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Power of Leslie matrices : Diagonalization and finite differences methods

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Abstract

This work presents a numerical comparison between the matrix diagonalization method and the finite differences method in the calculus of power Leslie matrix. Our aim is to determine the inherent advantages and disadvantages of each approach, fostering a deep understanding of the studied problem and guiding the choice of the most suitable method in different contexts.

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1. Introduction

Leslie matrices, introduced by Patrick Leslie in the mid-20th century [1], are fundamental tools in modeling age-structured populations, widely used in population biology and demography. They provide a compact framework for analyzing growth, stability, and distribution across age groups, with applications extending beyond biology [2], [3] to fields like epidemiology [4], ecology [5], and social sciences [6]. Their strength lies in encapsulating age-specific reproduction and survival patterns to project future population structures, aiding decisions in population planning and intervention analysis.

This study focuses on the computational analysis of Leslie matrices using two methods: matrix diagonalization and the finite differences method, the latter based on recurrence relations such as generalized Fibonacci sequences [7], [8]. We numerically compare both approaches, as discussed in [9], [10], aiming to highlight their respective advantages and limitations in different scenarios. Beyond theoretical insights, the findings support practical decision-making in contexts requiring demographic forecasting and strategic planning.

This paper is organized as follows: Section 2 introduces definitions and applications of Leslie matrices and presents both methods—diagonalization and the difference equations approach—for computing powers and analyzing asymptotic behavior. Section 3 details numerical simulations and presents the obtained results.

2. Leslie Matrix and Numerical Methods

2.1 *Power of a Leslie Matrix*

In this section, we delve into the Leslie Model, initially introduced in [1], which serves as a fundamental tool for studying the population growth of a given species. This model partitions the population into age cohorts and tracks the quantity of individuals over time. Unlike continuous models, the Leslie Model discretizes time, allowing the association of a sequence of vectors describing population variations within each age cohort.

The Leslie model is described by the equations:

$$\begin{cases} x_0(t+1) & = & \sum_{k=0}^{r-1} f_k x_k(t), \\ x_{k+1}(t+1) & = & p_k x_k(t). \end{cases} \tag{1}$$

Here, $x_k(t)$ represents the number of individuals in age cohort k at time t. The term $x_0(t+1)$ is determined by the weighted sum of individuals across all age cohorts k, based on the fecundity coefficients f_k. In contrast, $x_{k+1}(t+1)$ is obtained by multiplying the number of individuals in age cohort k at time t by the survival probability p_k. The constants p_k represent the survival probabilities in each age cohort k, where $p_k > 0$ for $k = 0,1,\ldots,r-2$. Fecundity rates f_k are non-negative coefficients representing the reproduction rate in each age cohort k, with $f_j \geq 0$ for $j = 0,1,\ldots,r-1$. The parameter r indicates the number of age cohorts in which the population has been divided, allowing for a detailed analysis of the population's age distribution. Time is discretized, with t values representing discrete time intervals rather than continuous time.

These equations describe the behavior of population growth over time, providing a fundamental mathematical model for studying population dynamics across different age cohorts.

Consider the vector $N(n) = (x_0(n), x_1(n), \ldots, x_{r-1}(n))$, describing the population distribution in each age cohort at time $t = n$. We can rewrite equations (1) in matrix form $N(n+1) = \mathbb{L}(f,p) \cdot N(n)$, where $\mathbb{L}(f,p)$ is known as the Leslie matrix and takes the following form:

$$\mathbb{L}(f,p) = \begin{bmatrix} f_0 & f_1 & \cdots & f_{r-2} & f_{r-1} \\ p_0 & 0 & \ldots & 0 & 0 \\ 0 & p_1 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & p_{r-2} & 0 \end{bmatrix}. \tag{2}$$

The Leslie matrix can be expressed more succinctly using the notation $\mathbb{L}(f,p)$, where f, p are the vectors given by $f = [f_0, \ldots, f_{r-1}]$ and $p = [p_0, \ldots, p_{r-2}]$. Thus, for $n = 1,2, \ldots, k$, we have

$$N(k) = (\mathbb{L}(f,p))^k \cdot N(0). \tag{3}$$

Therefore, to determine the population behavior at any time $k \in \mathbb{N}$, it is sufficient to know the matrix $(\mathbb{L}(f,p))^k$ and perform the product of this matrix with the vector representing the initial population $N(0)$.

In the next subsections, the investigation of the power of a Leslie matrix, $(\mathbb{L}(f,p))^n$, is divided into two cases: the finite case, where $n \in \mathbb{N}$, and the asymptotic case where $n \to \infty$. Both cases are assessed using the methods known as diagonalization and finite difference equations.

2.2 *Diagonalization method*

The diagonalization method for calculating the power of a Leslie matrix involves decomposing $\mathbb{L}(f,p) = MDM^{-1}$, where M is the matrix of eigenvectors of $\mathbb{L}(f,p)$ and D is a diagonal matrix with entries corresponding to the eigenvalues of $\mathbb{L}(f,p)$. Thus,

$$(\mathbb{L}(f,p))^n = MD^nM^{-1}. \tag{4}$$

Through equation (4), it can be observed that to compute the power of a Leslie matrix for $n \in \mathbb{N}$, it is necessary to calculate the eigenvalues and eigenvectors of the matrix $\mathbb{L}(f,p)$.

Further details and other important results related to the process of the diagonalization of a Leslie matrix are shown [11] and [12], [13].

