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Rings of continuous linear transformations on a topological vector space

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Abstract

A structural analysis of the ring $C_L(X)$ of all linear continuous transformations on a topological vector space X is presented here. It explores the algebraic cum topological properties of $C_L(X)$ in relation with the structural features of X. Specifically, we discussed the relations between left ideals, zero-sets and z-filters in a topological vector space X and proved that if X is a finite dimensional topological vector space, the left ideal generated by the Zero set of a left ideal I in $C_L(X)$ is I itself.

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1. Introduction

Linear algebra, a well known branch of mathematics, but truly is the \`mathematics of data'. Matrices and vectors are the language of data. This beautiful field of mathematics contains plenty of esoteric theories and findings with a lot of application in enormous areas of pure mathematics, applied mathematics and engineering [1, 2, 4, 5, 7, 10]. Characterization and generalization of many results in linear algebra can be acquired using the concepts from dynamical systems. Colonius and Kliemann in [3] provides an introduction to the interplay between linear algebra and dynamical systems.

Until the early 20th century, research on linear operators was mainly concerned with those acting between $FinDimS$[1] over the fields $\mathbb{R}$ and $\mathbb{C}$. The exploration of infinite-dimensional cases, as well as considerations involving more general fields, began with the work of Toeplitz [15]. Typically, when studying linear operators in infinite-dimensional spaces, continuity is assumed with respect to certain topologies.

A $\mathcal{TVS}$[2] is an algebraic structure that combines the properties of a vector space($\mathcal{VS}$) with those of a topological space($\mathcal{TS}$), enabling the concept of continuity within its framework. Continuous linear transformations too have received a lot of attention in the literature and is one of the area of interest in this paper.

Gillman and Jerison conducted an in-depth exploration of the structure and properties of the ring $C(X)$, which consists of all continuous real-valued functions defined on any given $\mathcal{TS}$ X. Their book *Rings of Continuous Functions* [6] is considered as one of the most important text in this field.

It is nice to observe that $C_L(X)$, set of all linear continuous transformations on a $\mathcal{TVS}$ X[3] into itself, is a ring. The purpose of this paper is a systematic analysis of the ring $C_L(X)$, in connection with the topological and algebraic properties of the ring $C_L(X)$ and the topology on the space X, especially by considering a special collection- left ideal in the ring $C_L(X)$ and examine some properties of the family of zero-sets of this left ideal. This paper also

[1] Finite Dimensional Space
[2] Topological Vector Space
[3] Topological Vector Space X

incorporates the findings of Ibrahim, on strongly α-$\mathcal{TVS}$s [8]. This study extends the understanding of $\mathcal{TVS}$s and their relationships with other structures.

We present this paper in the following sequel. We begin with some basic definitions and results which are essential for our purpose (See Section 2). The concept of ring of continuous linear transformations and related results are discussed in Section 3. In Section 4, we mainly focused on the connection between $z-filters$ and left ideals in both finite and infinite dimensional $\mathcal{TVS}$s.

2. Preliminaries

A linear $\mathcal{VS}$ over $\mathbb{C}$, equipped with a topology τ on X, is referred to as a $\mathcal{TVS}$ if the topology ensures that singleton sets in X are closed. Additionally, the vector space operations-namely, addition and scalar multiplication-must be continuous functions, where addition is defined as a mapping from $X \times X$ into X, and scalar multiplication as a mapping from $\mathbb{C} \times X$ into X.

By definition, it is evident that every $\mathcal{TVS}$ is a Hausdorff space. Furthermore, if X is a $\mathcal{TVS}$, then for any element $\alpha \in X$ and any nonzero scalar $\beta \in \mathbb{C}$, the translation operator $T_\alpha(x) = \alpha + x$ and the multiplication operator $M_\beta(x) = \beta x$ (for all $x \in X$) act as homeomorphisms on X. Consequently, every vector topology is translation invarient.

The following remarks showing some results from [12, 13, 14], which are frequently occurs in our discussion.

Remark 2.1: An important observation is that any FinDim $\mathcal{TVS}$ is homeomorphic to $\mathbb{C}^{\mathrm{N}}$ for some integer N. Additionally, any linear transformation between finite-dimensional $\mathcal{TVS}$s is necessarily continuous. Furthermore, a linear operator T mapping a $\mathcal{TVS}$ X into a FinDim $\mathcal{TVS}$ Y is continuous if and only if its kernel M is a closed subspace. It is also a well-established fact that every FinDim subspace of a $\mathcal{VS}$ is closed.

