be formulated as partial differential equations (PDEs) [1, 2, 3]. In the year 1999 a differential

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transformation was delineated as an approach to solve initial and boundary PDEs [4]. Furthermore, many studies have focused on differential transformation to find solutions to PDEs [5, 6, 7, 8, 9]. Düz and İlter, applied this transformation to solve complex partial differential equations (CPDEs) analytically and approximately [10]. In 2021, Düz introduced a formulation for the complex partial derivatives of the complex functions and used this formula to solve the CPDE of order n using the El-Zaki transformation method [11]. In addition, many researchers investigated this field using a variety of methods, including the two-dimensional differential transformation method [12, 13], SEE transformation approach [14], and the Aboodh transformation approach [15]. In this paper, the two-dimensional differential transformations for many complex functions of two variables are given, and then used to solve the complex partial differential equation of n^{th} order with the 2-dimensional differential transformation approach.

This work has been summarized into five Sections as: fundamental concepts of this paper are outlined, a table of two-dimensional differential transformations for elementary complex functions is given, a two-dimensional differential transformation is utilized to discover a recurrence relation related to the n^{th} order CPDE with two examples and the conclusions.

2. Fundamental concepts

This section consists the essential principles that are required in this work.

Definition 2.1: [4] Let $f(x, y)$ be an analytic function and continuously differentiable on the nonnegative integer. Then,

$$F(\zeta, \varpi) = \frac{1}{\zeta! \varpi!} \left. \frac{\partial^{\zeta+\varpi} f(x, y)}{\partial x^{\zeta} \partial y^{\varpi}} \right|_{x=y=0}, \quad (2.1)$$

is called the two dimensional differential transformation of $f(x, y)$. The inverse of transformed function given in (2.1) is:

$$f(x, y) = \sum_{\zeta=0}^{\infty} \sum_{\varpi=0}^{\infty} F(\zeta, \varpi) x^{\zeta} y^{\varpi}. \quad (2.2)$$

In the next theorem, some basic properties related with the two dimensional differential transformation for real functions are presented.

Theorem 2.1: [4] Let $v = v(t_1, t_2)$ be a real valued function, λ_1 and λ_2 are constants and $V(\vartheta, \rho)$ the 2-dimensional differential transformation of $v(t_1, t_2)$.

1. If $v = \lambda_1 g_1(t_1, t_2) \pm \lambda_2 g_2(t_1, t_2)$ then $V(\vartheta, \rho) = \lambda_1 G_1(\vartheta, \rho) \pm \lambda_2 G_2(\vartheta, \rho)$.
2. If $v = \frac{\partial g(t_1, t_2)}{\partial t_1}$ then $V(\vartheta, \rho) = (\vartheta + 1)G(\vartheta + 1, \rho)$.
3. If $v = \frac{\partial g(t_1, t_2)}{\partial t_2}$ then $V(\vartheta, \rho) = (\rho + 1)G(\vartheta, \rho + 1)$.
4. If $v = \frac{\partial^{k+l} g(t_1, t_2)}{\partial t_1^k \partial t_2^l}$ then $V(\vartheta, \rho) = (\vartheta + 1) \cdots (\vartheta + k)(\rho + 1) \cdots (\rho + l) G(\vartheta + k, \rho + l)$.
5. If $v = t_1^m t_2^n$ then $V(\vartheta, \rho) = \delta(\vartheta - m, \rho - n) = \begin{cases} 1, & \vartheta = m, \rho = n \\ 0, & \text{o.w.} \end{cases}$

Theorem 2.2: [11] If $w = w(z, \bar{z})$, then

$$\frac{\partial^{n+r} w}{\partial z^n \partial \bar{z}^r} = \frac{1}{2^{n+r}} \sum_{j=0}^n \sum_{l=0}^r (-i)^j i^l \binom{n}{j} \binom{r}{l} \frac{\partial^{n+r} w}{\partial x^{n+r-(j+l)} \partial y^{j+l}}$$

3. 2-dimensional differential transformation for important complex valued functions of 2 independent variables z and $\bar{z}$

This section starts with derivations of 2-dimensional differential transformations of real functions, followed by introducing a table of 2-dimensional differential transformations for complex elementary functions which is essential for solving CPDEs.