2.2.1 Asymptotic behavior

Consider a diagonalizable Leslie matrix $\mathbb{L}(f,p)$ with eigenvalues $\lambda_1 > |\lambda_2| \geq |\lambda_3| \geq \ldots |\lambda_r|$, where λ_1 is a dominant eigenvalue. Then $(\mathbb{L}(f,p))^n =$

MD^nM^{-1}, where $D = D(\lambda_1, \cdots, \lambda_r)$ is a diagonal matrix with eigenvalues entries and M is the matrix of eigenvectors of $\mathbb{L}(f,p)$. By using (3) and the previous relationship for $\mathbb{L}(f,p)$, we obtain

$$\lim_{k\to+\infty}\left(\frac{N(k)}{\lambda_1^k}\right) = c_1 v_1,$$

Therefore, according to the value of the eigenvalue $\lambda_1 > 0$, the population will exhibit the following behaviors:

1. If $0 < \lambda_1 < 1$, the population will decrease;
2. If $\lambda_1 = 1$, the population will converge to a certain value;
3. If $\lambda_1 > 1$, the population will continue to grow.

Observe that these cases are obtained for $\lambda_1 > |\lambda_k|$, with $k \neq 1$, but Lemma 2.3 only guarantees that $\lambda_1 \leq |\lambda_k|$. A Leslie matrix may have $\lambda_1 = |\lambda_k|$, for $k \neq 1$.

2.3 *Finite difference equations method*

The generalized Fibonacci equation, also known as a linear recurrence with constant coefficients, is defined by

$$v_{n+1} = a_0 v_n + \cdots + a_{r-1} v_{n-r+1}, \tag{5}$$

with $n \geq r-1$. The constants $a_0, \dots, a_{r-1}$ are the coefficients of this r-th order recurrence, and $v_0, \dots, v_{r-1}$ are the initial conditions that determine the uniqueness of the solution to this problem. Note that, similarly to equation (3), it is possible to rewrite (5) in matrix form:

$$\mathbb{V}_{n+1} = \mathbb{L}([a_0, \dots, a_{r-1}], [1, \dots, 1]) \cdot \mathbb{V}_n,$$

where $\mathbb{V}_n = (v_n, \dots, v_{n-r+1})$ and $\mathbb{L}([a_0, \dots, a_{r-1}], [1, \dots, 1])$ is the Leslie matrix as presented in (2).

Using the notation $\mathbb{A}[a_0, \dots, a_{r-1}] = \mathbb{L}([a_0, \dots, a_{r-1}], [1, \dots, 1])$, the goal is to determine the matrix $\mathbb{A}^n$ using the concepts of recurrence equations and from this determine the entries of the associated matrix $\mathbb{L}([a_0, \dots, a_{r-1}], [1, \dots, 1])$. For this, consider the family of sequences $\{v_n^{(s)}\}_{n\geq 0}$, with $0 \leq s \leq r-1$, defined by:

$$\begin{cases} v_{n+1}^{(s)} &= a_0 v_n^{(s)} + \cdots + a_{r-1} v_{n-r+1}^{(s)} \quad , \quad \text{for } n \geq r-1, \\ v_n^{(s)} &= \delta_{s,n} \quad , \quad \text{for } 0 \leq n \leq r-1, \end{cases} \tag{6}$$

where $\delta_{s,n}$ is the Kronecker symbol defined by $\delta_{s,n} = 0$ if $s \neq n$ and $\delta_{s,n} = 1$ if $s = n$. Using the notation employed in (6), it was introduced in [7] that the matrix $\mathbb{A}^n$ is given by

$$\mathbb{A}^n = \begin{pmatrix} v_{n+r-1}^{(r-1)} & \dots & v_{n+r-1}^{(0)} \\ \vdots & \ddots & \vdots \\ v_n^{(r-1)} & \dots & v_n^{(0)} \end{pmatrix}.$$

It also presents the relationship between the matrices $\mathbb{A}$ and $\mathbb{L}(f,p)$ through the matrix $\mathbb{H} = \mathrm{diag}(\omega_0, \omega_1, \dots, \omega_{r-1})$, where

$$\omega_s = \prod_{k=s}^{r-2} p_k \quad \text{and} \quad \omega_{r-1} = 1, \quad \text{with } 0 \le s \le r-2 \tag{7}$$

by the relation $\mathbb{L}(f,p) = \mathbb{H}^{-1}\mathbb{A}\mathbb{H}$ and the coefficients a_j have the following relationship:

$$a_0 = f_0, \quad a_1 = p_0 f_1, \quad a_2 = p_0 p_1 f_2, \quad \dots, \quad a_{r-1} = p_0 p_1 \dots p_{r-2} f_{r-1}. \tag{8}$$

Thus, obtaining $\mathbb{A}^n$ allows determining the entries of the matrix $(\mathbb{L}(f,p))^n$ as presented below:

$$(\mathbb{L}(f,p))^n = \begin{pmatrix} v_{n+r-1}^{(r-1)} & \frac{\omega_1}{\omega_0} v_{n+r-1}^{(r-2)} & \frac{\omega_2}{\omega_0} v_{n+r-1}^{(r-3)} & \cdots & \frac{\omega_{r-1}}{\omega_0} v_{n+r-1}^{(0)} \\ \frac{\omega_0}{\omega_1} v_{n+r-2}^{(r-1)} & v_{n+r-2}^{(r-2)} & \frac{\omega_2}{\omega_1} v_{n+r-2}^{(r-3)} & \cdots & \frac{\omega_{r-1}}{\omega_1} v_{n+r-2}^{(0)} \\ \frac{\omega_0}{\omega_2} v_{n+r-3}^{(r-1)} & \frac{\omega_1}{\omega_2} v_{n+r-3}^{(r-2)} & v_{n+r-3}^{(r-3)} & \cdots & \frac{\omega_{r-1}}{\omega_2} v_{n+r-3}^{(0)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\omega_0}{\omega_{r-1}} v_n^{(r-1)} & \frac{\omega_1}{\omega_{r-1}} v_n^{(r-2)} & \frac{\omega_2}{\omega_{r-1}} v_n^{(r-3)} & \cdots & v_n^{(0)} \end{pmatrix}. \tag{9}$$

