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Riemann-Liouville fractional operator on S-function

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Abstract

This paper provides a brief overview of the Riemann-Liouville fractional operator (R_l- operator) on S-Function. The key findings illustrate the impact of these operators on the parameters, showing that both the R_l- fractional operators of differential and integrals involving the S-Function can be represented using S-Function. These results can be used to derive specific cases by adjusting the parameters accordingly.

Subject Classification: *44C20, 44A20.*

Keywords: *Riemann-Liouville fractional operators, S-function, Modified beta function.*

1. Introduction

Fractional Calculus (F.C.) has gained attention in recent years, emerging as a rapidly expanding area of mathematical analysis. This field focuses on applied mathematics that involves arbitrary order derivatives and integrals. [1-3]. The F.C. operators that connect numerous special functions have become more vital and are applied in various branches of mathematical analysis. Applications of F.C. are prevalent in areas such as "astrophysics, turbulence, nonlinear biological systems, fluid dynamics, stochastic dynamical systems, plasma physics, nonlinear control theory, image processing, and quantum mechanics [4-5]. Over the past four decades, many researchers have extensively studied the

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properties, applications, and various extensions of different fractional calculus operators. Comprehensive discussions on generalized F.C. operators, their properties, and applications are observed in sources and research monographs [6-7].

Recently, several research publications have been published regarding generalized classical F.C. operators [8-9]. The R_l- operator [10] is a fundamental tool in fractional calculus, with numerous applications across physics, engineering, biology, and other fields where systems demonstrate memory and complex behaviour. The R_l- operator of integral of order $\alpha > 0$ for an integrable function $f: (0, \infty) \to R$ as

$$I_y^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds, t > 0 \tag{1}$$

R_l- operator of derivative of order $\alpha \in (n-1, n]$ for an integrable function $f: (0, \infty) \to R$ as

$$D_y^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t (t-s)^{n-\alpha+1} f(s) ds, t > 0 \tag{2}$$

The beta function [11] was introduced by Leonard Euler in the 18th century and is defined as

$$B(\lambda, \rho) = B(\rho, \lambda) = \frac{\Gamma(\lambda)\Gamma(\rho)}{\Gamma(\lambda+\rho)} \tag{3}$$

The modified beta function [12] is defined as:

$$\int_{\alpha}^{\beta} (\beta - x)^{\rho-1} (x-\alpha)^{\lambda-1} dx = (\beta-\alpha)^{\lambda+\rho-1} B(\lambda, \rho), R(\lambda) > 0, R(\rho) > 0 \tag{4}$$

Saxena and Daiya define the S-Function [13] as

$$S_{(\mathfrak{a},\mathfrak{b})}^{(\delta,\eta,\zeta,\beta,k)}[\mathfrak{p}_1 \ldots \mathfrak{p}_{\mathfrak{a}}; \mathfrak{q}_1 \ldots \mathfrak{q}_{\mathfrak{b}}; z] = \sum_{m=0}^{\infty} \frac{(\mathfrak{p}_1)_m \cdots (\mathfrak{p}_{\mathfrak{a}})_m (\zeta)_{m\beta,k} z^m}{(\mathfrak{q}_1)_m \cdots (\mathfrak{q}_{\mathfrak{b}})_m \Gamma_k(m\delta+\eta) m!} \tag{5}$$

$k \in R, \delta,\ \eta,\ \zeta,\ \beta \in C, Re(\delta) > 0, \mathfrak{p}_i (i = 1,2,3 \ldots \mathfrak{a}), \mathfrak{q}_j (j = 1,2,3 \ldots \mathfrak{b})$, $Re(\delta) > kRe(\beta)$, and $\mathfrak{a} < \mathfrak{b} + 1$. According to Diaz and Pariguan [14], the k-gamma function and the k-Pochhammer symbol [14] are defined as follows:

$$(\zeta)_{m,k} = \begin{cases} \frac{\Gamma_k(\zeta+mk)}{\Gamma_k(\zeta)},\ k \in R, \zeta \in C/\{0\}, k \in N \\ \zeta(\zeta+k) \ldots (\zeta+(m-1)k),\ m \in C, \zeta \in C \end{cases} \tag{6}$$