Theorem 3.1: Let $h = h(t_1, t_2)$ be a real valued function, λ, μ, σ and τ are constants.

1. If $h = e^{\lambda t_1 + \mu t_2}$, then $H(\vartheta, \rho) = \frac{\lambda^\vartheta \mu^\rho}{\vartheta! \rho!}$.
2. If $h = \sin(\lambda t_1 + \mu t_2)$, then $H(\vartheta, \rho) = \frac{\lambda^\vartheta \mu^\rho}{\vartheta! \rho!} \sin \frac{(\vartheta + \rho)\pi}{2}$.
3. If $h = \cos(\lambda t_1 + \mu t_2)$, then $H(\vartheta, \rho) = \frac{\lambda^\vartheta \mu^\rho}{\vartheta! \rho!} \cos \frac{(\vartheta + \rho)\pi}{2}$.

4. If $h = \sinh(\lambda t_1 + \mu t_2)$, then $H(\vartheta, \rho) = \begin{cases} \frac{\lambda^\vartheta \mu^\rho}{\vartheta! \rho!}, & \text{if } \vartheta + \rho \text{ is odd} \\ 0, & \text{if } \vartheta + \rho \text{ is even.} \end{cases}$
5. If $h = \cosh(\lambda t_1 + \mu t_2)$, then $H(\vartheta, \rho) = \begin{cases} \frac{\lambda^\vartheta \mu^\rho}{\vartheta! \rho!}, & \text{if } \vartheta + \rho \text{ is even} \\ 0, & \text{if } \vartheta + \rho \text{ is odd.} \end{cases}$
6. If $h = \ln(\lambda t_1 + \mu t_2 + \sigma)$, then $H(\vartheta, \rho) = \frac{(-1)^{\vartheta+\rho-1} \lambda^\vartheta \mu^\rho \sigma^{-\vartheta-\rho} (\vartheta+\rho-1)!}{\vartheta! \rho!}$.
7. If $h = e^{\lambda t_1 + \mu t_2} \sin(\sigma t_1 + \tau t_2)$, then

$$H(\vartheta, \rho) = \sum_{j_1=0}^{\vartheta} \sum_{j_2=0}^{\rho} \frac{\lambda^{\vartheta-j_1} \mu^{\rho-j_2} \sigma^{j_1} \tau^{j_2}}{j_1! j_2! (\vartheta-j_1)! (\rho-j_2)!} \sin \frac{(j_1+j_2)\pi}{2}.$$
8. If $h = e^{\lambda t_1 + \mu t_2} \cos(\sigma t_1 + \tau t_2)$, then

$$H(\vartheta, \rho) = \sum_{j_1=0}^{\vartheta} \sum_{j_2=0}^{\rho} \frac{\lambda^{\vartheta-j_1} \mu^{\rho-j_2} \sigma^{j_1} \tau^{j_2}}{j_1! j_2! (\vartheta-j_1)! (\rho-j_2)!} \cos \frac{(j_1+j_2)\pi}{2}.$$
9. If $h = e^{\lambda t_1 + \mu t_2} \sinh(\sigma t_1 + \tau t_2)$, then

$$H(\vartheta, \rho) = \frac{1}{2\vartheta! \rho!} [(\lambda + \sigma)^\vartheta (\mu + \tau)^\rho - (\lambda - \sigma)^\vartheta (\mu - \tau)^\rho].$$
10. If $h = e^{\lambda t_1 + \mu t_2} \cosh(\sigma t_1 + \tau t_2)$, then

$$H(\vartheta, \rho) = \frac{1}{2\vartheta! \rho!} [(\lambda + \sigma)^\vartheta (\mu + \tau)^\rho + (\lambda - \sigma)^\vartheta (\mu - \tau)^\rho].$$
11. If $h = (\lambda t_1 + \mu t_2)^n$, then $H(\vartheta, \rho) = \frac{\lambda^\vartheta \mu^\rho (\vartheta+\rho)!}{\vartheta! \rho!} \delta(\vartheta + \rho - n)$.
12. If $h = (\lambda t_1 + \mu t_2)^n e^{\sigma t_1 + \tau t_2}$, then