It is observed that to determine the matrix $(\mathbb{L}(f,p))^n$, it is sufficient to calculate the terms $v_i^{(j)}$ described by the recurrences in (6), where i and j vary according to the order in the matrix $\mathbb{L}(f,p)$ and the power being raised, as outlined in equation (10). Some of the methods for performing this calculation include computing term by term using the recurrence equation and its initial conditions, obtaining the general solution of the recurrence for each initial condition, and precisely determining the required terms. Another method is described in Theorem 2.2 of the article [14], where, by determining the last column of the matrix $(\mathbb{L}(f,p))^n$, it is demonstrated that the remaining columns can be determined through linear combinations of them.

Proposition 2.5: *1Let $\{\{v_n^{(s)}\}_{n\ge0}; 0 \le s \le r-1\}$ be the r-generalized Fibonacci fundamental system. Then, for each j with $1 \le j \le r-1$, the term $v_n^{(j)}$ is expressed in term of elements of the sequence $\{v_n^{(r-1)}\}_{n\ge0}$, as follows:*

$$\begin{aligned} v_n^{(0)} &= a_{r-1} v_{n-1}^{(r-1)}, \qquad for\ n \ge 1, \\ v_n^{(j)} &= a_{r-j-1} v_{n-1}^{(r-1)} + a_{r-j-2} v_{n-2}^{(r-1)} + \cdots + a_{r-1} v_{n-j+1}^{(r-1)}, \end{aligned}$$

for $n \ge j+1$.

Therefore, by using Proposition 2.5 we have the following form for the $(\mathbb{L}(f,p))^n$, in terms of elements of the sequence $\{v_n^{(r-1)}\}_{n\ge0}$.

Theorem 2.6: *2Let $\{\{v_n^{(s)}\}_{n\ge0}; 0 \le s \le r-1\}$ be the r-generalized Fibonacci fundamental system and the matrix $\mathbb{L}(f,p) = \mathbb{H}^{-1}\mathbb{A}\mathbb{H}$, where the coefficients a_j are given by (8). Then, the following matrix form holds:*

$$(\mathbb{L}(f,p))^n = \begin{pmatrix} v_{n+r-1}^{(r-1)} & \cdots & \frac{\omega_{r-2}a_{r-2}}{\omega_0} v_{n+r-2}^{(r-1)} + \frac{\omega_{r-2}a_{r-1}}{\omega_0} v_{n+r-1}^{(r-1)} & \frac{\omega_{r-1}a_{r-1}}{\omega_0} v_{n+r-2}^{(r-1)} \\ \frac{\omega_0}{\omega_1} v_{n+r-2}^{(r-1)} & \cdots & \frac{\omega_{r-2}a_{r-2}}{\omega_1} v_{n+r-3}^{(r-1)} + \frac{\omega_{r-1}a_{r-1}}{\omega_1} v_{n+r-2}^{(r-1)} & \frac{\omega_{r-1}a_{r-1}}{\omega_1} v_{n+r-3}^{(r-1)} \\ \vdots & \ddots & \vdots & \vdots \\ \frac{\omega_0}{\omega_{r-1}} v_n^{(r-1)} & \cdots & \frac{\omega_{r-2}a_{r-2}}{\omega_{r-1}} v_{n-1}^{(r-1)} + \frac{\omega_{r-2}a_{r-1}}{\omega_{r-1}} v_n^{(r-1)} & a_{r-1} v_{n-1}^{(r-1)} \end{pmatrix}. \tag{10}$$

Applying equation (10), an important result is obtained.

Theorem 2.7: 3*Let* $\mathbb{L}([f_0,\dots,f_{r-1}],[p_0,\dots,p_{r-2}])$ *be a Leslie matrix of order* r *such that* $f_i = 0$ *for* $i = 0,1,\dots,r-2$. *Then,* $(\mathbb{L}(f,p))^r = p_0 \cdot \ldots \cdot p_{r-2} \cdot f_{r-1} \cdot I_r$, *where* I_r *is the identity matrix of order* r.

Proof: Calculating the values of ω_s for $0 \le s \le r-1$ through the equation (7) yields:

$$\omega_0 = p_0 p_1 \cdot \ldots \cdot p_{r-2},$$
$$\omega_1 = p_1 \cdot \ldots \cdot p_{r-2},$$
$$\vdots$$
$$\omega_{r-2} = p_{r-2},$$
$$\omega_{r-1} = 1.$$

Using the relations (6) and (8), we obtain

$$v_{m+1}^{(0)} = f_0 \cdot v_m^{(0)} + p_0 f_1 \cdot v_{m-1}^{(0)} + p_0 p_1 f_2 \cdot v_{m-2}^{(0)} + \ldots + p_0 p_1 \ldots p_{r-2} f_{r-1} v_{m-r}^{(0)}$$
$$v_{m+1}^{(0)} = p_0 p_1 \ldots p_{r-2} f_{r-1} v_{m-r}^{(0)},$$

where $v_0^{(0)} = 1$ and $v_1^{(0)} = v_2^{(0)} = \ldots = v_{r-1}^{(0)} = 0$, and