Remark 2.2: Let X be a $\mathcal{TVS}$.

(a) If V is a neighborhood of 0 in X, and $\{r_n\}$ is an increasing sequence of positive real numbers such that $r_n \to \infty$ as $n \to \infty$, then the union of their scaled neighborhoods covers X:
$$X = \cup_{n=1}^{\infty} r_n V.$$
(b) The space X is metrizable if and only if it possesses a countable local base.

(c) Each neighborhood of 0 necessarily includes a balanced neighborhood.

(d) If Y is a subspace of X with a basis C, then X admits a basis B such that $C \subseteq B$.

Remark 2.3: Consider a Hilbert space H, and let Y be a closed subspace of H. The orthogonal complement $Y^\perp$ consists of all elements in H that are orthogonal to every element of Y. This complement is given by the null space of the orthogonal projection P that maps H onto Y, i.e., $Y^\perp = \mathcal{N}(P)$.

It can be noted that the algebraic local structural properties of a $\mathcal{TVS}$ is related with some specific families of subsets of X. Among those families; family of *zero-sets and* $z - filters$ [6] are very important in our study.

Let X be a $\mathcal{VS}$ and T is a linear transformation on X, $zero - set$ of a the linear transformation T, also known as the $null\ space$, is the set of vectors in X which are mapped to the zero vector [11]. ie; $Z(T) = Ker(T) = N(T) = \{x \in X : T(x) = 0\}$. Evidently it is a subspace of X. Now a subset Z of X is said to be a $Zero - set$ in X if it is the $Zero - set$ of some linear transformation T on X. Note that the $Zero - set$ of the zero transformation 0 is $\mathbf{Z}(0) = X$, and $Zero - set$ of the identity transformation I is $\mathbf{Z}(I) = \{\mathbf{0}\}$. Let $\mathfrak{P}$ be a family of linear transformations on X, then $Z(\mathfrak{P})$ denotes the collection of all $Zero - set$s $Z(f)$ such that $f \in \mathfrak{P}$.

A nonempty collection $\mathfrak{F}$ of $Zero - set$s in X is called a z-filter on X (see [6]) if it satisfies the following conditions:

(i) $\emptyset \notin \mathfrak{F}$.

(ii) The intersection of any two sets in $\mathfrak{F}$ is also in $\mathfrak{F}$, meaning that $\mathfrak{F}$ is closed under finite intersections.

(iii) Z belongs to $\mathfrak{F}$ and Z' is any $Zero - set$ in X with $Z \subseteq Z'$, then Z' must also be in $\mathfrak{F}$

These conditions ensure that a $z - filter$ behaves analogously to a filter in topology, but within the structure of $Zero - set$s.

3. Basics of ring of linear transformations on a $\mathcal{TVS}$

In this section we mainly consider an algebraic structure on the carrier set X, which is a $\mathcal{TVS}$ and we consider some of its substructures and reveals some important properties in connection with the topology on the carrier set.

Let us consider the collection $\mathcal{L}(X)$ of all linear transformations from a $\mathcal{VS}$ X into itself and define an addition and multiplication (composition) on $\mathcal{L}(X)$ by

$$(\varphi + \psi)(x) = \varphi(x) + \psi(x)$$

$$(\varphi\psi)(x) = \varphi(\psi(x)), \quad \forall\ x \in X \text{ and } \varphi, \psi \in \mathcal{L}(X)$$

Since the zero operator and identity operator are elements of $\mathcal{L}(X)$, and for each $\varphi \in \mathcal{L}(X)$ the function $-\varphi$ defined by $(-\varphi)(x) = -\varphi(x)$ is also in $\mathcal{L}(X)$, it is obvious that $\mathcal{L}(X)$ is a ring with unity. It is to be noted that the identity transformation is continuous, also the sum and composition of two continuous linear transformation is again continuous. More over if φ is a linear continuous map on X, then so is $-\varphi$. Thus the set $C_L(X)$ of all continuous linear transformations from a $\mathcal{TVS}$ X into itself is a subring of $\mathcal{L}(X)$ with unity.

Remark 3.1: If the $\mathcal{TVS}$ X is of finite dimension, by Remark 2.1 every linear transformation on X is continuous and hence $L(X) = C_L(X)$.

