Journal of Interdisciplinary Mathematics
ISSN: 0972-0502 (Print), ISSN: 2169-012X (Online)
Vol. 28 (2025), No. 4, pp. 1595–1605
DOI : 10.47974/JIM-2154

Solvability and factorization of mixed problem for the heat conduction equation

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Abstract

The study aims to analyse the solvability of mixed problems for the heat conduction equation. To achieve the objectives, analytical methods, methods of mathematical analysis of mixed problems and heat conduction equations were applied. Functional analysis and differential equation theory methods were also used to study the solvability. The study determined that the equation under consideration has a wide range of applications and plays an important role in describing heat transfer processes in various fields of science and technology. The study revealed that analysing mixed problems involving a variety of initial and boundary conditions provides a key tool for gaining a deeper understanding of heat transfer processes. Furthermore, properly chosen initial and boundary conditions have a significant impact on the long-term system operation. The study results have important implications for understanding and modelling heat conduction processes in a variety of physical systems. The results provide a deeper understanding of heat conduction processes and their mathematical description in a variety of systems, which opens new perspectives for improving the prediction and control of heat conduction phenomena. This study represents a significant contribution to the field of thermal conduction, not only expanding the theoretical understanding of this physical phenomenon but also providing a basis for practical applications.

Subject Classification (2010): *35K05, 35Pxx.*

Keywords: *Variable separation method, Spectral theory, Operator, Boundary conditions, Argument deviation.*

1. Introduction

The study of solvability and factorisation of the mixed problem for the heat conduction equation is crucial for optimizing thermal processes, reducing energy costs, and predicting system behaviour. This research has practical significance for industries such as transport, energy, electronics, and science, enhancing technology, efficiency, and sustainability, with positive impacts on the economy and quality of life. Despite its wide application in fields like engineering, geophysics, and medicine, issues regarding the solvability and analysis of mixed problems remain unresolved. Addressing these challenges enables more accurate modelling and control of thermal processes, fostering the development of efficient and sustainable systems.

Acharya [1] studied numerical modelling of heat and mass transfer in radiation-transported nanofluids over a curved surface using a spectral method, accounting for various convective conditions. Yuldashev and Kadirkulov [2] applied the fractional Hilfer operator to solve boundary problems for weakly nonlinear PDEs, proving existence and uniqueness theorems for regular spectral parameters and constructing Fourier series solutions for irregular cases. Gouari and Dahmani [3] explored the solvability of nonlocal singular fractional differential equations, offering insights relevant to complex heat conduction problems. Kumar et al. [4] introduced the Yang-Abdel-Aty-Cattani fractional operator to generalise classical heat conduction models. Their results demonstrated convergence of solutions using the generalised Mittag-Leffler function. Kalmenov et al. [5; 6] examined the completeness of root vectors in the Cauchy problem for first-order equations with delay, identifying conditions under which the operator's invariant subspaces are complete, despite the operator being non-self-adjoint [7-9].

The study aims to analyse the solution of mixed problems and to identify the peculiarities of the system behaviour under different initial and boundary conditions, which will allow a deeper understanding of the influence of these conditions on the heat conduction processes. To achieve the set goal, the following tasks were formed in the study of mixed problems for heat conduction equations: to consider theoretical approaches to the solution of mixed problems; to analyse the solvability of the formulated mixed problems.

2. Materials and Methods

In researching the solvability and factorisation of the mixed problem for the heat conduction equation, both theoretical and practical aspects of the subject were explored using appropriate materials and methods.

The definitions and properties associated with the heat conduction equation, as well as the concepts of functional analysis and the theory of differential equations, were used in this study. They were used to analyse the solvability of mixed problems, highlight important aspects related to boundary conditions, and develop factorization methods to solve such problems approximately. The heat conduction equation is one of the fundamental differential equations and describes the propagation of heat in time and space. The study of its definitions and properties has allowed us to better understand its behaviour and possible solutions:

$$Lu = u_t() + u_{xx}(x,t) = f(x,t), \tag{1}$$

or in as a differential:

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}, \tag{2}$$

where: α – heat transfer coefficient, which characterises the ability of a material to conduct heat; $\frac{\partial u}{\partial t}$ – time derivative of temperature; $\frac{\partial^2 u}{\partial x^2}$ – second derivative of temperature along the spatial coordinate.