$$v_{m+1}^{(1)} = f_0 \cdot v_m^{(1)} + p_0 f_1 \cdot v_{m-1}^{(1)} + p_0 p_1 f_2 \cdot v_{m-2}^{(1)} + \ldots + p_0 p_1 \ldots p_{r-2} f_{r-1} v_{m-r}^{(1)}$$
$$v_{m+1}^{(1)} = p_0 p_1 \ldots p_{r-2} f_{r-1} v_{m-r}^{(1)},$$

where $v_1^{(1)} = 1$ and $v_0^{(1)} = v_2^{(1)} = \ldots = v_{r-1}^{(1)} = 0$. Similarly, we obtain $v_{m+1}^{(s)} = 1$ if $m+1 = s$ and $v_{m+1}^{(s)} = 0$ otherwise, for $2 \le s \le r-1$. For each equation, we aim to obtain the terms $v_r^{(s)}, v_{r+1}^{(s)}, \ldots, v_{2r-1}^{(s)}$, with $s = 0,1,2,\ldots,r-1$. Constructing each sequence recursively, we have

$$v^{(0)} = \{\underbrace{1,0,0,\ldots,0}_{r-\text{terms}}, \underbrace{p_0 p_1 \ldots p_{r-2} f_{r-1}, 0,0,\ldots,0}_{r-\text{terms}}, (p_0 p_1 \ldots p_{r-2} f_{r-1})^2, 0, \ldots\}$$

$$v^{(1)} = \{\underbrace{0,1,0,\ldots,0}_{r-\text{terms}}, \underbrace{0, p_0 p_1 \ldots p_{r-2} f_{r-1}, 0,\ldots,0}_{r-\text{terms}}, 0, (p_0 p_1 \ldots p_{r-2} f_{r-1})^2, \ldots\}$$
$$\vdots$$
$$v^{(r-1)} = \{\underbrace{0,0,0,\ldots,1}_{r-\text{terms}}, \underbrace{0,0,0,\ldots, p_0 p_1 \ldots p_{r-2} f_{r-1}}_{r-\text{terms}}, 0,0,\ldots\}$$

Therefore, through equation (10), the sought Leslie matrix is given by

$$(\mathbb{L}(f,p))^r = \begin{pmatrix} v_{r+r-1}^{(r-1)} & \frac{\omega_1}{\omega_0} v_{r+r-1}^{(r-2)} & \frac{\omega_2}{\omega_0} v_{r+r-1}^{(r-3)} & \dots & \frac{\omega_{r-1}}{\omega_0} v_{r+r-1}^{(0)} \\ \frac{\omega_0}{\omega_1} v_{r+r-2}^{(r-1)} & v_{r+r-2}^{(r-2)} & \frac{\omega_2}{\omega_1} v_{r+r-2}^{(r-3)} & \dots & \frac{\omega_{r-1}}{\omega_1} v_{r+r-2}^{(0)} \\ \frac{\omega_0}{\omega_2} v_{r+r-3}^{(r-1)} & \frac{\omega_1}{\omega_2} v_{r+r-3}^{(r-2)} & v_{r+r-3}^{(r-3)} & \dots & \frac{\omega_{r-1}}{\omega_2} v_{r+r-3}^{(0)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\omega_0}{\omega_{r-1}} v_r^{(r-1)} & \frac{\omega_1}{\omega_{r-1}} v_r^{(r-2)} & \frac{\omega_2}{\omega_{r-1}} v_r^{(r-3)} & \dots & v_r^{(0)} \end{pmatrix},$$

or equivalent,

$$(\mathbb{L}(f,p))^r = p_0 p_1 \dots p_{r-2} f_{r-1} \cdot \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix} \Rightarrow (\mathbb{L}(f,p))^r = p_0 p_1 \dots p_{r-2} f_{r-1} \cdot I_r,$$

where I_r is the identity matrix of order r, allowing us to conclude the proof.

2.3.1 Asymptotic behaviour

The asymptotic behavior of a Leslie model can be separated into the following cases. Consider that $\gcd\{j+1, a_j > 0\} = 1$, with $0 \le j \le r-1$, and $a_0 + a_1 + \dots + a_{r-1} = 1$, which is equivalent to saying that the dominant eigenvalue is $\lambda_1 = 1$. Then we have

$$\lim_{n\to+\infty} \mathbb{A}^n = \begin{pmatrix} \Pi(0) & \dots & \Pi(r-1) \\ \vdots & \ddots & \vdots \\ \Pi(0) & \dots & \Pi(r-1) \end{pmatrix},$$

where the entries $\Pi(s)$, with $0 \le s \le r-1$, are given by $\Pi(s) = \Pi(s,1)$, and

$$\Pi(k,q) = \frac{\sum_{j=k}^{r-1} \frac{a_j}{q^{j+1}}}{\sum_{j=0}^{r-1} (j+1)\frac{a_j}{q^{j+1}}}. \quad (11)$$

Thus the matrix entries are given by

$$\Pi(s) = \frac{\sum_{j=s}^{r-1} a_j}{\sum_{j=0}^{r-1} (j+1) a_j}, \text{with} 0 \le s \le r-1. \quad (12)$$

In this way, the asymptotic behavior of the Leslie matrix exponentiation is

$$\lim_{n\to+\infty} (\mathbb{L}(f,p))^n = \begin{pmatrix} \Pi(0) & \frac{\omega_1}{\omega_0} \cdot \Pi(1) & \dots & \frac{\omega_{r-1}}{\omega_0} \cdot \Pi(r-1) \\ \frac{\omega_0}{\omega_1} \cdot \Pi(0) & \Pi(1) & \dots & \frac{\omega_{r-1}}{\omega_1} \cdot \Pi(r-1) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\omega_0}{\omega_{r-1}} \cdot \Pi(0) & \frac{\omega_1}{\omega_{r-1}} \cdot \Pi(1) & \dots & \Pi(r-1) \end{pmatrix}.$$

Now consider that $\gcd\{j+1, a_j > 0\} = 1$, but $a_0 + a_1 + \dots + a_{r-1} \ne 1$. It means that the dominant eigenvalue $\lambda_1 \ne 1$. In this case, the matrix will tend to zero if $0 < \lambda_1 < 1$ or its coefficients will tend to infinity if $\lambda_1 > 1$. Thus, the asymptotic behavior of the Leslie matrix is already described. It is interesting to determine the population proportion, i.e., the proportion matrix Q is determined such that $\frac{Qx_0}{||Qx_0||_1}$ indicates the asymptotic behavior of the percentage of the population by age group.

