

On fractional derivation and fractional convexity

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Abstract

Some novel applications of the conformable fractional derivative are presented. Indeed, we generalize the inverse function theorem and the Euler's theorem. We also propose a definition of fractional convexity for which we show certain properties and applications.

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1. Introduction

In recent years, the emergence of the conformable fractional derivative has ushered in a new area of mathematical tools that allow us to explore the inverse function theorem in more general and flexible settings. The conformable fractional derivative is a mathematical operator that extends

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the classical notions of differentiation by incorporating non-local and singular behavior [5, 6, 8, 10, 13].

This specific type of fractional derivative has gained attention in recent years. It's a derivative operator that preserves certain properties of standard calculus while still capturing the essence of fractional differentiation. It is defined in such a way that it maintains the Leibniz rule and the chain rule, making it a convenient tool for various applications. Among several other definitions, the conformable fractional derivative offers a more consistent and natural passage to fractional case. This makes it easier to integrate fractional calculus into existing mathematical frameworks and apply it to real-world problems. This concept has shown promise in modeling complex processes that exhibit fractal-like behavior, such as anomalous diffusion, viscoelasticity, and non-local phenomena.

In summary, fractional calculus introduces the concept of derivatives and integrals with non-integer orders, allowing for the analysis of systems and phenomena that exhibit fractional behavior. This makes it a powerful tool for understanding and modeling various complex processes in different scientific and engineering domains. One of the motivations behind the conformable fractional derivative is to provide a more intuitive and straightforward way to handle fractional differentiation. It can be particularly useful in modeling real-world phenomena involving irregularities or anomalous behavior, such as in the fields of physics, engineering, and finance.

The conformable fractional derivative was first proposed in [8] as an alternative approach to fractional calculus, and introduced as an attempt to address certain limitations and ambiguities associated with traditional fractional derivatives, such as the Riemann-Liouville and Caputo derivatives. These traditional derivatives are defined using integral operators and involve the gamma function and non-local properties, making their interpretation and application complex in some cases, see [10, 12] for example. Let's now recall this definition:

Definition 1.1: [8] The conformable α -derivative of $f: \mathbb{R}^+ \rightarrow \mathbb{R}$ is given for each $\alpha \in]0, 1[$ by

$$f^{(\alpha)}(z) := \lim_{\varepsilon \rightarrow 0} \frac{f(z + \varepsilon z^{1-\alpha}) - f(z)}{\varepsilon}, \quad \forall z > 0;$$

we take $f^{(\alpha)}(0) = \lim_{z \rightarrow 0^+} f^{(\alpha)}(z)$ if this limit exists.

In [6], we find a more general study involving the conformable fractional partial derivatives

$$\frac{\partial^{(\alpha)} f(x)}{\partial x_i^\alpha} := \lim_{\varepsilon \rightarrow 0} \frac{f(x_1, \dots, x_i + \varepsilon x_i^{1-\alpha}, \dots, x_n) - f(x_1, \dots, x_n)}{\varepsilon}, 1 \leq i \leq n$$

of a function with several variables

$$f : x = (x_1, \dots, x_n) \in [0, +\infty[^n \rightarrow f(x) \in \mathbb{R};$$

the conformable fractional Jacobian matrix $\left(\frac{\partial^{(\alpha)} f_i(x)}{\partial x_j^\alpha} \right)_{1 \leq i, j \leq n}$ of a vector valued function

$$F : x = (x_1, \dots, x_n) \in [0, +\infty[^n \rightarrow F(x) = (f_1(x), \dots, f_i(x), \dots, f_n(x)) \in \mathbb{R}^n$$

was also involved, when the associated continuous fractional partial derivatives exist. For more details on conformable fractional derivation and applications, we refer to [1, 3, 4, 6, 7, 8, 9, 11]

Definition 1.2: $f : [0, +\infty[^n \rightarrow \mathbb{R}$ is k -homogeneous with $k > 0$ iff $f(tx) = t^k f(x)$, $\forall t > 0$, $\forall x \in [0, +\infty[^n$.

The homogeneity is a very important notion because most economic production functions are homogeneous. The famous Euler's theorem stated below characterizes homogeneous functions and consequently their partial elasticities which have crucial economic interpretations.

Theorem 1.3: (Euler's theorem)

For $f : [0, +\infty[^n \rightarrow \mathbb{R}$, $k > 0$, one has :

$$f \text{ is } k\text{-homogeneous} \Leftrightarrow \sum_{i=1}^n x_i \frac{\partial f(x)}{\partial x_i} = k f(x).$$

Therefore, $\sum_{i=1}^n E_{x_i}(f) = k$, where $E_{x_i}(f) := \frac{x_i}{f(x)} \frac{\partial f(x)}{\partial x_i}$, $1 \leq i \leq n$ are the partial elasticities of f .

In all the following, I is an interval of $]0, +\infty[$ and $\alpha \in]0, 1]$.

2. Fractional inverse function theorem and fractional Euler's theorem

In this section, we recall fractional version of inverse function theorem proved in [2] and we establish the fractional Euler's theorem.