Functional analysis and operator methods were employed to determine solvability conditions and constraints for mixed problems, including non-linear and heterogeneous cases. These abstract methods allowed for the formulation of the equations in operator form and their systematic analysis. Additionally, the method of variable separation was applied to derive explicit solutions to differential equations, contributing to a deeper theoretical understanding of thermal processes and the limitations of the derived results.

3. Results

To better understand the problem of mixed problems for the heat conduction equations, the problem has been considered. Assume $\Omega \subset R^2$ a rectangle bounded by segments: $AB: 0 \le t \le T,\ x = 0;\ BC: 0 \le x \le l,\ t = T;\ CD: 0 \le t \le T,\ x = l;\ FA: 0 \le x \le l,\ t = 0.$ Using $\tilde{N}^{2.1}(\Omega)$ as a multitude of functions $u(x,t)$ twice continuously differentiable on x and once continuously differentiable by t in region Ω were denoted. Under the region boundary Ω of parts $\Gamma = AB \cup AD \cup CD.$ Find a solution to the equation (1). That meets boundary conditions:

$$u|_{x=0} + u|_{x=l} = u_x|_{x=0} + u_x|_{x=l} = 0, \tag{3}$$

$$u|_{t=0} = 0. \tag{4}$$

where: $f(x,t) \in L^2(\Omega)$.

To derive an analytical solution for the problem, the method of separation of variables was utilized. This method assumes that a differential equation describing a physical process can be divided into simpler, independent subproblems. For the heat conduction equation, this involves separating the heat propagation process into independent spatial and temporal components. The solution is then expressed as a product of two functions $u(x,t) = X(x)T(t)$:

$$X(x)T'(t) - X''(x)T(t) = f(x,t). \tag{5}$$

Acquired equation was split into $X(x)T(t)$:

$$\frac{X(x)}{X''(x)} = \left(\frac{1}{T'(t)}\right) f(x,t) = -\lambda, \tag{6}$$

where: λ – divisible constant.

Two separate differential equations are obtained. Equation for $X(x)$ with boundary conditions:

$$X''(x)+\lambda X(x)=0. \tag{7}$$

Equation for $T(t)$ with initial condition:

$$T'(t)+\lambda T(t)=f(x,t). \tag{8}$$

The study by Koiran [10] established the following statement.

Theorem 1: *The problem has infinitely many eigenvalues and their corresponding eigenfunctions, which formed the orthonormalized basis of the space* $L^2(\Omega)$, *where:* $\Omega=[0,l]\times[0,T]$:

$$\lambda_{mn}=(-1)^n\left(n+\frac{1}{2}\right)\frac{\pi}{T}-\left(\frac{m\pi}{l}\right)^2, m,n=0,1,2,\ldots, \tag{9}$$

$$u_{mn}(x,t)=\frac{2}{\sqrt{Tl}}\sin\frac{m\pi}{l}x\cdot\sin\left(n+\frac{1}{2}\right)\frac{\pi t}{T}, m,n=0,1,2,\ldots. \tag{10}$$

This theorem provides an important mathematical tool for analysing and solving a heat conduction problem with a given boundary and initial conditions. The minimal operator presented an abstract mathematical model that was used to study general principles of solvability and factorisation of heat conduction problems. This framework generalizes the fundamental characteristics of the heat conduction equation and serves as a foundation for analyzing more complex systems. In this study, the minimal operator in a problem with a deviating argument was considered. Within the space $H=L^2(0,1)$ a minimum operator was considered, generated by a first-order differentiable expression with a divergent argument:

$$L_0 y=y'(1-x), D(L_0)=C_0^\infty(0,1). \tag{11}$$

Regular expansion of the differentiation operator with a deviating argument L_0 is as follows [11]:

$$L_k u=u'(1-x)=f(x), \tag{12}$$

$$(1+k)u(0)-ku(1)=0, k\in, \tag{13}$$

where: $f(x)\in L^2(0,1)$.