To do this, dividing equation (10) by λ_1^n, we have:

$$\frac{(\mathbb{L}(f,p))^n}{\lambda_1^n} = \begin{pmatrix} \frac{v_{n+r-1}^{(r-1)}}{\lambda_1^n} & \frac{\omega_1}{\omega_0}\frac{v_{n+r-1}^{(r-2)}}{\lambda_1^n} & \frac{\omega_2}{\omega_0}\frac{v_{n+r-1}^{(r-3)}}{\lambda_1^n} & \cdots & \frac{\omega_{r-1}}{\omega_0}\frac{v_{n+r-1}^{(0)}}{\lambda_1^n} \\ \frac{\omega_0}{\omega_1}\frac{v_{n+r-2}^{(r-1)}}{\lambda_1^n} & \frac{v_{n+r-2}^{(r-2)}}{\lambda_1^n} & \frac{\omega_2}{\omega_1}\frac{v_{n+r-2}^{(r-3)}}{\lambda_1^n} & \cdots & \frac{\omega_{r-1}}{\omega_1}\frac{v_{n+r-2}^{(0)}}{\lambda_1^n} \\ \frac{\omega_0}{\omega_2}\frac{v_{n+r-3}^{(r-1)}}{\lambda_1^n} & \frac{\omega_1}{\omega_2}\frac{v_{n+r-3}^{(r-2)}}{\lambda_1^n} & \frac{v_{n+r-3}^{(r-3)}}{\lambda_1^n} & \cdots & \frac{\omega_{r-1}}{\omega_2}\frac{v_{n+r-3}^{(0)}}{\lambda_1^n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\omega_0}{\omega_{r-1}}\frac{v_n^{(r-1)}}{\lambda_1^n} & \frac{\omega_1}{\omega_{r-1}}\frac{v_n^{(r-2)}}{\lambda_1^n} & \frac{\omega_2}{\omega_{r-1}}\frac{v_n^{(r-3)}}{\lambda_1^n} & \cdots & \frac{v_n^{(0)}}{\lambda_1^n} \end{pmatrix}$$

Since $\lim_{n\to+\infty}\frac{v_n^{(s)}}{\lambda_1^n} = \frac{v_{n+1}^{(s)}}{\lambda_1^{n+1}} = \cdots = \frac{v_{n+r-1}^{(s)}}{\lambda_1^{n+r-1}} = \frac{\Pi(r-s-1;\lambda_1)}{\lambda_1^s}$, then the coefficients $L_{s;\lambda_1}$ are defined by

$$L_{s;\lambda_1} = \frac{\Pi(r-s-1;\lambda_1)}{\lambda_1^s}, \qquad 0 \le s \le r-1.$$

Thus, it is it possible to state that $\lim_{n\to+\infty}\frac{(\mathbb{L}(f,p))^n}{\lambda_1^n}$ converges to a matrix Q, such that

$$Q = \lim_{n\to+\infty}\frac{(\mathbb{L}(f,p))^n}{\lambda_1^n}$$

$$= \begin{pmatrix} \lambda_1^{r-1}L_{r-1;\lambda_1} & \lambda_1^{r-1}\frac{\omega_1}{\omega_0}L_{r-2;\lambda_1} & \lambda_1^{r-1}\frac{\omega_2}{\omega_0}L_{r-3;\lambda_1} & \cdots & \lambda_1^{r-1}\frac{\omega_{r-1}}{\omega_0}L_{0;\lambda_1} \\ \lambda_1^{r-2}\frac{\omega_0}{\omega_1}L_{r-1;\lambda_1} & \lambda_1^{r-2}L_{r-2;\lambda_1} & \lambda_1^{r-2}\frac{\omega_2}{\omega_1}L_{r-3;\lambda_1} & \cdots & \lambda_1^{r-2}\frac{\omega_{r-1}}{\omega_1}L_{0;\lambda_1} \\ \lambda_1^{r-3}\frac{\omega_0}{\omega_2}L_{r-1;\lambda_1} & \lambda_1^{r-3}\frac{\omega_1}{\omega_2}L_{r-2;\lambda_1} & \lambda_1^{r-3}L_{r-3;\lambda_1} & \cdots & \lambda_1^{r-3}\frac{\omega_{r-1}}{\omega_2}L_{0;\lambda_1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\omega_0}{\omega_{r-1}}L_{r-1;\lambda_1} & \frac{\omega_1}{\omega_{r-1}}L_{r-2;\lambda_1} & \frac{\omega_2}{\omega_{r-1}}L_{r-3;\lambda_1} & \cdots & L_{0;\lambda_1} \end{pmatrix}.$$

The other case consists of determining the asymptotic behavior of the Leslie model when $\gcd\{j+1, a_j > 0\} = d \ge 2$, with $0 \le j \le r-1$ and $\lambda_1 = 1$.