Theorem 3.1: *If X is an infinite dimensional first countable $\mathcal{TVS}$, then there is a discontinuous linear map from X into itself.*

Proof: Let X is first countable $\mathcal{TVS}$, then X has countable local base. By remark 2.2(c) X has a balanced local base $\{L_n\}$ such that

$$L_{n+1} + L_{n+1} + L_{n+1} + L_{n+1} \subset L_n \quad (n = 1,2,3,\dots)$$

Also by remark 2.2(a) every neighborhood of 0 contains a basis of X and hence for each $n \in \mathbb{N}$, L_n contains a basis of X. Thus we can construct a sequence $\{x_n\}$ such that $x_1 \in L_1$ and $x_n \in (L_n - Span\{x_1, x_2, \dots, x_{n-1}\})$ for $n = 2,3, \dots$. Then $\{x_n\}$ is an infinite linearly independent subset of X and it converges to 0. By remark 2.2(d), there exist a basis B of X containing all the terms of the sequence $\{x_n\}$.

Now define,

$$T(x) = \begin{cases} x_1 & \text{if } x \in S = \{x_n\} \quad \text{for} \quad n = 1,2,3,\dots \\ 0 & \text{if } x \in B \backslash S. \end{cases}$$

Since B is a basis of X, we can extend T linearly to all of X. Hence $T \in \mathcal{L}(X)$. Also note that $x_n \to 0$ as $n \to \infty$ but $T(x_n) = x_1 \neq 0$ for all n. Thus T is a discontinuous linear map on X.

In exploring the relationships between the structural properties of a $\mathcal{TVS}$ X and the function space $C_L(X)$, subsets of X of the form

$$\varphi^{-1}(y) = \{x \in X | \varphi(x) = y\}, \quad \varphi \in C_L(X), \qquad y \in X$$

play a significant role. Given that the space is Hausdorff, singletons are closed, which implies that the sets $\varphi^{-1}(y)$ are closed in X for all $\varphi \in C_L(X)$ and $y \in X$.

Now, consider the zero-set $\mathbf{Z}(\varphi)$ associated with a function $\varphi \in C_L(X)$. The mapping $\mathbf{Z}$ assigns to each function in $C_L(X)$ a corresponding zero-set in X, thereby establishing a connection between the algebraic structure of $C_L(X)$ and the topology of X.

Here onwards we denote $Z(C_L(X))$ by $Z(X)$ and an $z - filters$ $\mathfrak{F}$ is taken as a subfamily of $Z(X)$ unless otherwise specified. The following Proposition gives some basic results which trivially follows from the definition of $Zero - set$.

Proposition 3.1: Let (X, τ) be a $\mathcal{TVS}$ and $C_L(X)$ is the collection of all continuous linear transformations on X and $\psi, \chi \in C_L(X)$, then

1. $\mathbf{Z}(\psi) \neq \phi$
2. $\mathbf{Z}(\chi) \subseteq \mathbf{Z}(\psi \circ \chi)$.
3. If $n, m \in \mathbf{N}$ and $n \leq m$ then $\mathbf{Z}(\psi^n) \subseteq \mathbf{Z}(\psi^m)$ (ψ^n -composition of ψ, n times)
4. $\mathbf{Z}(\psi \circ \chi) = \chi^{-1}(B)$, where $B = \mathbf{Z}(\psi) \cap \mathbf{R}(\chi)$ and $\mathbf{R}(\chi) = \{\chi(x): x \in X\}$.
5. $\mathbf{Z}(\psi + \chi) \cap \mathbf{Z}(\psi - \chi) = \mathbf{Z}(\psi) \cap \mathbf{Z}(\chi)$.

Remark 3.2: It is interesting to see that the zero-sets in a $\mathcal{TVS}$ X corresponding to the elements in $C_L(X)$ are closed subspace of X. But in general not all the closed subspaces of X are zero-sets in X corresponding to some element in $C_L(X)$ [9].

4. Left ideals and $z - filters$

In this section, we examine the family of $Zero - sets$ and *z-filters*, highlighting how their properties contribute to the analysis of our $\mathcal{TVS}$. A subset I of a ring R is called a **left ideal** in R if it satisfies the following conditions: I is a subring, and for any $\varphi \in I$ and $\psi \in R$, the product $\psi\varphi$ belongs to I. Clearly, a left ideal I contains the unit element if and only if $I = R$. Furthermore, the intersection of any nonempty collection of left ideals remains a left ideal. The smallest left ideal containing a given family of left ideals is referred to as the left ideal generated by that family.

