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Planning a dynamical movement of patterns based on groups relations

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Abstract

Dynamical movement between different stages in a given product can be challenging. In this manuscript, we will show how groups relations indicate if exists dynamical movement between different stages in a given product defined by planner patterns, and what is the respective mechanism that defines this dynamical tiling. Our construction can be applied to folding tables (reduction and expansions), lightning systems (exposure, and hiding), and more.

Subject Classification (2010): *20A05, 70E17, 05B99.*

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1. Introduction

One of the primary objectives of industrial design is to foster innovation through the development of novel ideas and intricate details in products. This goal drives designers to explore and integrate advancements from various scientific disciplines, including materials engineering—particularly in the context of 3D printing prototypes [1]—artificial

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intelligence [2], mechanics [3], and other fields, all of which contribute to the design process. The planning step is crucial to well define the product properties for a given functional purpose. It turns out, that the question if such properties can be obtained in a given product with constraints may be just a theoretical mathematical question. This manuscript will show how group theory properties, can be a helpful tool for a designer to decide already in the planning step, if exists a mechanism that defines movement between two tilings that need to be assimilated in a given product.

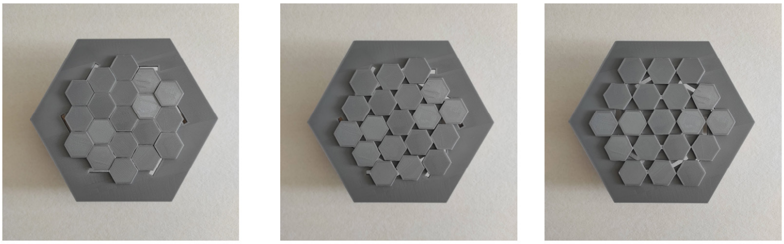
A good example has been given in [4], where discussed how the theoretical model of Lie group can help planning a design of mechanical systems. Also in [5], groups have been applied for investigation of large-deformation property of bar-hinge mechanisms with dihedral symmetry in $3D$ and that the system of reduced compatibility equations inherits the group equivariance from the original compatibility equations, which at last can judge whether the frame has a finite mechanism mode.

The set of Wang tiling are used as a tool to generate textures and patterns in computer graphics, [6] provide a linear algorithm to decide and solve the Wang tiling problem for an arbitrary planar region with holes.

This collaboration arose from an inadvertent encounter with distinctive folding tables[1], as described in [7, 8], which are associated with a dynamic system. This encounter led us to the artwork of Khushbu Kshirsagar, presented in the Bridges 2022 art gallery [9], which defines a dynamical system based on a hexagonal structure. While the designer was fascinated by the performance of the moving objects and their mechanisms, the mathematician posed the question: "Can the designer utilize the algebraic structure properties of repetitive patterns and tilings to determine when a dynamic product can emerge between two patterns in the plane".

In product design, repetitive patterns are typically employed during the finishing process and are primarily associated with the texture or aesthetic that the product aims to convey. This work extends the findings of [10] and explores how to determine whether a dynamic movement of repetitive patterns exists at the initial stages of product planning, based on mathematical structures. It is shown that group theory exploration (see section 2) provides a framework for justifying when a design can incorporate movement between patterns through the use of a dynamic mechanism, as illustrated in Figure 1.

[1] The intricate design of these tables results in a price exceeding 100k.



(a) Initial state $*632$ (b) Middle state 632 (c) Open stage $*632$

Figure 1

Our model defines a dynamic movement between tilings (constructed from PLA) using an appropriate mechanism. On the left side, we begin with a tiling featuring the signature $*632$. Through the mechanism, the tiling's signature first transitions to 632, and as it reaches the end of the rail, it generates an extended pattern with the signature $*632$.

From the perspective of the industrial designer, these patterns represent opposing states–two stages in terms of the mathematical formulation and three in terms of the pattern design. These states can be utilized to achieve various objectives, such as reductions and expansions, or exposure and concealment. With further development and the necessary budget, this approach could be applied to products like lighting, tables, and beyond. More importantly, it highlights the relevance of advanced theoretical mathematical concepts to the typical industrial designer, demonstrating their potential to be effectively applied when aligned with the right purpose.

2. Mathematical Background

Definition 1: A group $(G, \cdot)$ is a set G equipped with an operator $\cdot : G \cdot G \to G$, which satisfy the following group axioms

(1) Closure: If $x, y \in G$ then $xy \in G$.

(2) Associativity: For $x, y, z \in G$, $x(yz) = (xy)z$

(3) Identity: There exists an identity element 1 in G such that for each x in G, $x = 1 \cdot x = x \cdot 1$

(4) Inverse: For each $x \in G$, there is an inverse $x^{-1} \in G$ such that $xx^{-1} = x^{-1}x = 1$

We will mostly denote $(G, \cdot)$ as just G. The order of a finite group is the number of its elements. If a group is not finite, one says that its order is infinite, i.e., $|G| = \infty$.