Under these conditions, the limit of the sequence $((\mathbb{L}(f,p))^{nd+l})_{n\ge0}$ for $l = 1,2,\ldots,d$ is given by:

$$\lim_{n\to+\infty}(\mathbb{L}(f,p))^{nd+l} =$$
$$\begin{pmatrix} q_l(0,0) & \frac{\omega_1}{\omega_0}\cdot q_l(0,1) & \dots & \frac{\omega_{r-1}}{\omega_0}\cdot q_l(0,r-1) \\ \frac{\omega_0}{\omega_1}\cdot q_l(1,0) & q_l(1,1) & \dots & \frac{\omega_{r-1}}{\omega_1}\cdot q_l(1,r-1) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\omega_0}{\omega_{r-1}}\cdot q_l(r-1,0) & \frac{\omega_1}{\omega_{r-1}}\cdot q_l(r-1,1) & \dots & q_l(r-1,r-1) \end{pmatrix},$$

where ω_i, with $0 \le i \le r-1$, is given as presented in equation (7), and $q_l(i,j)$ is given by

$$q_l(i,j) = \begin{cases} d\cdot\Pi(j), & \text{if } j \equiv i+l \bmod d \\ 0, & \text{otherwise} \end{cases},$$

where $\Pi(j)$ is described in (12).

Finally, considering $\gcd\{j+1, a_j > 0\} = d \ge 2$ and $a_0 + a_1 + \cdots + a_{r-1} \ne 1$, the population will grow indefinitely if $\lambda_1 > 1$ or tend to zero if $0 < \lambda_1 < 1$. Thus, the behavior of the proportion between age groups of the population can be calculated using the matrix $Q_l = \lim_{n\to+\infty}\frac{(\mathbb{L}(f,p))^{nd+l}}{\lambda_1^{nd+l}}$ given by

$$Q_l = \lim_{n\to+\infty}\frac{(\mathbb{L}(f,p))^{nd+l}}{\lambda_1^{nd+l}}$$

$$= \begin{pmatrix} q_l(0,0) & \lambda_1\frac{\omega_1}{\omega_0}\cdot q_l(0,1) & \dots & \lambda_1^{r-1}\frac{\omega_{r-1}}{\omega_0}\cdot q_l(0,r-1) \\ \frac{\omega_0}{\lambda_1\omega_1}\cdot q_l(1,0) & q_l(1,1) & \dots & \lambda_1^{r-2}\frac{\omega_{r-1}}{\omega_1}\cdot q_l(1,r-1) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\omega_0}{\lambda_1^{r-1}\omega_{r-1}}\cdot q_l(r-1,0) & \frac{\omega_1}{\lambda_1^{r-2}\omega_{r-1}}\cdot q_l(r-1,1) & \dots & q_l(r-1,r-1) \end{pmatrix}.$$

Where ω_i, with $0 \le i \le r-1$, is given as presented in equation (7), and $q_l(i,j)$ is given by

$$q_l(i,j) = \begin{cases} d\cdot\Pi(j,\lambda_1), & \text{if } j \equiv i+l \bmod d \\ 0, & \text{otherwise} \end{cases},$$

where $\Pi(j,\lambda_1)$ is described in (11).

3. Simulation

To implement the diagonalization and difference equation methods, **Octave** was chosen for its efficiency in matrix operations, built-in mathematical functions, and MATLAB compatibility as free software, providing a solid base for comparing the approaches. This work presents three methods for computing the coefficients $v_i^{(j)}$: **RA1**, which computes them term by term using the recurrence; **RA2**, which solves the recurrence to obtain a closed-form expression and calculates only the necessary terms; and **RA3**, which computes one family of coefficients and derives the others through linear combinations, as discussed in [14]. Various Leslie matrices with randomly generated values are analyzed, varying the matrix order n and the exponent t, as shown in Table 1.

Table 1
The average compilation time in seconds according to the matrix order and exponent