$$H(\vartheta, \rho) = n! \sum_{j_1=0}^{\vartheta} \sum_{j_2=0}^{\rho} \frac{\lambda^{j_1} \mu^{j_2} \sigma^{\vartheta-j_1} \tau^{\rho-j_2}}{j_1! j_2! (\vartheta-j_1)! (\rho-j_2)!} \delta(n - j_1 - j_2).$$
13. If $h = (\lambda t_1 + \mu t_2)^n \sin(\sigma t_1 + \tau t_2)$, then

$$H(\vartheta, \rho) = n! \sin \frac{(\vartheta+\rho-n)\pi}{2} \sum_{k_1=0}^{\vartheta} \sum_{k_2=0}^{\rho} \frac{\lambda^{k_1} \mu^{k_2} \sigma^{\vartheta-k_1} \tau^{\rho-k_2}}{k_1! k_2! (\vartheta-k_1)! (\rho-k_2)!} \delta(n - k_1 - k_2).$$

14. If $h = (\lambda t_1 + \mu t_2)^n \cos(\sigma t_1 + \tau t_2)$, then

$$H(\vartheta, \rho) = n! \cos \frac{(\vartheta + \rho - n)\pi}{2} \sum_{k_1=0}^{\vartheta} \sum_{k_2=0}^{\rho} \frac{\lambda^{k_1} \mu^{k_2} \sigma^{\vartheta-k_1} \tau^{\rho-k_2}}{k_1! k_2! (\vartheta - k_1)! (\rho - k_2)!} \delta(n - k_1 - k_2).$$

15. If $h = (\lambda t_1 + \mu t_2)^n \sinh(\sigma t_1 + \tau t_2)$, then

$$H(\vartheta, \rho) = \begin{cases} n! \sum_{j_1=0}^{\vartheta} \sum_{j_2=0}^{\rho} \frac{\lambda^{j_1} \mu^{j_2} \sigma^{\vartheta-j_1} \tau^{\rho-j_2} \delta(n-j_1-j_2)}{j_1! j_2! (\vartheta - j_1)! (\rho - j_2)!}, & \vartheta + \rho - n \text{ is odd} \\ 0, & \vartheta + \rho - n \text{ is even.} \end{cases}$$

16. If $h = (\lambda t_1 + \mu t_2)^n \cosh(\sigma t_1 + \tau t_2)$, then

$$H(\vartheta, \rho) = \begin{cases} n! \sum_{j_1=0}^{\vartheta} \sum_{j_2=0}^{\rho} \frac{\lambda^{j_1} \mu^{j_2} \sigma^{\vartheta-j_1} \tau^{\rho-j_2} \delta(n-j_1-j_2)}{j_1! j_2! (\vartheta - j_1)! (\rho - j_2)!}, & \vartheta + \rho - n \text{ is even} \\ 0, & \vartheta + \rho - n \text{ is odd.} \end{cases}$$

$$\text{where } \delta(\vartheta + \rho - n) = \begin{cases} 1, & \vartheta + \rho = n \\ 0, & \text{o.w.} \end{cases} \text{ and } \delta(n - j_1 - j_2) = \begin{cases} 1, & j_1 + j_2 = n \\ 0, & \text{o.w.} \end{cases}$$

Proof: Some selected functions in Theorem 3.1 will be proved, and the other proofs are similar. To prove 2, Definition 2.1 yields the following after achieving the differentiations:

$$\begin{aligned} H(\vartheta, \rho) &= \frac{\lambda^{\vartheta} \mu^{\rho}}{\vartheta! \rho!} \sin\left(\frac{\rho\pi}{2} + \frac{\vartheta\pi}{2} + \lambda t_1 + \mu t_2\right) \Big|_{(0,0)} \\ &= \frac{\lambda^{\vartheta} \mu^{\rho}}{\vartheta! \rho!} \sin \frac{(\vartheta + \rho)\pi}{2}. \end{aligned}$$