Special Cases

When $\mathfrak{a} = \mathfrak{b} = 0$ the S-function simplifies to the generalized k-Mittag-Leffler function [15], as defined by Saxena et al.

$$S_{(0,0)}^{(\delta,\eta,\zeta,\beta,k)}[-;-;z] = \sum_{m=0}^{\infty} \frac{(\zeta)_{m\beta,k} z^m}{\Gamma_k(m\delta+\eta) m!} = E_{k,\delta,\eta}^{\zeta,\beta}(z), R\left(\frac{\delta}{k} - \beta\right) > \mathfrak{a} - \mathfrak{b} \tag{7}$$

When $k = \beta = 1$ the S-function simplifies to generalized K-function, as defined by Sharma [16]:

$$S_{(\mathfrak{a},\mathfrak{b})}^{(\delta,\eta,\zeta,1,1)}[\mathfrak{p}_1 \ldots \mathfrak{p}_{\mathfrak{a}}; \mathfrak{q}_1 \ldots \mathfrak{q}_{\mathfrak{b}}; z] = \sum_{m=0}^{\infty} \frac{(\mathfrak{p}_1)_m \cdots (\mathfrak{p}_{\mathfrak{a}})_m (\zeta)_m z^m}{(\mathfrak{q}_1)_m \cdots (\mathfrak{q}_{\mathfrak{b}})_m \Gamma(m\delta+\eta) m!}$$

$$= K_{(\mathfrak{a},\mathfrak{b})}^{(\delta,\eta,\zeta)}[\mathfrak{p}_1, \ldots, \mathfrak{p}_{\mathfrak{a}}; \mathfrak{q}_1, \ldots, \mathfrak{q}_{\mathfrak{b}}; z], R(\delta) > \mathfrak{a} - \mathfrak{b} \tag{8}$$

When $\beta = k = \zeta = 1$ the S-function simplifies to generalized M-series defined by Sharma and Jain [17]:

$$S_{(\mathfrak{a},\mathfrak{b})}^{(\delta,\eta,1,1,1)}[\mathfrak{p}_1 \dots \mathfrak{p}_\mathfrak{a}; \mathfrak{q}_1 \dots \mathfrak{q}_\mathfrak{b}; z] = \sum_{m=0}^{\infty} \frac{(\mathfrak{p}_1)_m \cdots (\mathfrak{p}_\mathfrak{a})_m z^m}{(\mathfrak{q}_1)_m \cdots (\mathfrak{q}_\mathfrak{b})_m \Gamma(m\delta+\eta) m!}$$

$$= M_{(\mathfrak{a},\mathfrak{b})}^{(\delta,\eta)}[\mathfrak{p}_1, \dots, \mathfrak{p}_\mathfrak{a}; \mathfrak{q}_1, \dots, \mathfrak{q}_\mathfrak{b}; z], R(\delta) > \mathfrak{a} - \mathfrak{b} - 1 \tag{9}$$

2. Riemann-Liouville Fractional Operator of S-Function

In this section, we will discuss the results related to the S-Function when subjected to the R_l- operator of fractional derivatives and integrals.

Theorem 2.1: *Let* $\alpha > 0,\ k \in R, \delta, \eta, \zeta, \beta \in C, Re(\delta) > 0$ *and* I_y^α *be the* R_l-*operator of fractional integral then there holds the relation:*

$$I_y^\alpha S_{(\mathfrak{a},\mathfrak{b})}^{(\delta,\eta,\zeta,\beta,k)}(y) = \frac{y^\alpha}{\Gamma(1+\alpha)} S_{(\mathfrak{a}+1,\mathfrak{b}+1)}^{(\delta,\eta,\zeta,\beta,k)}[\mathfrak{p}_1 \dots \mathfrak{p}_\mathfrak{a}, 1; \mathfrak{q}_1 \dots \mathfrak{q}_\mathfrak{b}, (1+\alpha); y] \tag{10}$$