Theorem 2.1: [2] If $f : I \rightarrow \mathbb{R}$ is continuous and strictly monotonic then, f is bijective, f^{-1} is also continuous and strictly monotonic. If moreover $f^{(\alpha)}(x)$ exists and $f'(x) \neq 0$ then,

$$(f^{-1})^{(\alpha)}(y) = \frac{(f(f^{-1}(y)))^{1-\alpha}}{f'(f^{-1}(y))}, y = f(x).$$

The particular case $\alpha = 1$ is the classical inverse function theorem.

Proof: See [2, Theorem 3.1]

Example 1: $f(x) = x^2$, $\forall x > 0$. f is continuous and strictly increasing. $f'(x) = 2x \neq 0$ and by [8], $f^{(\alpha)}(x) = 2x^{2-\alpha}$. By the previous theorem

$$\begin{aligned} (f^{-1})^{(\alpha)}(f(x)) &= \frac{(f(x))^{1-\alpha}}{f'(x)} \\ &= \frac{x^{2-2\alpha}}{2x} \\ &= \frac{1}{2}x^{1-2\alpha}, \quad \forall x > 0. \end{aligned}$$

Theorem 2.2: For $f : [0, +\infty[^n \rightarrow \mathbb{R}$, $k > 0$ one has :

$$f \text{ is } k\text{-homogeneous} \Leftrightarrow \sum_{i=1}^n x_i^\alpha \frac{\partial^{(\alpha)} f(x)}{\partial x_i^\alpha} = kf(x).$$

Therefore, $\sum_{i=1}^n E_{x_i}^{(\alpha)}(f) = k$, where $E_{x_i}^{(\alpha)}(f) := \frac{x_i^\alpha}{f(x)} \frac{\partial^{(\alpha)} f(x)}{\partial x_i^\alpha}$, $1 \leq i \leq n$ are the fractional partial elasticities of f .

Proof: The case $\alpha = 1$ is the known Euler's theorem (Theorem 1.3).

Let $\alpha \in]0, 1[$ and note $x = (x_1, \dots, x_n) \in [0, +\infty[^n$. By [6], the conformable fractional partial derivatives of f are given by :

$$\frac{\partial^{(\alpha)} f(x)}{\partial x_i^\alpha} = x_i^{1-\alpha} \frac{\partial f(x)}{\partial x_i}, 1 \leq i \leq n.$$

$$\text{Thus, } \sum_{i=1}^n x_i^\alpha \frac{\partial^{(\alpha)} f(x)}{\partial x_i^\alpha} = \sum_{i=1}^n x_i^{\alpha+1-\alpha} \frac{\partial f(x)}{\partial x_i} = \sum_{i=1}^n x_i \frac{\partial f(x)}{\partial x_i}.$$

The demonstration is completed by application of theorem 1.3 because f is k -homogeneous.

Consequently,

$$\sum_{i=1}^n E_{x_i}^{(\alpha)}(f) = \sum_{i=1}^n \frac{x_i^\alpha}{f(x)} \frac{\partial^{(\alpha)} f(x)}{\partial x_i^\alpha} = k \frac{f(x)}{f(x)} = k.$$

Example 2: Set $f(x, y) = x^2 + y^2, \forall (x, y) \in \mathbb{R}_+^2$. f is k -homogeneous with $k = 2$.

$$x^\alpha \frac{\partial^{(\alpha)} f(x, y)}{\partial x^\alpha} + y^\alpha \frac{\partial^{(\alpha)} f(x, y)}{\partial y^\alpha} = 2(x^2 + y^2) = 2f(x, y).$$

$$E_x^{(\alpha)}(f) + E_y^{(\alpha)}(f) = \frac{2x^\alpha x^{2-\alpha}}{x^2 + y^2} + \frac{2y^\alpha y^{2-\alpha}}{y^2 + y^2} = 2.$$

3. Fractional convexity

Convexity is a fundamental concept of mathematics and applications, notably in convex analysis and optimization of various problems. Here we present the notion of fractional convexity as a generalization of classical convexity. Some of its properties are then shown and applications to inequalities are illustrated.

Definition 3.1: $f: \mathbb{R}^+ \rightarrow \mathbb{R}$ is α -convex on $\mathbb{R}^+$ iff

$$\forall x, y \in \mathbb{R}^+, \forall \mu \in [0, 1], \quad f((\mu x^\alpha + (1-\mu)y^\alpha)^{\frac{1}{\alpha}}) \leq \mu f(x) + (1-\mu)f(y).$$

f is α -concave on $\mathbb{R}^+$ if and only if $(-f)$ is α -convex on $\mathbb{R}^+$.

Example 3:

- (1) For $\alpha = 1$ we recognize the classical convexity.
- (2) For any $\alpha \in]0, 1[$ and $c \in \mathbb{R}$, the function f defined on $\mathbb{R}^+$ by

$$f(x) = x^\alpha + c$$

is obviously α -convex.