For 11, Definition 2.1 provides us the following:

$$H(\vartheta, \rho) = n(n-1) \cdots (n - \vartheta - \rho + 1) \frac{\lambda^{\vartheta} \mu^{\rho}}{\vartheta! \rho!} (\lambda t_1 + \mu t_2)^{n - \vartheta - \rho} \Big|_{(0,0)}. \quad (3.1)$$

Equation (3.1) implies that:

$$H(\vartheta, \rho) = \frac{\lambda^{\vartheta} \mu^{\rho} (\vartheta + \rho)!}{\vartheta! \rho!} \text{ if } n = \vartheta + \rho, \text{ and } H(\vartheta, \rho) = 0 \text{ if } n \neq \vartheta + \rho.$$

Then, 11 of Theorem 3.1 is obtained.

For 12, Binomial theorem and Leibniz rule yield:

$$\begin{aligned}
H(\vartheta, \rho) &= \frac{1}{\vartheta! \rho!} \frac{\partial^{\vartheta+\rho}}{\partial t_1^\vartheta \partial t_2^\rho} \sum_{k=0}^n \binom{n}{k} (\lambda t_1)^k (\mu t_2)^{n-k} e^{\sigma t_1 + \tau t_2} \Big|_{(t_1, t_2) = (0,0)} \\
&= \frac{1}{\vartheta! \rho!} \sum_{k=0}^n \binom{n}{k} \lambda^k \mu^{n-k} [D_{t_1}^\vartheta t_1^k e^{\sigma t_1}]_{t_1=0} [D_{t_2}^\rho t_2^{n-k} e^{\tau t_2}]_{t_2=0}, \quad D_{t_1} = \frac{d}{dt_1}, D_{t_2} = \frac{d}{dt_2} \\
&= \frac{1}{\vartheta! \rho!} \sum_{k=0}^n \binom{n}{k} \lambda^k \mu^{n-k} \left[\sum_{j_1=0}^\vartheta \binom{\vartheta}{j_1} D_{t_1}^{j_1} t_1^k D_{t_1}^{\vartheta-j_1} e^{\sigma t_1} \right]_{t_1=0} \left[\sum_{j_2=0}^\rho \binom{\rho}{j_2} D_{t_2}^{j_2} t_2^{n-k} D_{t_2}^{\rho-j_2} e^{\tau t_2} \right]_{t_2=0}.
\end{aligned}$$

If $k = j_1$ and $n - k = j_2$, then

$$H(\vartheta, \rho) = n! \sum_{k=0}^n \sum_{j_1=0}^\vartheta \sum_{j_2=0}^\rho \frac{\lambda^{j_1} \mu^{j_2} \sigma^{\vartheta-j_1} \tau^{\rho-j_2}}{j_1! j_2! (\vartheta-j_1)! (\rho-j_2)!},$$

and otherwise $F(\vartheta, \rho) = 0$.

Hence, 12 of Theorem 3.1 is achieved.