Now, consider an operator $\varphi \in C_L(X)$ and its associated $Zero - set$, denoted by $Z(\varphi)$. The $Zero - set$ of a collection $\mathfrak{F} \subseteq C_L(X)$ is defined as

$$Z(\mathfrak{F}) = \cap_{\varphi \in \mathfrak{F}} Z(\varphi),$$

which is typically represented as $Z_{\mathfrak{F}}$. It is worth noting that for each $\varphi \in C_L(X)$, the set $Z(\varphi)$ forms a subspace of X. Consequently, for any subset $\mathfrak{F} \subseteq C_L(X)$, the set $Z_{\mathfrak{F}}$ is also a subspace of X, since the intersection of any collection of subspaces in a $\mathcal{VS}$ remains a subspace.

Theorem 4.1*: If Z is a subspace of X, then $I_Z = \{\varphi \in C_L(X) \ : \ \varphi(x) = 0 \ \forall \ x \in Z\}$ is a left ideal in $C_L(X)$.*

Proof: First note that I_Z is subring of $C_L(X)$. Also for any $\psi \in C_L(X)$ and $\varphi \in I_Z$, $(\psi\varphi)(x) = \psi(\varphi(x)) = \psi(0) = 0$ for all $x \in Z$. Thus $\psi\varphi \in I_Z$. Hence, I_Z is a left ideal in $C_L(X)$.

Remark 4.1: Consider the collection $Z(\mathcal{L}(X))$. Since $0 \in Z(\varphi)$ for every $\varphi \in \mathcal{L}(X)$, and given any nonempty subspace V of X, there exists some $\varphi \in \mathcal{L}(X)$ such that $Z(\varphi) = V$, it follows that $Z(\mathcal{L}(X))$ forms a z-filter on X. Moreover, if X is a separable Banach space or a Hilbert space, then the family $Z(X)$ also constitutes a z-filter on X [9].

Theorem 4.2: *If $\mathfrak{F}$ is a z-filter on X, then the set*

$$\mathbf{Z}^{\leftarrow}[\mathfrak{F}] = \{\varphi \in C_L(X) | \mathbf{Z}(\varphi) \in \mathfrak{F}\}$$

forms a left ideal in $C_L(X)$.

Proof: Let $\varphi, \psi \in \mathbf{Z}^{\leftarrow}[\mathfrak{F}]$ and $\chi \in C_L(X)$. Then $\mathbf{Z}(\varphi)$ and $\mathbf{Z}(\psi)$ are contained in $\mathfrak{F}$. Now by Proposition 3.1 $\mathbf{Z}(\varphi - \psi) \in \mathfrak{F}$ and $\mathbf{Z}(\chi\varphi) \in \mathfrak{F}$. Hence $\varphi - \psi \in \mathbf{Z}^{\leftarrow}[\mathfrak{F}]$ and $\chi\varphi \in \mathbf{Z}^{\leftarrow}[\mathfrak{F}]$. Thus, $\mathbf{Z}^{\leftarrow}[\mathfrak{F}]$ is a left ideal in $C_L(X)$.

Remark 4.2: In particular, by subspace property of a $\mathcal{VS}$, if $\{0\} \in \mathfrak{F} \Rightarrow \mathfrak{F} = \boldsymbol{Z}(X)$. Therefore $\boldsymbol{Z}^{\leftarrow}[\mathfrak{F}] = \boldsymbol{Z}^{\leftarrow}[\boldsymbol{Z}(X)] = C_L(X)$.

Remark 4.3: It can be noted that an $z - filter$ $\mathfrak{F}$ on a $\mathcal{TVS}$ X is a sub lattice of the lattice $P(X)$ of all subsets of X. As a lattice the $z - filter$ $\mathfrak{F}$ has a largest element precisely the join of all the elements in $\mathfrak{F}$, but it may or may not has a least element. If $\mathfrak{F}$ has a minimal element, then it is unique and it must be the least element of $\mathfrak{F}$. Also if X is an n dim$\mathcal{VS}$ then the length of any strictly decreasing chain in the $z - filter$ $\mathfrak{F}$ is atmost $n + 1$.

Theorem 4.3: *An $z - filter$ $\mathfrak{F}$ on X has a least element if $\mathfrak{F}$ is of finite cardinality or X is a $FinDimS$.*

Proof: Suppose that $\mathfrak{F}$ is of finite cardinality. Then $\bigcap_{Z\in\mathfrak{F}} Z \in \mathfrak{F}$, since $\mathfrak{F}$ is closed under finite intersections. Obviously $\bigcap_{Z\in\mathfrak{F}} Z$ is the least element in $\mathfrak{F}$.

If X is finite dimensional, then by Remark 4.3 the length of any strictly decreasing chain in the $z - filter$ $\mathfrak{F}$ is atmost $n + 1$ and hence $\mathfrak{F}$ has a minimal element, precisely the intersection of all elements in such a chain. Hence again from Remark 4.3 that $\mathfrak{F}$ has a least element.