Definition 2: Given $(G,\cdot)$, a subset H of G is called a subgroup of G if H also forms a group under the operation $\cdot$, and we denote $H \leq G$.

For example, the dihedral group is the group of symmetries of a regular polygon, which includes rotations and reflections. The group of rotations is a subgroup.

Adding the dihedral group translation leads to the following definition.

Definition 3: A tessellation or tiling is the covering of a surface, often a plane, using one or more geometric shapes, called tiles, with no overlaps and no gaps.

Different tilings are obtained by a fundamental area that is being copied in infinite times by a combination of reflections, rotations, and translation. In the plane these tilings are called *Wallpaper groups*.

In [11], Conway has denoted the different symmetries as follows:

Notation 1:

(1) We write $*N$ to denote a point in the plane which through this point crosses N reflections. If there is a single reflection $* = *1$.

(2) We write just N, if a given point define a symmetry of N rotation, i.e., the angle of rotation is $\frac{2\pi}{N}$.

(3) A glide reflection[2], which will be denoted by $\times$ is the composition of a reflection across a line and a translation parallel to the line.

(4) A pattern that is constructed by pure translation (and tiles the plane), will be defined by two vectors in different directions, which a combination of them span the plane. we will denote these patterns by $\circ$.

See different examples in Fig. 2.

Since the pattern is repetitive, we can refer only to unique points of symmetry, which all together define the symmetry of the pattern[3] as follows.

Definition 4: The signature of the repetitive pattern is defined by the number of unique points of reflections, rotations, glides reflection or pure translation.

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[2] Can be called 'Miracle'

[3] There are a few different ways to denote these groups, such as Crystallographic and Coxeter. We choose Conway notations, which in our opinion is the most intuitive notation.

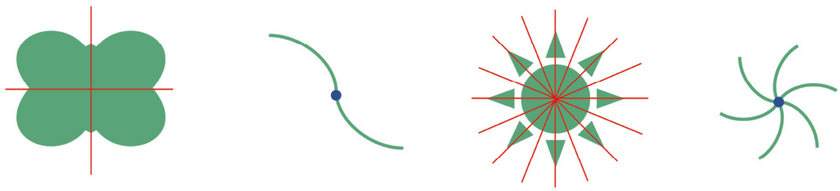


Figure 2

Different symmetries and the respective notations by Conway.

It turns out, that only 17 different signatures exists, i.e., 17 planar patterns. In Fig. 1a, the pattern is obtained by the hexagonal regular lattice, which is obtained by reflection in three different unique points, so the respective signature is $*632$, while Fig. 1b is obtained by three rotations (and translations), so the respective signature is 632.

This extension of Dihedral groups to wallpaper groups (plane repeating patterns), leads to the following 17 signatures.

Table 1

Planar signatures

*632	*442	*333	*2222	**
			2 * 22	* ×
	4 * 2	3 * 3	22 *	× ×
			22 ×	
632	442	333	2222	∘

Table 1, leads to the following group's relations.

- $632 \leq *632$
- $442 \leq 4*2 \leq *442$
- $333 \leq 3*3 \leq *333$
- $2222 \leq 22\times \leq 22* \leq 2*22 \leq *2222$
- $\times\times \leq *\times \leq **$

Since $\circ$ is defined only by translation, then it is a subgroup of all the other signatures.

3. Our Results

These relations between groups and subgroups are obtained by omitting symmetries as reflections. By these mathematical rules, we can deduce if given two patterns, there is a natural transformation from $G \to H$, where $H \leq G$. These groups' relations between tilings indicate, that in product design, exists a respective mechanism that will transform a pattern with signature X, to a pattern with signature Y, such that $X \leq Y$. We will show how this dynamical movement is obtained between two stages or more. All parts of the desired model/result will be printed in 3D printer.

The design process began with an understanding of the basic geometric form that would facilitate working with moving objects. After exploring several polygons that were prominent in repetitive patterns, the hexagon was selected due to its ease of use and simplified movement. This choice resulted in the respective signature $*632$.

Fig. 3 illustrates the initial step in translating our mathematical concept into design. By arranging seven regular hexagons–one at the center and six surrounding neighbors–a reflection symmetry is defined. Simultaneously sliding these six neighboring hexagons along the edges of the central hexagon creates gaps between them, resulting in a single point of rotation at a specific angle. This experiment demonstrates that a movement between the signatures $*632$ and 632 $(632 \leq *632)$ is indeed achievable.