Proof: Consider the R_l- operator of fractional integral (with the lower limit set $a = 0$ to the variable y) applied to the S-Function.

$$I_y^\alpha S_{(\mathfrak{a},\mathfrak{b})}^{(\delta,\eta,\zeta,\beta,k)}(y) = \frac{1}{\Gamma(\alpha)} \int_0^y (y-t)^{\alpha-1} S_{(\mathfrak{a},\mathfrak{b})}^{(\delta,\eta,\zeta,\beta,k)} t dt \tag{11}$$

On using definition (5) on eq (11) we get,

$$= \frac{1}{\Gamma(\alpha)} \int_0^y (y-t)^{\alpha-1} \sum_{m=0}^{\infty} \frac{(\mathfrak{p}_1)_m \cdots (\mathfrak{p}_\mathfrak{a})_m (\zeta)_{m\beta,k} t^m}{(\mathfrak{q}_1)_m \cdots (\mathfrak{q}_\mathfrak{b})_m \Gamma_k(m\delta+\eta) m!} dt \tag{12}$$

subsequently altering the integration and summation order,

$$= \frac{1}{\Gamma(\alpha)} \sum_{m=0}^{\infty} \frac{(\mathfrak{p}_1)_m \cdots (\mathfrak{p}_\mathfrak{a})_m (\zeta)_{m\beta,k}}{(\mathfrak{q}_1)_m \cdots (\mathfrak{q}_\mathfrak{b})_m \Gamma_k(m\delta+\eta) m!} \int_0^y (y-t)^{\alpha-1} t^m dt \tag{13}$$

By using definition (3) and (4)

$$= \frac{1}{\Gamma(\alpha)} \sum_{m=0}^{\infty} \frac{(\mathfrak{p}_1)_m \cdots (\mathfrak{p}_\mathfrak{a})_m (\zeta)_{m\beta,k}}{(\mathfrak{q}_1)_m \cdots (\mathfrak{q}_\mathfrak{b})_m \Gamma_k(m\delta+\eta) m!} y^{m+\alpha} B(m+1, \alpha) \tag{14}$$

$$= y^\alpha \sum_{m=0}^{\infty} \frac{(\mathfrak{p}_1)_m \cdots (\mathfrak{p}_\mathfrak{a})_m (\zeta)_{m\beta,k}}{(\mathfrak{q}_1)_m \cdots (\mathfrak{q}_\mathfrak{b})_m \Gamma_k(m\delta+\eta) m!} y^m \frac{\Gamma(m+1)}{\Gamma(m+1+\alpha)} \tag{15}$$

$$= \frac{y^\alpha}{\Gamma(1+\alpha)} \sum_{m=0}^{\infty} \frac{(\mathfrak{p}_1)_m \cdots (\mathfrak{p}_\mathfrak{a})_m (\zeta)_{m\beta,k} (1)_m y^m}{(\mathfrak{q}_1)_m \cdots (\mathfrak{q}_\mathfrak{b})_m \Gamma_k(m\delta+\eta) (1+\alpha)_m m!} \tag{16}$$

$$= \frac{y^\alpha}{\Gamma(1+\alpha)} S_{(\mathfrak{a}+1,\mathfrak{b}+1)}^{(\delta,\eta,\zeta,\beta,k)}[\mathfrak{p}_1 \dots \mathfrak{p}_\mathfrak{a}, 1; \mathfrak{q}_1 \dots \mathfrak{q}_\mathfrak{b}, (1+\alpha); y] \tag{17}$$

This demonstrates that the R_l- operator of fractional integral of the S-function results in another S-function with indices $\mathfrak{a}+1$ and $\mathfrak{b}+1$.

This concludes the proof of Theorem 2.1.