Corollary 4.4: *Let X is a $FinDimS$. If $\mathfrak{F}$ is a $z - filter$ on X, then $\mathbf{Z}^{\leftarrow}[\mathfrak{F}] = I_{Z_0} = \{\varphi \in C_L(X) \ : \varphi(x) = 0 \ \ \forall x \in \boldsymbol{Z}_0\}$, where $\boldsymbol{Z}_0$ is the least element in $\mathfrak{F}$.*

Proof: By Theorem 4.3 $z - filter$ $\mathfrak{F}$ on X has a least element Z_0. Now consider $I_{Z_0} = \{\varphi \in C_L(X) \ : \varphi(x) = 0 \ \ \forall x \in \mathbf{Z}_0\}$. Note that if $\varphi \in I_{Z_0}$ then $Z(\varphi) \supseteq Z_0$ and hence $Z(\varphi) \in \mathfrak{F}$. Thus $\varphi \in \mathbf{Z}^{\leftarrow}[\mathfrak{F}]$. Conversely $\varphi \in \mathbf{Z}^{\leftarrow}[\mathfrak{F}]$ implies that $Z(\varphi) \supseteq Z_0$ and hence $\varphi \in I_{Z_0}$.

Let I be a left ideal in $C_L(X)$, denote the family $\{\mathbf{Z}(\varphi) \ : \ \ \varphi \in I\}$ by $\mathbf{Z}[I]$ and the set $\bigcap_{\varphi\in I} Z(\varphi)$ by $\mathbf{Z}_I$ (called the zero-set of the left ideal I) . It can be noted that since arbitrary intersection of subspaces of a $\mathcal{VS}$ is a subspace, $Z_I = \bigcap_{\varphi\in I} Z(\varphi)$ is a subspace of X. Hence by Theorem 4.2 $I_{Z_I} = \{\varphi \in C_L(X) \mid \varphi(x) = 0 \ \ \forall x \in Z_I\}$ is a left ideal in $C_L(X)$. By the construction of I_{Z_I}, it is obvious that $I \subseteq I_{Z_I}$.

Example 4.1: Let $\{e_1, e_2, \ldots.\}$ be an orthonormal basis of an infinite dimensional separable Hilbert space H. Consider the left ideal $I = \{\varphi \in C_L(H) : dim(R(\varphi)) < \infty\}$. Let $\varphi_i(e_j) = e_j$ for $j \leq i$ and $\varphi_i(e_j) = 0$ for $j > i$. Each φ_i is a projection function on H. By remarks 2.1, 2.3 and the construction of I, $\varphi_i \in I$ for $i = 1,2, \ldots$ Also $Z(\varphi_i)$ is the subspace generated by $\{e_{i+1}, e_{i+2}, \ldots\}$, that is $Z(\varphi_i) =< e_{i+1}, e_{i+2}, \ldots >$. Therefore $\bigcap_{i\in N} Z(\varphi_i) = \{0\}$. Each $\varphi_i \in I$ for $i = 1,2, \ldots$ implies $Z_I = \bigcap_{\varphi\in I} Z(\varphi) \subseteq \bigcap_{i\in N} Z(\varphi_i) = \{0\}$. Hence $Z_I = \{0\}$. The left ideal $I_{Z_I} = Z^{\leftarrow}[Z_I] = Z^{\leftarrow}[\{0\}] = C_L(H)$. Hence I_{Z_I} certainly contains I. However, $I_{Z_I} \neq I$. For instance, the identity transformation $\mathfrak{I} \in I_{Z_I} - I$, where $\mathfrak{I}(x) = x$ for all $x \in H$.

Here naturally arises the following questions about the relationship between these sets.

1. When $\mathbf{Z}[I]$ is an $z-filter$ on X?.
2. When a left ideal I satisfies the relation $I = I_{Z_I}$?.

The forthcoming discussions tried to answer these questions.

Theorem 4.5: *Let X be a $\mathcal{TVS}$, and let I be a left ideal in $C_L(X)$ satisfying $I = I_{Z_I}$. If the $Zero-set$ $\boldsymbol{Z}(X)$ is closed under finite intersections, then the collection $\boldsymbol{Z}[I]$ forms a z-filter on X.*

Proof: We verify the conditions required for $\mathbf{Z}[I]$ to be a z-filter:

(i) Since the empty set ϕ does not belong to $\mathbf{Z}[X]$, it follows that $\phi \notin \mathbf{Z}[I]$.