n	t	Simulation				
		Direct form	**RA1**	**RA2**	**RA3**	**Diagonal**
3	5	$1.0014 \cdot 10^{-5}$	$5.1856 \cdot 10^{-4}$	$1.3885 \cdot 10^{-3}$	$4.4560 \cdot 10^{-4}$	$6.0439 \cdot 10^{-5}$
	10	$1.0967 \cdot 10^{-5}$	$8.7512 \cdot 10^{-4}$	$1.3345 \cdot 10^{-3}$	$7.1847 \cdot 10^{-4}$	$5.3883 \cdot 10^{-5}$
	50	$1.3113 \cdot 10^{-5}$	$3.9464 \cdot 10^{-3}$	$1.3379 \cdot 10^{-3}$	$3.0919 \cdot 10^{-3}$	$6.0081 \cdot 10^{-5}$
5	5	$1.0967 \cdot 10^{-5}$	$1.1598 \cdot 10^{-3}$	$4.5094 \cdot 10^{-3}$	$9.0158 \cdot 10^{-4}$	$6.8903 \cdot 10^{-5}$
	10	$1.1921 \cdot 10^{-5}$	$2.0511 \cdot 10^{-3}$	$4.3194 \cdot 10^{-3}$	$1.5479 \cdot 10^{-3}$	$6.6996 \cdot 10^{-5}$
	50	$1.4067 \cdot 10^{-5}$	$9.6911 \cdot 10^{-3}$	$4.3640 \cdot 10^{-3}$	$6.9020 \cdot 10^{-3}$	$8.2493 \cdot 10^{-5}$
10	5	$1.2875 \cdot 10^{-5}$	$4.0996 \cdot 10^{-3}$	$1.6953 \cdot 10^{-1}$	$2.8830 \cdot 10^{-3}$	$1.2648 \cdot 10^{-4}$
	10	$1.9073 \cdot 10^{-5}$	$1.0949 \cdot 10^{-2}$	$2.4942 \cdot 10^{-1}$	$7.3301 \cdot 10^{-3}$	$1.7595 \cdot 10^{-4}$
	50	$1.9073 \cdot 10^{-5}$	$4.2047 \cdot 10^{-2}$	$1.9502 \cdot 10^{-1}$	$2.7878 \cdot 10^{-2}$	$1.5497 \cdot 10^{-4}$
20	5	$1.3828 \cdot 10^{-5}$	$1.5064 \cdot 10^{-2}$	3.6665	$9.8335 \cdot 10^{-3}$	$2.2650 \cdot 10^{-4}$
	10	$1.6093 \cdot 10^{-5}$	$2.8265 \cdot 10^{-2}$	3.5991	$1.7650 \cdot 10^{-2}$	$2.1601 \cdot 10^{-4}$
	50	$2.0504 \cdot 10^{-5}$	$1.3502 \cdot 10^{-1}$	3.6680	$8.3117 \cdot 10^{-2}$	$2.2638 \cdot 10^{-4}$
50	5	$5.6028 \cdot 10^{-5}$	$1.1976 \cdot 10^{-1}$	$2.4513 \cdot 10$	$8.0204 \cdot 10^{-2}$	$1.7340 \cdot 10^{-3}$
	10	$7.7963 \cdot 10^{-5}$	$1.9722 \cdot 10^{-1}$	$2.4776 \cdot 10$	$1.3175 \cdot 10^{-1}$	$1.6160 \cdot 10^{-3}$
	50	$9.1791 \cdot 10^{-5}$	$8.9644 \cdot 10^{-1}$	$2.4868 \cdot 10$	$5.4971 \cdot 10^{-1}$	$1.4720 \cdot 10^{-3}$
100	5	$1.1060 \cdot 10^{-4}$	1.2090	$6.0438 \cdot 10^{2}$	$6.9305 \cdot 10^{-1}$	$3.2881 \cdot 10^{-2}$
	10	$9.2983 \cdot 10^{-4}$	1.6457	$7.0271 \cdot 10^{2}$	$9.9044 \cdot 10^{-1}$	$2.4294 \cdot 10^{-2}$
	50	$2.0559 \cdot 10^{-3}$	9.4563	$7.5332 \cdot 10^{2}$	4.0480	$2.0163 \cdot 10^{-2}$

The obtained results, summarized in a table for various matrix sizes (n = 3, 20, 100) and powers (t = 5, 10, 50), provide a detailed view of each method's computational efficiency. The **Direct Multiplication Method (MD)** showed consistently low computational cost across all cases. The **Recurrence Approaches (RA1, RA2, RA3)** were more variable: for t = 5 and t = 10, the order of efficiency was RA2 < RA3 < RA1, while for t = 50, RA2 and RA3 performed similarly, though RA3's cost increased sharply with n. The **Matrix Diagonalization Method (DM)** showed intermediate performance. As n increases, all methods demand more computation, but MD remains effective and consistent, while RA methods, especially RA3, are more sensitive to this increase. Overall, MD stands out as the most efficient method for smaller t, and DM is a reliable intermediate option.

The next algorithms aim to analyze the asymptotic behavior of the Leslie matrix, i.e., $\lim_{n\to+\infty}(\mathbb{L}(f,p))^n$. Since repeated multiplication does not guarantee convergence, decomposition methods such as diagonalization or difference equations are required.

3.1 *Simulation of asymptotic behavior*

As shown previously, the matrix converges only when the dominant eigenvalue is $\lambda_1 = 1$; if $\lambda_1 = 1$, the population grows indefinitely; if $0 < \lambda_1 < 1$, it tends to zero. Thus, we analyze the matrix representing the relative proportion of each age group.

3.1.1 Case 1: $\lambda_1 = 1$ and $\gcd\{i, a_i > 0\} = 1$

This case studies the compilation time to determine $\lim_{n\to\infty}(\mathbb{L}(f,p))^n$. when the dominant eigenvalue is $\lambda_1 = 1$, implying convergence. Simulations were performed for matrices of size *n* = 3, 5, 10 using diagonalization and difference equation methods. The results are shown in Table 2.

The data analysis reveals that, in all examined cases, the Diagonalization Method is more effective in terms of compilation time than the Difference Equations Method. These results highlight the inherent efficiency of the Diagonalization Method, suggesting that it is a faster option for analyzing the asymptotic behavior of Leslie matrices with a dominant eigenvalue of $\lambda_1 = 1$.

Furthermore, a consistent pattern of increasing compilation time for both methods as the matrix order grows was observed. It is noteworthy that the Difference Equations Method exhibits a more pronounced growth in time, indicating higher computational complexity when dealing with matrices of higher orders.

Table 2

Compilation time for calculating the asymptotic behavior of the matrix $\mathbb{L}(\boldsymbol{f}, \boldsymbol{p})$ with eigenvalue $\boldsymbol{\lambda_1 = 1}$.

n	Diagonalization Method (seconds)	Finite Difference Method (seconds)
3	$1.7190 \cdot 10^{-4}$	$2.6298 \cdot 10^{-4}$
20	$1.5698 \cdot 10^{-3}$	$9.4429 \cdot 10^{-3}$
1000	$1.0570 \cdot 10$	$4.6155 \cdot 10$

The data analysis reveals that, in all examined cases, the Diagonalization Method is more effective in terms of compilation time than the Difference Equations Method. These results highlight the inherent efficiency of the Diagonalization Method, suggesting that it is a faster option for analyzing the asymptotic behavior of Leslie matrices with a dominant eigenvalue of $\lambda_1 = 1$.

