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New type of fractional Langevin equations of sequential derivatives

Ouafaa Benhamida *
Department of Sciences Exacts
Teacher Education College of Mostaganem
(Labo LAMDA-RO)
University of Mostaganem
Mostaganem 27000
Algeria

Zoubir Dahmani †
Department of Mathematics
(Labo LAMDA-RO)
University of Blida 1
Blida 09000
Algeria

Abstract

The aim of this work is in using Mittag-Leffler functions to study a new type of Langevin problems. We shall study the question of existence of unique solutions for a new type of fractional Langevin type equations of sequential derivatives. Using fixed point theory on Banach spaces, two main results are discussed. At the end, some examples are presented.

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1. Introduction

Over the last years, fractional calculus theory had an applications in many phenomena of sciences, see [1], [30], [31], [34], [41], [44], [46], [50].

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* *E-mail:* **w.benhamida@yahoo.com;**
ouafaa.benhamida@univ-mosta.dz (Corresponding Author)
† *E-mail:* **zzdahmani@yahoo.fr;**
dahmani_zoubir@univ-blida.dz

Langevin equation is a stochastic differential equation. The first proposed formula for this equation is found by Langevin in 1908 for describe the evolution of physical phenomena in fluctuating environments, see [9], [56]. After that, various versions of this kind of equation have been issued for application in different area of real word problems, see [11],[38],[40],[42]. The application of this equation in fractional version has been appear in different fields, for more details, see [10],[29],[36] [39].

For other developments of the fractional Langevin equation concerning the existence theory of solution, many researches investigated different existence and uniqueness results, see, for example [2], [3], [4], [5], [6], [7], [8], [12], [13], [15], [16], [17], [18], [19], [22], [23], [24], [27], [28], [35], [37], [45], [46], [48], [49], [51], [53], [54], [55].

In this sense, in [20], the authors were focus to find extremals solutions of the following fractional Langevin equation

$$\begin{cases} \mathcal{D}^{\mathrm{u}}(\mathcal{D}^{\mathrm{v}}+\wp)\aleph(\iota)-\mathcal{G}(\iota,\aleph(\iota))=0 \quad \mathrm{u},\mathrm{v}\in]0,1],\iota\in[0,\mathbb{T}] \\ \mathcal{F}(\aleph(0),\aleph(\mathbb{T}))=0,\mathcal{D}^{\mathrm{u}}\aleph(0)=\aleph_{\mathrm{u}}, \end{cases}$$

noting that $\mathcal{D}^{\mathrm{u}},\mathcal{D}^{\mathrm{v}}$ are two derivatives in Caputo sense.

Then, H. Fazli et al. [21] presented specific findings regarding the existence and uniqueness results for the following Langevin equation involving the Caputo derivative.

$$\begin{cases} \mathcal{D}^{\mathrm{u}}(\mathcal{D}^{\mathrm{v}}+\wp)\aleph(\iota)-f(\iota,\aleph(\iota))=0 \quad \mathrm{u},\mathrm{v}\in]0,1],\iota\in[0,\mathbb{T}] \\ \aleph^{(i)}(0)=\alpha_i \quad 0\le i<l \\ \aleph^{(i+\alpha)}(0)=\zeta_i \quad 0\le i<\mathrm{n}, \end{cases}$$

For $\mathrm{u}\in]\mathrm{m}-1,\mathrm{m}]$, $\mathrm{v}\in]\mathrm{n}-1,\mathrm{n}]$, and l is the biggest value between m and n, for any natural numbers m, and n, $\mathcal{D}^{\mathrm{u}},\mathcal{D}^{\mathrm{v}}$ are the Caputo-type derivatives.

Also, in [20], the authors were focused on a specific problem involving Caputo derivative operators, as follow

$$\begin{cases} \mathcal{D}^{\mathrm{u}}(\mathcal{D}^{\mathrm{v}}(D^{\delta}\mathrm{w}(\iota)))+\mathcal{F}(\iota,\mathrm{w}(\iota),\mathcal{D}^{p}\mathrm{w}(\iota))+\mathcal{G}(\iota,\mathrm{w}(\iota),\mathcal{I}^{q}\mathrm{w}(\iota)+\mathcal{R}(\iota,\mathrm{w}(\iota))=l(\iota) \\ \mathrm{w}(0)=\varepsilon \\ \mathrm{w}(1)=\int_0^{\nu}\mathrm{w}(s)ds,0<\nu<1 \\ \mathcal{I}^{e}\mathrm{w}(\theta)=D^{\mu}\mathrm{w}(1),0<\mu<1 \\ 0<\mathrm{u},\mathrm{v},\delta,p\le 1 \end{cases}$$