Table 1
2-dimensional differential transformation for some complex functions

No.	Function	$D_2\{\text{Function}\}$
1	$e^{\lambda z}$	$\frac{i^\rho \lambda^{\vartheta+\rho}}{\vartheta! \rho!}$
2	$e^{\lambda \bar{z}}$	$\frac{(-i)^\rho \lambda^{\vartheta+\rho}}{\vartheta! \rho!}$
3	$\cosh \lambda z$	$\begin{cases} \frac{i^\rho \lambda^{\vartheta+\rho}}{\vartheta! \rho!}, & \text{if } \vartheta+\rho \text{ is even} \\ 0, & \text{if } \vartheta+\rho \text{ is odd} \end{cases}$
4	$\ln(\lambda \bar{z} + \sigma)$	$\frac{(-1)^{\vartheta-1} i^\rho \lambda^{\vartheta+\rho} \sigma^{-\vartheta-\rho} (\vartheta+\rho-1)!}{\vartheta! \rho!}$
5	$e^{\lambda z} \sin \sigma z$	$i^\rho \sum_{j_1=0}^\vartheta \sum_{j_2=0}^\rho \frac{\lambda^{\vartheta+\rho-j_1-j_2} \sigma^{j_1+j_2}}{j_1! j_2! (\vartheta-j_1)! (\rho-j_2)!} \sin \frac{(j_1+j_2)\pi}{2}$
6	$e^{\lambda z} \cos \sigma \bar{z}$	$i^\rho \sum_{j_1=0}^\vartheta \sum_{j_2=0}^\rho \frac{(-1)^{j_2} \lambda^{\vartheta+\rho-j_1-j_2} \sigma^{j_1+j_2}}{j_1! j_2! (\vartheta-j_1)! (\rho-j_2)!} \cos \frac{(j_1+j_2)\pi}{2}$
7	$e^{\lambda \bar{z}} \cosh \sigma \bar{z}$	$\frac{(-i)^\rho}{2\vartheta! \rho!} [(\lambda + \sigma)^{\vartheta+\rho} + (\lambda - \sigma)^{\vartheta+\rho}]$
8	z^n	$\frac{i^\rho (\vartheta+\rho)!}{\vartheta! \rho!} \delta(\vartheta + \rho - n)$

Based on the previous derivations for selected functions, the above table 1 introduced a large class of complex valued functions that are required for many applications and in solving CPDEs via the two dimensional differential transformation method. The above table 1 introduced is presented based on specific constant values λ, μ, σ and τ in Theorem 3.1. For example $\mu = \lambda i$ in 1, and $\mu = \lambda i, \tau = -\sigma i$ in 8 of Theorem 3.1 are chosen to correspond to Nos. 1 and 6 respectively, in Table 1 when $x = t_1$ and $y = t_2$.

4. Solution of the nonhomogeneous n^{th} order CPDEs with constant coefficients using two- dimensional differential transformation method

This section exploits a 2-dimensional differential transformation for solving the nonhomogeneous complex PDE where the coefficients are constants.

Theorem 4.1: Consider the following complex PDE of order n :

$$\sum_{m=0}^n \sum_{r=0}^{n-m} C_{n-m-r,r} \frac{\partial^{n-m} \varphi(z, \bar{z})}{\partial z^{n-m-r} \partial \bar{z}^r} = f(z, \bar{z}). \quad (4.1)$$

with initial conditions:

$$\varphi_{y^s}(x, 0) = d_s(x), \quad s = 0, 1, \dots, n-1, \quad (4.2)$$

and at least one of the constants $C_{n,0}, C_{n-1,1}, \dots, C_{0,n}$ is not identical to zero. The CIVP (4.1)-(4.2) has a solution of the formula:

$$\varphi(z, \bar{z}) = \sum_{\rho=0}^{n-1} \sum_{\vartheta=0}^{\infty} \Phi(\vartheta, \rho) x^{\vartheta} y^{\rho} + \sum_{\rho=0}^{\infty} \sum_{\vartheta=0}^{\infty} \Phi(\vartheta, \rho+n) x^{\vartheta} y^{\rho+n}, \quad (4.3)$$

such that:

$$\begin{aligned} \Phi(\vartheta, \rho+n) = & \frac{(-i)^n}{(\rho+1) \cdots (\rho+n) \sum_{r=0}^n (-1)^{n-r} C_{n-r,r}} \left[2^n F(\vartheta, \rho) - C_{n,0} \sum_{j=0}^{n-1} (-1)^j i^j \binom{n}{j} (\vartheta+1) \right. \\ & \cdots (\vartheta+n-j)(\rho+1) \cdots (\rho+j) \Phi(\vartheta+n-j, \rho+j) - \sum_{r=1}^{n-1} C_{n-r,r} \sum_{j=0}^{n-r-1} (-1)^j i^{j+r} \\ & \binom{n-r}{j} (\vartheta+1) \cdots (\vartheta+n-j-r)(\rho+1) \cdots (\rho+j+r) \Phi(\vartheta+n-j-r, \rho+j+r) \\ & - \sum_{r=1}^n C_{n-r,r} \sum_{j=0}^{n-r} (-1)^j \binom{n-r}{j} \sum_{l=0}^{r-1} i^{j+l} \binom{r}{l} (\vartheta+1) \cdots (\vartheta+n-j-l)(\rho+1) \cdots (\rho+j+l) \\ & \Phi(\vartheta+n-j-l, \rho+j+l) - \sum_{m=1}^n 2^m \sum_{r=0}^{n-m} C_{n-m-r,r} \sum_{j=0}^{n-m-r} (-1)^j \binom{n-m-r}{j} \sum_{l=0}^r i^{j+l} \binom{r}{l} \\ & \left. (\vartheta+1) \cdots (\vartheta+n-m-j-l)(\rho+1) \cdots (\rho+j+l) \Phi(\vartheta+n-m-j-l, \rho+j+l) \right], \end{aligned} \quad (4.4)$$

where $\rho = 0, 1, \dots, n-1$ and $\sum_{\vartheta=0}^{\infty} \Phi(\vartheta, \rho) x^{\vartheta}$ is expansion of

$$\frac{1}{\rho!} d_{\rho}(x), \quad \rho = 0, 1, \dots, n-1. \quad (4.5)$$

Proof: Let $\Phi(\vartheta, \rho)$ and $F(\vartheta, \rho)$ are 2-dimensional differential transformation of $\varphi(z, \bar{z})$ and $f(z, \bar{z})$ respectively. Applying Theorem 2.2 to equation (4.1) and then using 4 of Theorem 2.1 and simplifying the result relation to obtain $\Phi(\vartheta, \rho + n)$, then the iterated formula (4.4) is obtained.

Furthermore, the conditions in equation (4.2) make it clear that:

$$\sum_{\vartheta=0}^{\infty} \Phi(\vartheta, s) x^{\vartheta} = \frac{1}{s!} d_s(x), \quad s = 0, 1, \dots, n-1. \quad (4.6)$$

4.1 Illustrations for Solving Linear CPDEs:

This section presents exact solutions for CPDEs using Theorem 4.1.

Illustration 4.1. Consider the non-homogeneous second order CIVP as follows:

$$\varphi_{z\bar{z}} + \varphi_{\bar{z}} = 1 + z, \quad (4.7)$$

$$\varphi(x, 0) = x^2, \quad \varphi_y(x, 0) = 0. \quad (4.8)$$

Here, the analytical solution is $\varphi(z, \bar{z}) = z\bar{z}$.

In comparison with Theorem 4.1 and equation (4.2), then: $n = 2$, $C_{1,1} = 1$, $C_{0,1} = 1$, $f(z, \bar{z}) = 1 + z$. Also:

$$\sum_{\vartheta=0}^{\infty} \Phi(\vartheta, 0) x^{\vartheta} = x^2, \quad (4.9)$$

$$\sum_{\vartheta=0}^{\infty} \Phi(\vartheta, 1) x^{\vartheta} = 0 \quad (4.10)$$

Table 1 yields that $F(\vartheta, \rho) = \frac{i^{\rho}(\vartheta + \rho)!}{\vartheta! \rho!} (\delta(\vartheta + \rho) + \delta(\vartheta + \rho - 1))$.

Equation (4.4) provides the following recurrence relation for $\vartheta, \rho = 0, 1, 2, \dots$:

$$\begin{aligned} \Phi(\vartheta, \rho + 2) = & \frac{1}{(\rho + 1)(\rho + 2)} \left[\frac{4i^{\rho}(\vartheta + \rho)!}{\vartheta! \rho!} (\delta(\vartheta + \rho) + \delta(\vartheta + \rho - 1)) - (\vartheta + 1) \right. \\ & \left. (\vartheta + 2)\Phi(\vartheta + 2, \rho) - 2(\vartheta + 1)\Phi(\vartheta + 1, \rho) - 2i(\rho + 1)\Phi(\vartheta, \rho + 1) \right], \end{aligned} \quad (4.11)$$

Putting $\rho = 0$ into equation (4.11), implies:

$$\Phi(\vartheta, 2) = \frac{1}{2}[4(\delta(\vartheta) + \delta(\vartheta - 1)) - (\vartheta + 1)(\vartheta + 2)\Phi(\vartheta + 2, 0) - 2(\vartheta + 1)\Phi(\vartheta + 1, 0)],$$

Hence $\Phi(0, 2) = 1$, $\Phi(\vartheta, 2) = 0$ for $\vartheta \neq 0$.

