

Solution of Volterra fuzzy integral equations by using double fuzzy Aboodh transform

Murtadha A. Kadhim [‡]

Department of Mathematics

Al-Furat Al-Awsat Technical University (ATU)

Najaf 56001

Iraq

Samer Thaaban Abaas Alshibley [†]

Department of Mathematics

Faculty of Education

University of Kufa

Najaf 56001

Iraq

Alaa M. Abed [§]

Department of Mathematics

The General Directorate of Education

Diwaniyah 54001

Iraq

Shaymaa Maki Kadham ^{*}

Department of Mathematics

Faculty of Computer Science and Mathematics

University of Kufa

Najaf 56001

Iraq

[‡] ORCID-ID: 0000-0003-2664-1669

[§] ORCID-ID: 0009-0004-1222-0219

^{*} ORCID-ID: 0000-0001-5613-6255

[‡] E-mail: murtadha.abduljawad.iku@atu.edu.iq

[†] E-mail: samir.alshebly@uokufa.edu.iq

[§] E-mail: Alaamuhsien3030@gmail.com

^{*} E-mail: Shaimam.altaee@uokufa.edu.iq (Corresponding Author)

Abstract

In this study, we resolve fuzzy convolution in two dimensions. By using the double fuzzy Aboodh transform method, Volterra integral equations are solved. Using the double FT-Aboodh method, when applied to the problem it converts into an algebraic problem. A detailed investigation about convolution along with its properties and theorem and an accompanying proof is presented. We provide some first findings using the double fuzzy Aboodh transform technique. Lastly, an example is used to illustrate the method's functionality and highlight its potential.

Subject Classification (2020): 26A18, 28Dxx.

Keywords: Fuzzy aboodh transform-transform, Solve ordinary linear differential, Double fuzzy aboodh transform.

1. Introduction

Applied mathematics is vital in giving the means to comprehend and evaluate unclear occurrences in a world where ambiguity and complexity are the norm. Fuzzy integral equations are one such instrument that has grown in significance during the last several decades. The aforementioned equations embody the union of integral theory with fuzzy set theory, providing a sturdy mathematical structure for managing data and information that are intrinsically ambiguous or inaccurate. These components are combined in fuzzy integral equations to provide an analytical tool that can handle the complexity and imprecision that researchers often run across in a variety of domains. The integral transforms have their grandfather in the Fourier transform, which is used to express functions in a finite interval. Several works address the theory and applications of integral transforms, such as Laplace and Mellin transforms [6-10]. Later, the idea of integral transforms was extended such that finite intervals were not required. The Aboodh transform is a novel integral transform suggested by Khalid [11-19]. Fuzzy Aboodh transform [1-5] is among the most modern techniques for solving issues represented in a fuzzy environment. Numerous types of fuzzy differential and integral equations have been solved using FAT [6]. The issue is transformed to an algebraic problem, which is considerably easier to answer, by utilizing FAT. This is under the category of operational calculus, and within it, fuzzy Laplace transform is regarded as one of the most effective methods.

2. Double Fuzzy Aboodh Transform

This section introduces the definition of the double fuzzy Aboodh transform. We also provide various characteristics and theorems related to the double fuzzy Aboodh transform.

Definition 2.1 [10]: Let $\dot{U}(x)$ be a continuous fuzzy-valued function. Assume that $\frac{1}{p}\dot{U}(x)e^{-px}$ is an inappropriate fuzzy Riemann-integrable on $[0, \infty)$, then $\frac{1}{p}\int_0^\infty \dot{U}(x)e^{-px} dx$ is being called *fuzzy Aboodh transform* and is referred to as

$$\begin{aligned}\widehat{A}[\dot{U}(x)] &= \mathfrak{S}(p) = \frac{1}{p}\int_0^\infty \dot{U}(x)e^{-px} dx, (p > 0 \text{ and integer}) \\ \frac{1}{ps}\int_0^\infty \dot{U}(x)e^{-px} dx &= \left(\frac{1}{p}\int_0^\infty \underline{\dot{U}}(x, \mu)e^{-px} dx, \frac{1}{p}\int_0^\infty \overline{\dot{U}}(x, \mu)e^{-px} dx\right)\end{aligned}$$