Furthermore, a consistent pattern of increasing compilation time for both methods as the matrix order grows was observed. It is noteworthy that the Difference Equations Method exhibits a more pronounced growth in time, indicating higher computational complexity when dealing with matrices of higher orders.

3.1.2 Case 2: $\lambda_1 \neq 1$ and $\gcd\{i, a_i > 0\} = 1$

Similarly to the case where $\lambda_1 = 1$ and $\gcd\{i, a_i > 0\} = 1$, in this scenario where $\lambda_1 \neq 1$, the diagonalization and difference equations methods are also compared in terms of compilation time to assess their advantages and disadvantages based on the matrix order. The results obtained are presented in table 3.

Table 3

Compilation time for calculating the asymptotic behavior of the matrix $\mathbb{L}(f, p)$ with eigenvalue $\lambda_1 \neq 1$.

n	Diagonalization Method (seconds)	Finite Difference Method (seconds)
3	$1.3781 \cdot 10^{-4}$	$3.2294 \cdot 10^{-4}$
20	$1.4662 \cdot 10^{-3}$	$6.0564 \cdot 10^{-3}$
1000	9.2742	$5.1521 \cdot 10$

Extending the analysis to the case where $\lambda_1 \neq 1$, the results obtained showed a similar pattern to those observed previously when $\lambda_1 = 1$. In both scenarios, the Diagonalization Method proved more effective, outperforming the Difference Equations Method in terms of compilation time.

It is also noteworthy that as the matrix order increases, both methods exhibit a growth pattern in compilation time. However, in the case where $\lambda_1 \neq 1$, the diagonalization method shows even less pronounced growth since, in this case, it is only necessary to determine the eigenvector associated with λ_1 to assess its asymptotic behavior.

3.1.3 Case 3: $\lambda_1 = 1$ and $\gcd\{i, a_i > 0\} = d > 1$

The third case deals with Leslie matrices having the dominant eigenvalue $\lambda_1 = 1$, similarly to case 1, but with $d = \gcd\{i, a_i > 0\} > 1$, leading to convergence of $\lim_{n\to\infty}(\mathbb{L}(f, p))^{nd+l}$, with $l = 0,1,\ldots,d-1$.

Matrices with dimensions n equal to three, five, and ten that satisfy the conditions of this case are used. The results generated through simulations are shown in table 4.

Table 4

Compilation time for calculating the asymptotic behavior of the matrix $\mathbb{L}(f, p)$ with eigenvalue $\lambda_1 = 1$ and $d > 1$.

n	Diagonalization Method (seconds)	Finite Difference Method (seconds)
3	$2.0599 \cdot 10^{-4}$	$4.1902 \cdot 10^{-4}$
20	$1.7897 \cdot 10^{-3}$	$9.5879 \cdot 10^{-3}$
1000	8.4571	$6.1039 \cdot 10$

Analyzing the graphs, it is once again evident that the diagonalization method outperforms the method of difference equations, exhibiting shorter compilation times for all studied orders. It consistently showed an increase in compilation time with the matrix order, with the difference equations method demonstrating a more pronounced increase.

3.1.4 Case 4: $\lambda_1 \neq 1$ and $\gcd\{i, a_i > 0\} = d > 1$

Finally, the fourth case addresses Leslie matrices with a dominant eigenvalue $\lambda_1 \neq 1$ and $d = \gcd\{i, a_i > 0\} > 1$. This implies that the matrices $\lim_{n\to\infty}(\mathbb{L}(f,p))^{nd+l}$, with $l = 0,1,\ldots,d-1$, will not converge. In other words, they will grow indefinitely if $\lambda_1 > 1$ or tend to zero if $0 < \lambda_1 < 1$. In this case, similar to case 2, the goal is to determine the proportion that each age group approaches concerning the total population over time. The simulation results for this study are found in Table 5.

Table 5

Compilation time for calculating the asymptotic behavior of the matrix $\mathbb{L}(\boldsymbol{f},\boldsymbol{p})$ with eigenvalue $\boldsymbol{\lambda_1 \neq 1}$ and $\boldsymbol{d > 1}$.

n	Diagonalization Method (seconds)	Finite Difference Method (seconds)
3	$1.7190 \cdot 10^{-4}$	$2.6298 \cdot 10^{-4}$
20	$1.6005 \cdot 10^{-3}$	$9.5001 \cdot 10^{-3}$
1000	9.2242	$7.7443 \cdot 10$

Once again, the Diagonalization Method exhibited shorter compilation times when compared to the Method of Difference Equations, proving to be a more robust approach for calculating the asymptotic behavior of Leslie matrices.

4. Conclusion

In this exploration of Leslie matrices, we delve into the practical applications of matrix diagonalization and difference equations. The groundwork lays a theoretical foundation covering the application of matrix diagonalization and generalized Fibonacci recurrence relations in Leslie matrices.

In practical applications, our focus sharpens on calculating Leslie matrix powers and understanding their asymptotic behavior. Through simulations, we uncover computational efficiencies, with the direct multiplication method shining in finite scenarios.

A key takeaway is the discernment that different methods offer specific advantages based on the analytical context. Diagonalization stands out for efficiency, while difference equations prove potent for specific Leslie matrix forms as shown in matrices of form $\mathbb{L}([0,0,\ldots,0,f_{n-1}],[p_0,\ldots,p_{n-2}])$.

This research underscores the invaluable roles of matrix diagonalization and difference equations in Leslie matrix analysis, paving the way for future explorations. The interplay of theory and application unveils mathematical intricacies and opens new frontiers in scientific inquiry.

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