(ii) Let $\mathbf{Z}_1, \mathbf{Z}_2 \in \mathbf{Z}[I]$. Then, there exist $\varphi, \psi \in I$ such that $\mathbf{Z}_1 = \mathbf{Z}(\varphi)$ and $\mathbf{Z}_2 = \mathbf{Z}(\psi)$. Given that $\mathbf{Z}_I = \bigcap_{\varphi \in I} Z(\varphi)$, we have $\mathbf{Z}_I \subseteq \mathbf{Z}_1 \cap \mathbf{Z}_2$. Since $\mathbf{Z}(X)$ is closed under finite intersections, it follows that $\mathbf{Z}_1 \cap \mathbf{Z}_2 \in \mathbf{Z}(X)$. Consequently, there exists some $\chi \in C_L(X)$ such that $\mathbf{Z}(\chi) = \mathbf{Z}_1 \cap \mathbf{Z}_2$. From $\mathbf{Z}_I \subseteq \mathbf{Z}_1 \cap \mathbf{Z}_2$, we conclude that $\chi(x) = 0$ for all $x \in \mathbf{Z}_I$. Therefore, $\chi \in I_{Z_I} = I$, implying that $\mathbf{Z}_1 \cap \mathbf{Z}_2 = \mathbf{Z}(\chi) \in \mathbf{Z}[I]$.

(iii) Suppose $\mathbf{Z} \in \mathbf{Z}[I]$ and $\mathbf{Z}' \in \mathbf{Z}(X)$ satisfy $\mathbf{Z} \subset \mathbf{Z}'$. Since $\mathbf{Z}' \in \mathbf{Z}(X)$, there exists some $\varphi \in C_L(X)$ such that $\mathbf{Z}(\varphi) = \mathbf{Z}'$. Given that $\mathbf{Z} \subset \mathbf{Z}'$, we see that $\varphi(x) = 0$ for all $x \in \mathbf{Z}$. Since $\mathbf{Z} \in \mathbf{Z}[I]$, there exists $\psi \in I$ such that $\mathbf{Z}(\psi) = \mathbf{Z}$. This gives $\mathbf{Z}_I \subseteq \mathbf{Z} \subseteq \mathbf{Z}'$, implying $\varphi(x) = 0$ for all $x \in \mathbf{Z}_I$. Therefore, $\varphi \in I_{Z_I} = I$, which leads to $\mathbf{Z}' \in \mathbf{Z}[I]$.

Thus, we conclude that $\mathbf{Z}[I]$ is a z-filter on X.

Lemma 4.1: *Let X be an n-dimensional $\mathcal{TVS}$, and let I be a left ideal in $C_L(X)$. If there exists an element $\varphi \in I$ such that $dim(\boldsymbol{Z}_I) = dim(\boldsymbol{Z}(\varphi))$, then $I = I_{Z_I}$.*

Proof: Assume that $\dim(\mathbf{Z}_I) = \dim(\mathbf{Z}(\varphi)) = k$. Since $\varphi \in I$, it follows that $\mathbf{Z}_I$ is a subspace of $\mathbf{Z}(\varphi)$. Given that they share the same dimension, we conclude that $\mathbf{Z}_I = \mathbf{Z}(\varphi)$. Furthermore, since $\dim(\mathbf{Z}(\varphi)) = k$, we deduce that $\dim(R(\varphi)) = n - k$.

Let $\{e_1, e_2, \dots, e_k\}$ be a basis for $\mathbf{Z}(\varphi)$, and let $\{a_1, a_2, \dots, a_{n-k}\}$ be a basis for $R(\varphi)$. It follows that the preimages $\{\varphi^{-1}(a_1) = b_1, \varphi^{-1}(a_2) = b_2, \dots, \varphi^{-1}(a_{n-k}) = b_{n-k}\}$ are linearly independent. Consequently, the set $\{e_1, e_2, \dots, e_k, b_1, b_2, \dots, b_{n-k}\}$ forms a basis for X.

Additionally, we can extend $\{a_1, a_2, \dots, a_{n-k}\}$ to a basis $\{a_1, a_2, \dots, a_{n-k}, d_1, d_2, \dots, d_k\}$ for X. Now, define a linear transformation $\psi: X \to X$ by setting $\psi(a_i) = b_i$ for $i = 1,2, \dots, n-k$ and $\psi(d_i) = 0$ for $i = 1,2, \dots, k$.