Definition 2.2: Let $\dot{U}(x, y)$ be a continuous fuzzy-valued function. Assume that $\frac{1}{p}\frac{1}{u}\dot{U}(x, y)e^{-px-uy}$ is an improper fuzzy Riemann-integrable on $[0, \infty) \times [0, \infty)$, then $\frac{1}{p}\frac{1}{u}\int_0^\infty \int_0^\infty \dot{U}(x, y)e^{-px-uy} dx dy$ is being called *double fuzzy Aboodh transform* and is referred to as

$$\widehat{A}_2[\dot{U}(x, y)] = \mathfrak{S}(p, u) = \frac{1}{p}\frac{1}{u}\int_0^\infty \int_0^\infty \dot{U}(x, y)e^{-px-uy} dx dy, \\ (p, u > 0 \text{ and integer}),$$

$$\begin{aligned}\frac{1}{p}\frac{1}{u}\int_0^\infty \int_0^\infty \dot{U}(x, y)e^{-px-uy} dx dy &= \\ \left(\frac{1}{p}\frac{1}{u}\int_0^\infty \int_0^\infty \underline{\dot{U}}(x, y, \mu)e^{-px-uy} dx dy, \frac{1}{p}\frac{1}{u}\int_0^\infty \int_0^\infty \overline{\dot{U}}(x, y, \mu)e^{-px-uy} dx dy\right).\end{aligned}$$

Furthermore, possible to display parametrically as follows

$$\widehat{A}_2[\dot{U}(x, y)] = (A_2 \underline{\dot{U}}(x, y, \mu), A_2 \overline{\dot{U}}(x, y, \mu)).$$

Where double fuzzy inverse Aboodh transform can be written as the formula

$$\widehat{A}_2^{-1}[\mathfrak{S}(p, u)] = [\dot{U}(x, y)] = [(A_2^{-1} \underline{\dot{U}}(x, y, \mu), A_2^{-1} \overline{\dot{U}}(x, y, \mu))]$$

Theorem 3.3: If $\dot{U}(x, y)$ is a continuous fuzzy valued function and $\widehat{A}_2[\dot{U}(x, y)] = \mathfrak{S}(p, u)$ then:

$\widehat{A}[\dot{U}(x, y)e^{ax+by}] = \mathfrak{S}(p-a, (u-b))$, where e^{ax+by} is real value function and $(p-a, (u-b)) > 0$.

Proof: $\widehat{A}_2[\dot{U}(x, y)] = \mathfrak{S}(p, u)$

$$\widehat{A}_2[\dot{U}(x, y)] = \frac{1}{p}\frac{1}{u}\int_0^\infty \int_0^\infty \dot{U}(x, y)e^{-px-uy} dx dy$$

$$\begin{aligned}\widehat{A}_2[\dot{U}(x, y)] &= \\ \left(\frac{1}{p}\frac{1}{u}\int_0^\infty \int_0^\infty \underline{\dot{U}}(x, y, \mu)e^{-px-uy} dx dy, \frac{1}{p}\frac{1}{u}\int_0^\infty \int_0^\infty \overline{\dot{U}}(x, y, \mu)e^{-px-uy} dx dy\right)\end{aligned}$$

$$\begin{aligned}\widehat{A}_2[\underline{U}(\mathfrak{x}, y)e^{ax+by}] &= \left(\frac{1}{\mathcal{P}}\frac{1}{\mathcal{U}}\int_0^\infty\int_0^\infty \underline{U}(\mathfrak{x}, y, \mu)e^{-\mathcal{P}\mathfrak{x}-\mathcal{U}y}e^{ax+by}d\mathfrak{x}dy,\right. \\ &\quad \left.\frac{1}{\mathcal{P}}\frac{1}{\mathcal{U}}\int_0^\infty\int_0^\infty \overline{U}(\mathfrak{x}, y, \mu)e^{-\mathcal{P}\mathfrak{x}-\mathcal{U}y}e^{ax+by}d\mathfrak{x}dy\right) \\ \widehat{A}_2[\underline{U}(\mathfrak{x}, y)e^{ax+by}] &= \left(\frac{1}{\mathcal{P}}\frac{1}{\mathcal{U}}\int_0^\infty\int_0^\infty \underline{U}(\mathfrak{x}, y, \mu)e^{-((\mathcal{P}-a)\mathfrak{x}+(-\mathcal{U}+b)y)}d\mathfrak{x}dy,\right. \\ &\quad \left.\frac{1}{\mathcal{P}}\frac{1}{\mathcal{U}}\int_0^\infty\int_0^\infty \overline{U}(\mathfrak{x}, y, \mu)e^{-((\mathcal{P}-a)\mathfrak{x}-(\mathcal{U}-b)y)}d\mathfrak{x}dy\right) \\ \widehat{A}_2[\underline{U}(\mathfrak{x}, y)e^{ax+by}] &= \xi(\mathcal{P}-a, (\mathcal{U}-b))\end{aligned}$$

3. Regarding fuzzy convolution in two dimensions Integral equations of Volterra and double fuzzy Aboodh transform

The two-dimensional fuzzy convolution the definition of second-kind integral equations is

$$\underline{U}(\mathfrak{x}, y) = \mathfrak{p}(\mathfrak{x}, y) + \int_0^y \int_0^x K(\mathfrak{x} - \mathfrak{H}, y - \phi) \underline{U}(\mathfrak{H}, \phi) d\mathfrak{H}d\phi$$