Proposition 3.2: Suppose that f, g are α -convex on $\mathbb{R}^+$. Then $(f + g)$ and $\max(f, g)$ are also α -convex on $\mathbb{R}^+$.

Proof: It follows immediately from the definitions of $(f + g)$, $\max(f, g)$ and α -convexity of f and g .

Proposition 3.3: Let's suppose $f: \mathbb{R}^+ \rightarrow \mathbb{R}$ is α -convex on $\mathbb{R}^+$, $g: f(\mathbb{R}^+) \rightarrow \mathbb{R}$ is convex and increasing on $f(\mathbb{R}^+)$. Then $g \circ f$ is also α -convex on $\mathbb{R}^+$.

Proof: Let $\mu \in [0, 1]$, $x, y \in \mathbb{R}^+$. By α -convexity of f ,

$$f((\mu x^\alpha + (1-\mu)y^\alpha)^{\frac{1}{\alpha}}) \leq \mu f(x) + (1-\mu)f(y).$$

g being increasing on $f(\mathbb{R}^+)$, we deduce that

$$g(f((\mu x^\alpha + (1-\mu)y^\alpha)^{\frac{1}{\alpha}})) \leq g(\mu f(x) + (1-\mu)f(y)).$$

Therefore, convexity of g leads to

$$(g \circ f)((\mu x^\alpha + (1-\mu)y^\alpha)^{\frac{1}{\alpha}}) \leq \mu(g \circ f)(x) + (1-\mu)(g \circ f)(y),$$

the result is therefore obtained.

Corollary 3.4: For all $\mu \in [0, 1]$, $\alpha \in]0, 1]$, $n \in \mathbb{N}^*$ and $x, y \in \mathbb{R}^+$, this inequality is verified:

$$(\mu x^\alpha + (1-\mu)y^\alpha)^n \leq \mu x^{n\alpha} + (1-\mu)y^{n\alpha}.$$

In particular, the identity function is $\frac{1}{n}$ -convex on $\mathbb{R}^+$, for any $n \in \mathbb{N}^*$.

Proof: Let $\mu \in [0, 1]$, $\alpha \in]0, 1]$, $n \in \mathbb{N}^*$ and $x, y \in \mathbb{R}^+$. The function $f(t) = t^\alpha$ is α -convex on $\mathbb{R}^+$ (example 3), and the function $g(t) = t^n$ is obviously convex and increasing on $f(\mathbb{R}^+) = \mathbb{R}^+$. Therefore, $g \circ f$ is α -convex on $\mathbb{R}^+$ according to the proposition 3.3. That is

$$(g \circ f)((\mu x^\alpha + (1-\mu)y^\alpha)^{\frac{1}{\alpha}}) \leq \mu(g \circ f)(x) + (1-\mu)(g \circ f)(y).$$

It follows that

$$(\mu x^\alpha + (1-\mu)y^\alpha)^n \leq \mu x^{n\alpha} + (1-\mu)y^{n\alpha},$$

because $g \circ f(t) = t^{n\alpha}$. In particular, for $\alpha = \frac{1}{n}$ we get

$$(\mu x^{\frac{1}{n}} + (1-\mu)y^{\frac{1}{n}})^n \leq \mu x + (1-\mu)y,$$

which translates that the identity function is $\frac{1}{n}$ -convex on $\mathbb{R}^+$ for any $n \in \mathbb{N}^*$.

Proposition 3.5: Let's suppose that $f: \mathbb{R}^+ \rightarrow \mathbb{R}$ is continuous, strictly increasing and α -convex on $\mathbb{R}^+$.

Then for any $\mu \in [0, 1]$ and any $t_1, t_2 \in I \subset \mathbb{R}^+$ with $z_1 = f(t_1)$ and $z_2 = f(t_2)$ one has

$$\mu(f^{-1}(z_1))^\alpha + (1-\mu)(f^{-1}(z_2))^\alpha \leq (f^{-1}(\mu z_1 + (1-\mu)z_2))^\alpha.$$

Proof: Let $\mu \in [0,1]$, $t_1, t_2 \in I$, $z_1 = f(t_1)$, $z_2 = f(t_2)$.

By α -convexity of f we get

$$f((\mu t_1^\alpha + (1-\mu)t_2^\alpha)^{\frac{1}{\alpha}}) \leq \mu f(t_1) + (1-\mu)f(t_2).$$

Thus we deduce according to theorem 2.1 that

$$(\mu t_1^\alpha + (1-\mu)t_2^\alpha)^{\frac{1}{\alpha}} \leq f^{-1}(\mu f(t_1) + (1-\mu)f(t_2)).$$

Finally, f is bijective by theorem 2.1, and therefore $t_1 = f^{-1}(z_1)$, $t_2 = f^{-1}(z_2)$. Replacing each variable by its value we obtain the result :

$$\mu(f^{-1}(z_1))^\alpha + (1-\mu)(f^{-1}(z_2))^\alpha \leq (f^{-1}(\mu z_1 + (1-\mu)z_2))^\alpha.$$

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