Since I is a left ideal in $C_L(X)$ and $\varphi \in I$, it follows that $\chi = \psi \circ \varphi \in I$. Furthermore, we note that $\chi(e_i) = 0$ for all $i = 1,2,\dots,k$ and $\chi(b_i) = b_i$ for $i = 1,2,\dots,n-k$. Thus, $\dim(\mathbf{Z}(\chi)) = k$, implying that $\mathbf{Z}(\chi) = \mathbf{Z}_I$.

Now, consider any $\psi^* \in I_{Z_I}$. By definition, $\psi^*(e_i) = 0$ for all $i = 1,2,\dots,k$ and $\psi^*(b_i) = b_i$ for $i = 1,2,\dots,n-k$. This implies that $\psi^* \circ \chi = \psi^*$, leading to the conclusion that $\psi^* \in I$. Hence, $I_{Z_I} \subseteq I$. Since the reverse inclusion holds by definition, we obtain $I = I_{Z_I}$.

Theorem 4.6: *If X is an n-dimensional $\mathcal{TVS}$ and I is a left ideal in $C_L(X)$, then $I = I_{Z_I}$.*

Proof: By the construction of I_{Z_I}, it is evident that $I \subseteq I_{Z_I}$.

If $\dim(\mathbf{Z}_I) = n$, then $\mathbf{Z}_I = X$, which implies that $I = \{0\}$ and consequently, $I_{Z_I} = \{0\}$. Thus, in this case, we have $I = I_{Z_I}$.

For the case where $\dim(\mathbf{Z}_I) = k < n$, consider any $x \in X\backslash\mathbf{Z}_I$. By definition, there exists some $\varphi \in I$ such that $\varphi(x) \neq 0$. Given that $\varphi \in I$, it follows that $\dim(\mathbf{Z}(\varphi)) \geq k$.

If $\dim(\mathbf{Z}(\varphi)) = k$, then by Lemma 4.1, it follows that $I = I_{Z_I}$. Suppose instead that $\dim(\mathbf{Z}(\varphi)) = m$, where $k < m < n$. Choose a basis $\{e_1, e_2, \dots, e_k\}$ for $\mathbf{Z}_I$. Since $\mathbf{Z}_I \subset \mathbf{Z}(\varphi)$, we can extend this basis to a basis for $\mathbf{Z}(\varphi)$, denoted as $\{e_1, e_2, \dots, e_k, e_{k+1}, e_{k+2}, \dots, e_m\}$.

By the rank-nullity theorem, we have $\dim(R(\varphi)) = n - m$. Let $\{a_1, a_2, \dots, a_{n-m}\}$ be a basis for $R(\varphi)$. Then the set $\{e_1, e_2, \dots, e_m, \varphi^{-1}(a_1) = b_1, \dots, \varphi^{-1}(a_{n-m}) = b_{n-m}\}$ forms a basis for X.

Following a similar approach as in Lemma 4.1, we can construct $\varphi_1 \in I$ such that $\varphi_1(e_i) = 0$ for $i = 1,\dots,m$ and $\varphi_1(b_i) = b_i$ for $i = 1,\dots,n-m$. Since $e_{k+1} \notin \mathbf{Z}_I$, it follows that $e_{k+1} \in X\backslash\mathbf{Z}_I$. Thus, there exists a nonzero vector $c \in X$ and a function $\psi \in I$ such that $\psi(e_{k+1}) = c$.

Extend $\{c\}$ to form a basis of X, say $\{c, c_1, c_2, \dots, c_{n-1}\}$. Then there exists a function $\chi \in C_L(X)$ such that $\chi(c) = e_{k+1}$ and $\chi(c_i) = 0$ for $i = 1,2,\dots,n-1$. Define $\chi_1 = \chi \circ \psi$. Then $\chi_1(e_{k+1}) = e_{k+1}$, and since $\psi \in I$, it follows that $\chi_1 \in I$.

Next, define $\varphi_2 \in C_L(X)$ as $\varphi_2 = \varphi_1 + \chi_1$. Since both φ_1 and χ_1 belong to I, we conclude that $\varphi_2 \in I$. Moreover,

$$\varphi_2(e_{k+1}) = \varphi_1(e_{k+1}) + \chi_1(e_{k+1}) = e_{k+1}$$

and

$$\varphi_2(b_i) = \varphi_1(b_i) + \chi_1(b_i) = b_i + \alpha_i e_{k+1}, \quad \text{for} i = 1,2,\dots,n-m,$$

where each α_i is a scalar from the underlying field of X. Since the set $\{b_1, b_2, \dots, b_{n-m}, e_{k+1}\}$ is linearly independent, it follows that the set $\{e_{k+1}, b_i + \alpha_i e_{k+1} | i = 1,\dots,n-m\}$ is also linearly independent and contained in $R(\varphi_2)$. Consequently, we have

$$\dim(R(\varphi_2)) \geq n - m + 1.$$

If $\dim(R(\varphi_2)) = n - k$, then by Lemma 4.1, we conclude that $I = I_{Z_I}$. Otherwise, we can construct a function $\varphi^* \in I$ in a finite number of steps such that $\dim(R(\varphi^*)) = n - k$.