Putting $\rho = 1$ into equation (4.11), implies:

$$\Phi(\vartheta, 3) = \frac{1}{6}[4i(\vartheta + 1)\delta(\vartheta) - 4i\Phi(\vartheta, 2)], \quad \vartheta = 0, 1, 2, \dots$$

which gives $\Phi(\vartheta, 3) = 0$, $\forall \vartheta$.

If $\rho = 2, 3, \dots$ is placed into equation (4.11), then:

$$\Phi(\vartheta, 4) = \Phi(\vartheta, 5) = \dots = 0, \quad \forall \vartheta.$$

Thus, through equation (4.3) and values of $\Phi(\vartheta, \rho)$; the solution of the CIVP (4.7)-(4.8) is:

$$\begin{aligned} \varphi(z, \bar{z}) &= \Phi(2, 0)x^2 + \Phi(0, 2)y^2 \\ &= z\bar{z} \end{aligned}$$

Illustration 4.2: Consider the non-homogeneous fifth order CIVP as follows:

$$\varphi_{z^4\bar{z}} - \varphi_{z^2\bar{z}} + \varphi_{z^2} + \varphi = 2\sin z, \quad (4.12)$$

$$\varphi(x, 0) = x \sin(x),$$

$$\varphi_y(x, 0) = i(x \cos(x) - \sin(x)),$$

$$\varphi_{y^2}(x, 0) = x \sin(x) + 2\cos(x),$$

$$\varphi_{y^3}(x, 0) = i(x \cos(x) - 3\sin(x)),$$

$$\varphi_{y^4}(x, 0) = x \sin(x) + 4\cos(x).$$

Here, the analytical solution is $\varphi(z, \bar{z}) = \bar{z} \sin z$.

In comparison with Theorem 4.1 and equation (4.2), we obtain that: $n = 5$, $C_{4,1} = C_{2,0} = C_{0,0} = 1$, $C_{2,1} = -1$ and $f = 2\sin z$. Also:

$$\sum_{\vartheta=0}^{\infty} \Phi(\vartheta, 0)x^{\vartheta} = x^2 - \frac{x^4}{3!} + \frac{x^6}{5!} - \frac{x^8}{7!} + \dots, \quad (4.13)$$

$$\sum_{\vartheta=0}^{\infty} \Phi(\vartheta, 1)x^{\vartheta} = -\frac{2ix^3}{3!} + \frac{4ix^5}{5!} - \frac{6ix^7}{7!} + \frac{8ix^9}{9!} - \dots \quad (4.14)$$

$$\sum_{\vartheta=0}^{\infty} \Phi(\vartheta, 2)x^{\vartheta} = 1 - \frac{x^4}{4!} + \frac{2x^6}{6!} - \frac{3x^8}{8!} + \dots, \quad (4.15)$$

$$\sum_{\vartheta=0}^{\infty} \Phi(\vartheta, 3)x^{\vartheta} = \frac{-2ix}{3!} + \frac{14ix^5}{7!} - \frac{2ix^7}{3.7!} + \cdots, \quad (4.16)$$

and

$$\sum_{\vartheta=0}^{\infty} \Phi(\vartheta, 4)x^{\vartheta} = \frac{1}{3!} - \frac{5x^2}{5!} + \frac{x^6}{12(6!)} - \frac{2x^8}{12(8!)} + \cdots \quad (4.17)$$

Theorem 3.1 yields that $F(\vartheta, \rho) = \frac{2i^{\rho}}{\vartheta! \rho!} \sin \frac{(\vartheta+\rho)\pi}{2}$.