Where

- $\underline{U}(\mathfrak{x}, y)$ is the unidentified fuzzy function that requires our solution.
- $\mathfrak{p}(\mathfrak{x}, y)$ is an established fuzzy function.
- $K(\mathfrak{x} - \mathfrak{H}, y - \phi)$ is the fuzzy kernel function, which defines how the values of $\underline{U}(\mathfrak{x}, y)$ at different points interact.
- The convolution operation Multiplication in the kernel $K(\mathfrak{x} - \mathcal{P}, y - \mathcal{U}) \underline{U}(\mathcal{P}, \mathcal{U})$ reflects the interdependence of the function values over the domain.
-

Double fuzzy application Aboodh changes on both sides, giving us

$$\widehat{A}_2[\underline{U}(\mathfrak{x}, y)] = \widehat{A}_2\left[\mathfrak{p}(\mathfrak{x}, y) + \int_0^y \int_0^x K(\mathfrak{x} - \mathfrak{H}, y - \phi) \underline{U}(\mathfrak{H}, \phi) d\mathfrak{H}d\phi\right]$$

Case (1): If $\mathfrak{p}(\mathfrak{x}, y)$ and $K(\mathfrak{x} - \mathfrak{H}, y - \phi)$ are positive, then

$$\begin{aligned}A_2[\underline{U}(\mathfrak{x}, y, \mu)] &= A_2\left[\underline{\mathfrak{p}}(\mathfrak{x}, y, \mu)\right] + \mathcal{P}\mathcal{U} A_2[K(\mathfrak{x}, y) A_2[\underline{U}(\mathfrak{x}, y, \mu)]] \\ A_2[\overline{U}(\mathfrak{x}, y, \mu)] &= A_2[\overline{\mathfrak{p}}(\mathfrak{x}, y, \mu)] + \mathcal{P}\mathcal{U} A_2[K(\mathfrak{x}, y) A_2[\overline{U}(\mathfrak{x}, y, \mu)]]\end{aligned}$$

Case (2): If $\mathfrak{p}(\mathfrak{x}, y)$ and $K(\mathfrak{x} - \mathfrak{H}, y - \phi)$ are negative, then

$$\begin{aligned}A_2[\overline{U}(\mathfrak{x}, y, \mu)] &= A_2[\overline{\mathfrak{p}}(\mathfrak{x}, y, \mu)] + \mathcal{P}\mathcal{U} A_2[K(\mathfrak{x}, y) A_2[\overline{U}(\mathfrak{x}, y, \mu)]] \\ A_2[\underline{U}(\mathfrak{x}, y, \mu)] &= A_2[\underline{\mathfrak{p}}(\mathfrak{x}, y, \mu)] + \mathcal{P}\mathcal{U} A_2[K(\mathfrak{x}, y) A_2[\underline{U}(\mathfrak{x}, y, \mu)]]\end{aligned}$$

Case (3): If $\mathfrak{p}(\mathfrak{x}, y)$ is positive and $K(\mathfrak{x} - \mathfrak{h}, y - \phi)$ is negative, then

$$\begin{aligned} A_2[\overline{U}(\mathfrak{x}, y, \mu)] &= A_2[\overline{p}(\mathfrak{x}, y, \mu)] + \mathcal{PU} A_2[K(\mathfrak{x}, y) A_2[\underline{U}(\mathfrak{x}, y, \mu)]] \\ A_2[\underline{U}(\mathfrak{x}, y, \mu)] &= A_2[\underline{p}(\mathfrak{x}, y, \mu)] + \mathcal{PU} A_2[K(\mathfrak{x}, y) A_2[\overline{U}(\mathfrak{x}, y, \mu)]] \end{aligned}$$

Case (3): If $\mathfrak{p}(\mathfrak{x}, y)$ is negative and $K(\mathfrak{x} - \mathfrak{h}, y - \phi)$ is positive, then

$$\begin{aligned} A_2[\underline{U}(\mathfrak{x}, y, \mu)] &= A_2[\underline{p}(\mathfrak{x}, y, \mu)] + \mathcal{PU} A_2[K(\mathfrak{x}, y) A_2[\overline{U}(\mathfrak{x}, y, \mu)]] \\ A_2[\overline{U}(\mathfrak{x}, y, \mu)] &= A_2[\overline{p}(\mathfrak{x}, y, \mu)] + \mathcal{PU} A_2[K(\mathfrak{x}, y) A_2[\underline{U}(\mathfrak{x}, y, \mu)]] \end{aligned}$$

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Received November, 2024