Corollary 4.7: *If X is a FinDim $\mathcal{TVS}$, then every left ideal I in $C_L(X)$ form a $z - filter$ on X.*

Continuing our study of the relation between I_{Z_I} and $z - filter$, we now introduce a weak topology on $C_L(X)$. Let $x_o \in X$. Define $I_{x_o} = \{\varphi \in C_L(X) \mid \varphi(x_o) = 0\}$ and $S_{x_o} = {I_{x_o}}^c = C_L(X) - I_{x_o} = \{\varphi \in C_L(X) \mid \varphi(x_o) \neq 0\}$. Then we define weak topology τ on $C_L(X)$ as the topology generated by the family $\{S_x \mid x \in X\}$.

Lemma 4.2: *Let X be a $\mathcal{TVS}$, and let $\{x_1, x_2, \dots, x_n\}$ be a finite nonempty subset of X. Suppose that I is a left ideal in $C_L(X)$ such that, for each $i = 1, \dots, n$, there exists a function $\psi_i \in I$ satisfying $\psi_i(x_i) \neq 0$. Then, there exists a function $\chi \in I$ such that $\chi(x_i) \neq 0$ for all $i = 1,2, \dots, n$.*

Proof: We construct χ inductively. Since each $\psi_i \in I$ satisfies $\psi_i(x_i) \neq 0$, we can select scalars $k_m \in F$ (where F is the field associated with the $\mathcal{TVS}$ X) such that

$$\sum_{i=1}^{m-1} \psi_i(x_j) \neq -\sum_{i=m}^{n} k_m \psi_i(x_i), \quad \forall j = 1, \dots, n.$$

Define the function $\chi_i: X \to X$ by $\chi_i(x) = k_i x$, and set

$$\chi = \sum_{i=1}^{n} (\chi_i \circ \psi_i).$$

Since I is a left ideal in $C_L(X)$, it follows that $\chi \in I$. Furthermore, by the chosen construction, we ensure that $\chi(x_i) \neq 0$ for all $i = 1,2, \dots, n$.

Theorem 4.8: *Let X be a $\mathcal{TVS}$, and let I be a left ideal in $C_L(X)$ that is closed in $C_L(X)$ with respect to the weak topology τ. Then, we have $I = I_{Z_I}$.*

Proof: By definition of I_{Z_I}, it is clear that $I \subseteq I_{Z_I}$.

Now, let $\varphi \in I_{Z_I}$ and define $N_\varphi = \{x \in X | \varphi(x) \neq 0\}$. Consider Λ as the collection of all finite subsets of N_φ. Since I is closed in the weak topology, let B_φ be a basic open set in $C_L(X)$ that contains φ. Then, there exists a finite subset $G = \{x_1, x_2, \dots, x_n\} \in \Lambda$ such that

$$B_\varphi = \{\psi \in C_L(X) | \psi(x_i) \neq 0 \text{forall} i = 1, \dots, n\}.$$

Since $\varphi(x_i) \neq 0$ for each $i = 1,2, \dots, n$, we conclude that $x_i \notin Z_I$ for all i. By the definition of Z_I, there exist functions $\psi_i \in I$ such that $\psi_i(x_i) \neq 0$ for each i.

Applying Lemma 4.2, we obtain a function $\chi \in I$ satisfying $\chi(x_i) \neq 0$ for all $i = 1,2, \dots, n$. Thus, $\chi \in B_\varphi$, meaning that every basic open set containing φ also contains an element of I. Since I is closed in the weak topology, it follows that $\varphi \in I$.

This implies $I_{Z_I} \subseteq I$, and hence $I = I_{Z_I}$.

Corollary 4.9: *Let X be a $\mathcal{TVS}$, and let I be a left ideal in $C_L(X)$ that is closed in $C_L(X)$ with respect to the weak topology τ. If the zero-set $\boldsymbol{Z}(X)$ of X is closed under finite intersections, then the family $\boldsymbol{Z}[I]$ forms a z-filter on X.*

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