Reverse operator:

$$u(x)=L_k^{-1}f(x)=\frac{\int_0^x f(1-t)dt}{2}-\frac{\int_x^1 f(1-t)dt}{2}-\upsilon\left(x-\frac{1}{2}\right)+x\cdot\int_0^1 f(t)\,dt+k\int_0^1 f(t)\,dt$$
$$=\frac{1}{2}\left[\int_0^x f(1-t)dt-\int_x^1 f(1-t)dt\right]+\left(k+\frac{1}{2}\right)\cdot\int_0^1 f(t)\,dt. \qquad (14)$$

Lemma 1: *Operator L_k^{-1} – inverse to the regular expansion of the minimal operator L_0, generated by the operation of differentiation with a deviating argument, continuous in space $H=L^2(0.1)$.*

Proof: From the representation (12):

$$|u(x)|\le\frac{1}{2}\int_0^1|f(t)|\,dt+\left|k+\frac{1}{2}\int_0^1|f(t)|\,dt\right|\le\left(\frac{1}{2}+\left|k+\frac{1}{2}\right|\right)\cdot\int_0^1|f(t)|\,dt, \qquad (15)$$

$$|u(x)|^2\le\left(\frac{1}{2}+\left|k+\frac{1}{2}\right|\right)^2\left(\int_0^1|f(t)|dt\right)^2\le\left(\frac{1}{2}+\left|k+\frac{1}{2}\right|\right)^2\cdot\int_0^1|f(t)|^2dt, \qquad (16)$$

$$\|u\|^2\le\left(\frac{1}{2}+\left|k+\frac{1}{2}\right|\right)^2\|f\|^2, \|u\|\le\left(\frac{1}{2}+\left|k+\frac{1}{2}\right|\right)\|L_K u\|, \qquad (17)$$

then:

$$L_k u=u'(1-k)\Rightarrow\|u'\|^2=\|u'(1-x)\|^2=\|L_K u\|^2, \qquad (18)$$

afterwards:

$$\|u\|_1^2=\|u\|_0^2+\|\dot{u}\|_0^2\le\left(\frac{1}{2}+\left|k+\frac{1}{2}\right|\right)^2\|L_K u\|^2+L_K u^2\le\left[\left(\frac{1}{2}+\left|k+\frac{1}{2}\right|\right)^2\right]\|L_K u^2\|. \qquad (19)$$

The last equation shows that the operator L_k^{-1} displays a bounded set of $R(L_k)$ into a compact one, which means that the operator L_k^{-1} is continuous. According to the well-known Hilbert's theorem [12] any non-zero point of the spectrum of a completely continuous operator A is its normal point. Consequently, such an operator has at most a countable number of points of the spectrum, which can have as its limit only a point $\lambda=0$. The last one always (at $\dim H=\infty$) falls into the spectrum A.

Functional analysis provided a framework for analysing solvability conditions and developing methods for approximate solutions. It established the function spaces in which solutions exist and ensured the correctness of the mathematical model. Operator methods allowed analysis of eigenvalues and eigenfunctions, identifying spectral properties and their impact on solvability. Spectral theory elucidated critical frequencies and wave numbers influencing heat conduction solutions. Differential theory, particularly the spectral theory of differential equations with deviating arguments, was pivotal in analysing operators and solving mixed problems with inhomogeneous or non-linear conditions [13-15]. The analysis determined the initial and boundary conditions under which the heat conduction equation has a stable, unique solution. This is crucial in designing systems involving heat transfer, such as thermal devices, material science applications, and geothermal energy [16-19]. Mathematical methods addressed various practical initial and boundary conditions, including temperature profiles, heat flux, and other parameters affecting heat propagation [20, 21]. These findings enhance the theoretical understanding of heat conduction while providing practical insights for optimising systems where heat transfer is critical.