The second step is planning the mechanism to Fig. 3, which will define the dynamic movement according to the symmetry rule of and it respective subgroup. In Fig 4., the desired result for central symmetry (which will be generalized to a given tiling).

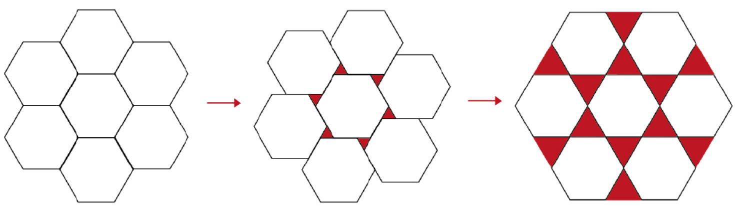
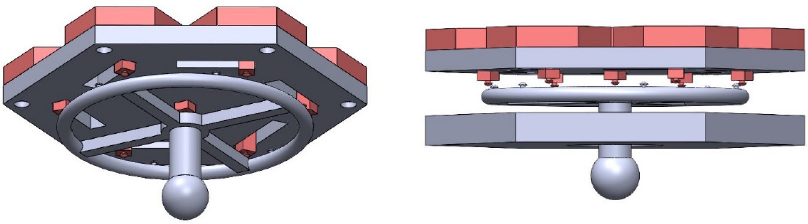


Figure 3

First sketch examining the feasibility of our idea. Given seven regular hexagons, where the center of mass is the point of $*6$ symmetry. In the second image by a rotation respective to the center of mass, leads to gaps and a bigger area and 6 symmetry. In the last image, the open stage defines a new pattern and a signature $*6$.



(a) (b)

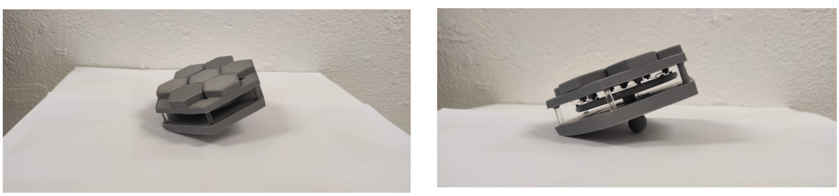
Figure 4

Planning/imaging the mechanism of the model for a center symmetry (solid software).

The different stages need to be obtained by a respective mechanism. We considered several mechanisms, but the group relations point us that the movement will best obtained by a mechanism that is defined by a wheel (which is respective to rotation).

The first realized model of the hexagonal movement is based on the sliding of six hexagons along the sides of the central hexagon they surround. These hexagons slide simultaneously while maintaining the symmetry of the pattern. To ensure both the symmetrical spacing between the six hexagons and the requirement for simultaneous movement, rails were constructed to guide the hexagons' motion.

The mechanism operates as follows: seven hexagons are arranged on a surface such that the central hexagon remains stationary, while the six surrounding hexagons are able to move across the surface. Each of the six surrounding hexagons is equipped with a square pin located at the center, which protrudes downwards beneath the surface. These square pins are mounted on tracks, allowing them to move back and forth along the track.



(a) Central Hexagon movement (b) Side view of the mechanism

Figure 5

Our first working model, the symmetry is defined as a single regular hexagon, surrounded by six regular hexagons, which gives us an approximation for the respective tiling.

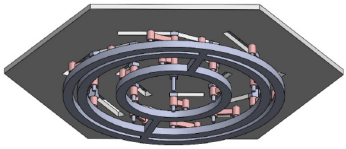


Figure 6

Bottom view to the mechanism: the rails, the pins and the wheels will change the signature of the pattern. It connects signature ∗632 and 632.

The model in Fig. 5 is obtained by assembling all parts that have been printed in $3D$. This model gives us the ability to imagine how a mechanism will connect signature $*632$ and 632.

The generalization of the tiling model with respective signature $*632$ works as follows: choose a center hexagon that will represent the center of mass of the finite model. Define the hexagon layers respective to this center, i.e., in the first layer there exist six hexagons (which can be dynamical by the previous mechanism), in the second layer there are 12 hexagons (and the sequence increasing times two), so connecting the second layer with a weal to the weal of the first layer is our induction which define the mechanism, see Fig. 6.

This concept can be generalized to multiple layers of hexagons, with each layer having its respective track. The track is designed such that when all the hexagons are at the starting position, a closed pattern with the $*632$ signature is formed. The second stage is achieved through dynamic movement along the tracks, resulting in a 632 signature. Finally, when all the hexagons reach the end of the tracks, an open pattern of $*632$ is formed. The outcome is the model depicted in Fig. 1, which features these three stages. To observe the model in action, refer to the video link in [12].