Equation (4.4) implies the following recurrence relation:

$$\begin{aligned} \Phi(\vartheta, \rho+5) = & \frac{-i}{(\rho+1)(\rho+2)(\rho+3)(\rho+4)(\rho+5)} \left[\frac{64i^{\rho}}{\vartheta! \rho!} \sin \frac{(\vartheta+\rho)\pi}{2} + 3(\vartheta+1)(\rho+1)(\rho+2) \right. \\ & (\rho+3)(\rho+4)\Phi(\vartheta+1, \rho+4) + 2i(\vartheta+1)(\vartheta+2)(\rho+1)(\rho+2) \\ & (\rho+3)\Phi(\vartheta+2, \rho+3) + 2(\vartheta+1)(\vartheta+2)(\vartheta+3)(\rho+1)(\rho+2) \\ & \Phi(\vartheta+3, \rho+2) + 3i(\vartheta+1)(\vartheta+2)(\vartheta+3)(\vartheta+4)(\rho+1)\Phi(\vartheta+4, \\ & \rho+1) - (\vartheta+1)(\vartheta+2)(\vartheta+3)(\vartheta+4)(\vartheta+5)\Phi(\vartheta+5, \rho) - 4i(\rho \\ & +1)(\rho+2)(\rho+3)\Phi(\vartheta, \rho+3) + 4(\vartheta+1)(\rho+1)(\rho+2)\Phi(\vartheta+1, \rho \\ & +2) - 4i(\vartheta+1)(\vartheta+2)(\rho+1)\Phi(\vartheta+2, \rho+1) + 4(\vartheta+1)(\vartheta+2) \\ & (\vartheta+3)\Phi(\vartheta+3, \rho) + 8(\rho+1)(\rho+2)\Phi(\vartheta, \rho+2) + 16i(\vartheta+1) \\ & (\rho+1)\Phi(\vartheta+1, \rho+1) - 8(\vartheta+1)(\vartheta+2)\Phi(\vartheta+2, \rho) - 32\Phi(\vartheta, \rho), \\ & \left. \vartheta=0, 1, 2, \dots \right] \end{aligned} \quad (4.18)$$

We realized the following cases using equation (4.18) for $r_1, r_2=0, 1, \dots$:

1. $\vartheta = 2r_1, \rho = 2r_2$ gives $\Phi(2r_1, 2r_2+5) = 0$.
2. $\vartheta = 2r_1+1, \rho = 2r_2+1$ gives $\Phi(2r_1+1, 2r_2+6) = 0$.
3. $\vartheta = 2r_1+1, \rho = 2r_2$, gives $\Phi(2r_1+1, 2r_2+5) = \frac{[2r_1-(2r_2+4)]i(-1)^{r_1}}{(2r_1+1)!(2r_2+5)!}$.
4. $\vartheta = 2r_1, \rho = 2r_2+1$, gives $\Phi(2r_1, 2r_2+6) = \frac{[-2r_1+(2r_2+6)](-1)^{r_1}}{(2r_1)!(2r_2+6)!}$.

Using the above cases and equations (4.13)- (4.17), equation (4.3) provides following solution of the CIVP (4.12) :

$$\begin{aligned} \varphi(z, \bar{z}) = & \sum_{\rho=0}^4 \sum_{\vartheta=0}^{\infty} \Phi(\vartheta, \rho)x^{\vartheta}y^{\rho} + \sum_{\rho=5}^{\infty} \sum_{\vartheta=0}^{\infty} \Phi(\vartheta, \rho)x^{\vartheta}y^{\rho} \\ = & x\left[z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots\right] - iy\left[z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots\right] \\ = & \bar{z} \sin z. \end{aligned}$$

5. Conclusions

The results presented in this paper can be used to calculate the 2-dimensional differential transformation for an oversize of complex functions. Furthermore, we present an iterated relation for solving the complex partial differential equation of n^{th} order using the 2-dimensional differential transformation method, which makes solving any nonhomogeneous linear complex partial differential equation with constant coefficients easier and simpler.

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