Theorem 2.2: *Let* $\alpha > 0,\ k \in R, \delta, \eta, \zeta, \beta \in C, Re(\delta) > 0$ *and* D_y^α *be the* R_l-*operator of fractional derivative then there holds the relation:*

$$D_y^\alpha S_{(\mathfrak{a},\mathfrak{b})}^{(\delta,\eta,\zeta,\beta,k)}(y) = \frac{y^{-\alpha}(m+n-\alpha)!}{\Gamma(n-\alpha+1)} \times \ S_{(\mathfrak{a}+1,\mathfrak{b}+1)}^{(\delta,\eta,\zeta,\beta,k)}[\mathfrak{p}_1 \dots \mathfrak{p}_\mathfrak{a}, 1; \mathfrak{q}_1 \dots \mathfrak{q}_\mathfrak{b}, (n-\alpha+1); y] \tag{18}$$

Proof: Consider the R_l- operator of fractional derivative (with the lower limit set $a = 0$ to the variable y applied to the S-Function.

$$D_y^\alpha S_{(\mathfrak{a},\mathfrak{b})}^{(\delta,\eta,\zeta,\beta,k)}(y) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dy}\right)^n \int_0^y (y-t)^{n-\alpha-1}\, S_{(\mathfrak{a},\mathfrak{b})}^{(\delta,\eta,\zeta,\beta,k)}(t)dt \tag{19}$$

On using definition (5) on eq (19) we get,

$$= \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dy}\right)^n \int_0^y (y-t)^{n-\alpha-1} \sum_{m=0}^{\infty} \frac{(\mathfrak{p}_1)_m \cdots (\mathfrak{p}_\mathfrak{a})_m (\zeta)_{m\beta,k} t^m}{(\mathfrak{q}_1)_m \cdots (\mathfrak{q}_\mathfrak{b})_m \Gamma_k(m\delta+\eta) m!} dt \tag{20}$$

subsequently altering the integration and summation order,

$$= \frac{1}{\Gamma(n-\alpha)} \sum_{m=0}^{\infty} \frac{(\mathfrak{p}_1)_m \cdots (\mathfrak{p}_\mathfrak{a})_m (\zeta)_{m\beta,k}}{(\mathfrak{q}_1)_m \cdots (\mathfrak{q}_\mathfrak{b})_m \Gamma_k(m\delta+\eta) m!} \left(\frac{d}{dy}\right)^n \int_0^y (y-t)^{n-\alpha-1} t^m dt \tag{21}$$

By using definition (3) and (4)

$$= \frac{1}{\Gamma(n-\alpha)} \sum_{m=0}^{\infty} \frac{(\mathfrak{p}_1)_m \cdots (\mathfrak{p}_\mathfrak{a})_m (\zeta)_{m\beta,k}}{(\mathfrak{q}_1)_m \cdots (\mathfrak{q}_\mathfrak{b})_m \Gamma_k(m\delta+\eta) m!} \left(\frac{d}{dy}\right)^n y^{m+n-\alpha} B(m+1, n-\alpha) \tag{22}$$

Now differentiating n times, the term $y^{m+n-\alpha}$ and using definition (6) we get

$$= \sum_{m=0}^{\infty} \frac{(\mathfrak{p}_1)_m \cdots (\mathfrak{p}_\mathfrak{a})_m (\zeta)_{m\beta,k}\,(m+n-\alpha)!}{(\mathfrak{q}_1)_m \cdots (\mathfrak{q}_\mathfrak{b})_m \Gamma_k(m\delta+\eta) m!} y^{m-\alpha} \frac{(1)_m}{(n-\alpha+1)_m} \tag{23}$$

$$= \frac{y^{-\alpha}(m+n-\alpha)!}{\Gamma(n-\alpha+1)} \sum_{m=0}^{\infty} \frac{(\mathfrak{p}_1)_m \cdots (\mathfrak{p}_\mathfrak{a})_m (\zeta)_{m\beta,k}(m+n-\alpha)!(1)_m y^m}{(\mathfrak{q}_1)_m \cdots (\mathfrak{q}_\mathfrak{b})_m \Gamma_k(m\delta+\eta)(n-\alpha+1)_m m!} \tag{24}$$

$$= \frac{y^{-\alpha}(m+n-\alpha)!}{\Gamma(n-\alpha+1)} S_{(\mathfrak{a}+1,\mathfrak{b}+1)}^{(\delta,\eta,\zeta,\beta,k)}[\mathfrak{p}_1 \ldots \mathfrak{p}_\mathfrak{a}, 1; \mathfrak{q}_1 \ldots \mathfrak{q}_\mathfrak{b}, (n-\alpha+1); y] \tag{25}$$