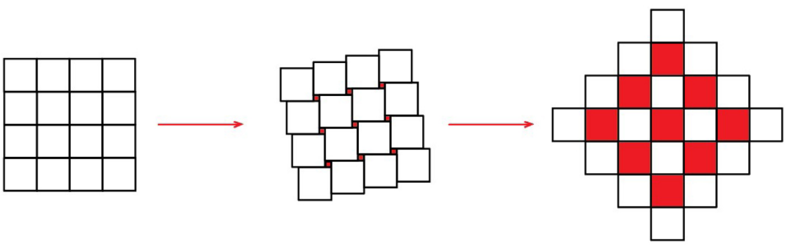
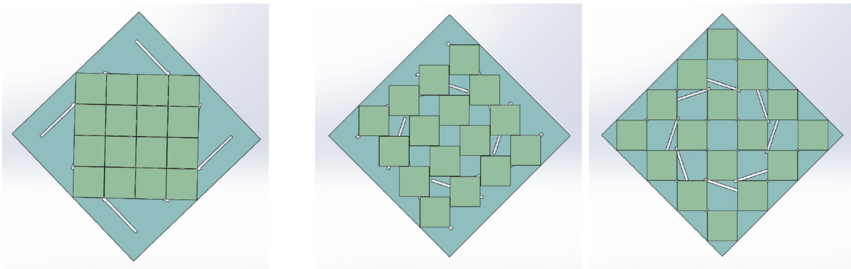


Figure 7

A sketch for a second model which defines another group relation, in this case, ∗442 and 442.



(a) (b) (c)

Figure 8

Planning the mechanism which will define a transformation. The white slots are the rails which sliding along them simultaneously, will define a dynamic movement between the patterns.

Figure 9

The printed mechanism respective to signature $*442$ and 442, a bottom view.

Our idea which shows when a dynamical tiling exists, and how to construct the mechanism by the relations between the group and the sub-groups, can be achieved for any repetitive planar pattern, where in some of them planning the mechanism may be more challenging. Fig. 7, is another example, which presents a feasibility model for dynamical tiling defined by $442 \leq *442$. The mechanism for this case is similar to $*632$, with a slight change, see the imagining in Fig. 8 and the respective printed mechanism in Fig. 9. After attaching all components which have been printed in $3D$ printer, the result is Fig. 10, which have three different stages which are obtained by the group relations.

(a) (b) (c)

Figure 10

All parts of the model are 3*D* printed. Starting with an initial pattern having the signature ∗442, the mechanism transforms it into the 442 pattern. In the final, open stage, the pattern defines a signature ∗442, where the pattern is distinct from the initial state, exhibiting a larger area, but retaining the same signature.

4. Industrial Designer Perspective

When I first encountered the concept of symmetry, I initially saw it as a tool primarily for aesthetic purposes. It was about making things look good, balanced, and pleasing to the eye. However, as I delved deeper into the mathematical intricacies of symmetry groups and the relationships between different patterns, my perspective began to shift. I started to realize that symmetry is not just about static beauty; it's about the dynamic interplay between elements, the way patterns can transform and evolve.

This realization prompted me to embark on a journey of exploration. I began researching and experimenting with 3D printing technology to create prototypes that would test the boundaries of symmetry in motion. With 3*D* printing, I found the flexibility to design intricate mechanisms that could seamlessly transition between different patterns. Collaborating with mechanical engineers was crucial in ensuring that these mechanisms were not just visually appealing but also functional and precise.

As I watched these prototypes in action, witnessing the mesmerizing spectacle of patterns shifting and transforming, I couldn't help but think about the potential applications beyond mere aesthetics. It struck me that this dynamic interplay of symmetry could be harnessed to create products that offer not just visual delight but also interactive experiences.

One such application that captured my imagination is an interactive light fixture. Imagine a lamp where users can manipulate the patterns themselves, adjusting the light and shadow to suit their mood or preference. By incorporating dynamic symmetry into product design, we can offer consumers not just static objects but engaging experiences that evolve and adapt to their needs and desires.

5. Conclusions and Future Work

We have demonstrated that algebraic structures can assist designers in the planning stages by determining whether a dynamic transformation of components can be achieved through a corresponding mechanism that facilitates movement between different patterns or arrangements, each serving a distinct purpose. This approach can be applied to various design tasks, such as folding tables (for reductions and expansions), lighting systems (for exposure and concealment), and other product designs. We intend to generalize this result not only for planar patterns but also for spherical patterns and especially geodesic domes. To be functional the dome and lacks of space, the mechanism needs to be chosen carefully, such that it will not fill the interior of the dome.

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- Consent to participate: Yes
- Consent for publication: Yes
- Availability of data and materials: N/A
- Code availability: N/A
- Authors' contributions: Equal contribution

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