The results obtained can find practical application in various fields. For example, in engineering, they can be used to optimise thermal processes in various devices and systems. In science, the results of the study can help to better understand the physical processes associated with heat conduction, including in biological systems. They can also be applied to energy and other industries. Thus, this study represents an important contribution to the development of heat conduction theory and its practical applications. The results and methods obtained in the course of the study can be further developed and adapted to solve specific problems in various fields of science and engineering.

4. Discussion

Simoncelli et al. [22] addressed the generalisation of Fourier's law to the case of viscous (viscoelastic) processes in the framework of heat conduction and developed a theory of heat conduction that considers viscosity and viscoelastic processes in materials. The study provides a valuable contribution to the theoretical development of mathematical research in heat conduction and mechanics of materials, as the results of the work allow a broader class of materials and physical scenarios to be considered. The similarity of this study with the work of the researchers is that both works are relevant to heat conduction and extend the theoretical framework for analysing heat transfer under different conditions, as well as enrich the theoretical apparatus and methods of analysis in this field.

Makhnei [23] noted that heat transfer modelling often involves boundary value problems with piecewise continuous coefficients or coefficients that are generalized derivatives of discontinuous functions. The author proposed a scheme for solving the mixed problem using reduction methods and the Fourier method. Mixed problems are a class of problems involving both boundary and initial conditions, which is a common situation in heat conduction and diffusion modelling [24]. The proposed schemes for solving the problem using reduction methods and the Fourier method provide effective tools for analysing and solving such problems. Reduction methods and the Fourier method are widely used in mathematical analysis and reduce the complexity of the problem. The similarity of this study with the work of the researcher is in the general interest in solving problems related to heat transfer and modelling such processes. Both studies face problems with boundary conditions, including cases with piecewise continuous coefficients or derivatives of discontinuous functions.

Boussaada et al. [25] explored second-order differential equations with delay, analysing solution multiplicity and its implications for system control. Delayed equations are relevant in modelling systems in control theory, biology, and telecommunications. Their analysis is significant across disciplines where time-lagged effects play a role, sharing a conceptual connection with the current study's focus on differential equations in applied contexts.

Thus, solving mixed problems for the heat conduction equations is critical in understanding and controlling thermal processes in various physical systems and technological applications. This research not only enhances the theoretical understanding of physical processes but also has great practical significance for a multitude of industries and applications including industry, transport, energy, electronics, and science. The development of solution factorisation methods, solvability analysis and the study of system behaviour under different conditions are important steps in improving existing technologies and creating new, more efficient, and resilient systems. This, in turn, contributes to the development of modern technologies, increasing the efficiency of systems and the sustainability of production processes.

5. Conclusions

This study confirms the importance of investigating the solvability of mixed problems for the heat conduction equation. By applying the spectral theory of differential equations with a deviating argument, conditions ensuring the existence and uniqueness of solutions were established. The analysis of eigenvalues and eigenfunctions of associated operators provided insights into the structure and properties of solutions, thereby expanding the theoretical framework for heat conduction problems. Functional analysis was employed to study the solvability of mixed problems, offering a generalised and abstract approach suitable for various mathematical models, including nonlinear and inhomogeneous cases. This method enhances the flexibility of applying mathematical tools to a wide range of scientific and engineering challenges.

The findings have practical significance in engineering, energy systems, and biomedicine, where accurate modelling of thermal processes is essential. In engineering, they support the design of more efficient heat transfer systems; in biomedicine, they aid in modelling heat distribution in biological tissues. This research lays the foundation for further studies, particularly in extending the theory to nonlinear problems involving phase transitions and variable heat capacities. Future work may also focus on developing numerical methods grounded in the present analytical results to enhance the efficiency of solving applied thermal problems.

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Received June, 2024