This demonstrates that the R_l- operator of fractional derivative of the S-function results in another S-function with indices $\mathfrak{a}+1$ and $\mathfrak{b}+1$.

This concludes the proof of Theorem 2.2.

3. Special Cases

3.1. If we put $\beta = k = \zeta = 1$ in theorem 2.1 then we obtain

$$I_y^\alpha K_{(\mathfrak{a},\mathfrak{b})}^{(\delta,\eta,\zeta)}(y) = \frac{y^\alpha}{\Gamma(1+\alpha)} K_{(\mathfrak{a}+1,\mathfrak{b}+1)}^{(\delta,\eta,\zeta)}[\mathfrak{p}_1 \ldots \mathfrak{p}_\mathfrak{a}, 1; \mathfrak{q}_1 \ldots \mathfrak{q}_\mathfrak{b}, (1+\alpha); y] \tag{26}$$

exact result for R_l- operator of fractional integral for K-Function in [16].

3.2. If we put $k = \beta = 1$ in theorem 2.1 then we obtain

$$I_y^\alpha M_{(\mathfrak{a},\mathfrak{b})}^{(\delta,\eta)}(y) = \frac{y^\alpha}{\Gamma(1+\alpha)} M_{(\mathfrak{a}+1,\mathfrak{b}+1)}^{(\delta,\eta)}[\mathfrak{p}_1 \ldots \mathfrak{p}_\mathfrak{a}, 1; \mathfrak{q}_1 \ldots \mathfrak{q}_\mathfrak{b}, (1+\alpha); y] \tag{27}$$

exact result for R_l- operator of fractional integral for M-series in [18].

4. Corollary

4.1. If we put $\beta = k = \zeta = 1$ in theorem 2.2 then we obtain

$$D_y^\alpha K_{(\mathfrak{a},\mathfrak{b})}^{(\delta,\eta,\zeta)}(y) = \frac{y^{-\alpha}(m+n-\alpha)!}{\Gamma(n-\alpha+1)} K_{(\mathfrak{a}+1,\mathfrak{b}+1)}^{(\delta,\eta,\zeta)}[\mathfrak{p}_1 \ldots \mathfrak{p}_\mathfrak{a}, 1; \mathfrak{q}_1 \ldots \mathfrak{q}_\mathfrak{b}, (n-\alpha+1); y] \tag{28}$$

as a result, for R_l- operator of fractional derivative for K-Function.

4.2. If we put $k = \beta = 1$ in theorem 2.2 then we obtain

$$D_y^{\alpha} M_{(\mathfrak{a},\mathfrak{b})}^{(\delta,\eta)}(y) = \frac{y^{-\alpha}(m+n-\alpha)!}{\Gamma(n-\alpha+1)} M_{(\mathfrak{a}+1,\mathfrak{b}+1)}^{(\delta,\eta)}[\mathfrak{p}_1 \ldots \mathfrak{p}_{\mathfrak{a}}, 1; \mathfrak{q}_1 \ldots \mathfrak{q}_{\mathfrak{b}}, (n-\alpha+1); y] \quad (29)$$

as a result, for R_l- operator of fractional derivative for M-series.

Conclusion

The study has established some theorems using the R_l- operator of the S-function. Some special cases of the R_l- operator. As mentioned, this function has been extensively studied in literature. Therefore, the results presented in this paper can be readily transformed into a comparable form of novel and intriguing integrals with different arguments through various suitable parametric substitutions. It is expected that some of the findings will be useful in addressing certain fractional order differential and integral equations encountered in physical sciences and engineering problems.

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