We can also cite the paper of Y. Gouari et al. [25], where the authors gave some existence results for the following nonlinear (Langevin type) differential problem with nonlocal integral conditions and convergence series:

$$\begin{cases} {}_C D^{\alpha_1}[{}_C D^{\alpha_2}[{}_C D^{\alpha_3}...[{}_C D^{\alpha_n}[{}_{RL}D^{\beta}u(s)]]...]] = \Lambda_1 f(\iota, u(\iota), {}_{RL}D^{\gamma}u(\iota)) \\ +\Lambda_2 f(\iota, u(\iota), I^{\varepsilon}u(\iota)) + \sum_{j=1}^{\infty}\phi_j(\iota) I^{\varepsilon}[h_j(\iota, u(\iota)) + l_j], \iota \in [0,1] \\ u(0) = 0 \\ {}_C D^{\alpha_2}[{}_C D^{\alpha_3}[{}_C D^{\alpha_4}...[{}_C D^{\alpha_n}[{}_{RL}D^{\beta}u(0)]]...]] = \theta, \theta > 0 \\ {}_C D^{\alpha_3}[{}_C D^{\alpha_4}[{}_C D^{\alpha_5}...[{}_C D^{\alpha_n}[{}_{RL}D^{\beta}u(0)]]...]] = 0 \\ {}_C D^{\alpha_4}[{}_C D^{\alpha_5}[{}_C D^{\alpha_6}...[{}_C D^{\alpha_n}[{}_{RL}D^{\beta}u(0)]]...]] = 0 \\ \quad . \\ \quad . \\ \quad . \\ {}_C D^{\alpha_{n-1}}[{}_C D^{\alpha_n}[{}_{RL}D^{\beta}u(0)]] = 0, \\ {}_C D^{\alpha_n}[{}_{RL}D^{\beta}u(0)]] = \int_0^{\eta}u(s)ds, \ \eta \in]0,1[\\ {}_{RL}D^{\beta}u(1) = {}_{RL}D^{\varphi}u(\tau), \ \tau \in]0,1[, \end{cases}$$

where $J := [0,1], 0 \le \beta, \varphi, \alpha_i \in [0,1[; \gamma < \alpha_i, i = \overline{1,n},$ and $\gamma, \Lambda_1, \Lambda_2 > 0.$

Very recently, K. Bensassaa and Z. Dahmani in [5] were focused with the study of the following system involving sequential Caputo derivatives:

$$\begin{cases} D^{\alpha_1}D^{\alpha_2}D^{\alpha_3}D^{\alpha_4}\mathbf{x}(\iota) = H_1(\iota, \mathbf{x}(\iota), \mathbf{y}(\iota)) + a_1 f_1(\mathbf{x}(\iota)) + b_1 g_1(D^{\alpha_1}D^{\alpha_2}\mathbf{x}(\iota)), \\ D^{\beta_1}D^{\beta_2}D^{\beta_3}D^{\beta_4}\mathbf{y}(\iota) = H_2(\iota, \mathbf{x}(\iota), \mathbf{y}(\iota)) + a_2 f_2(\mathbf{y}(\iota)) + b_2 g_2(D^{\beta_1}D^{\beta_2}\mathbf{y}(\iota)) \\ \mathbf{x}(0) = \mathbf{x}(1) = D^{\alpha_1}D^{\alpha_2}\mathbf{x}(1) = D^{\alpha_4}\mathbf{x}(0) = 0 \quad \forall \iota \in [0,1] \\ \mathbf{y}(0) = \mathbf{y}(1) = D^{\beta_1}D^{\beta_2}\mathbf{y}(1) = D^{\beta_4}\mathbf{y}(0) = 0, \end{cases}$$

where, D^{α_1}, D^{α_2}, D^{α_3}, D^{α_4}, D^{β_1}, D^{β_2}, D^{β_3}, D^{β_4}, are Caputo fractional derivative, $(\alpha_i, \beta_i) \in]0,1] \times]0,1]$, $i = 1, ..., 4$, $\alpha_2 + \alpha_1 < \alpha_4$, $\beta_2 + \beta_1 < \beta_4$, $f_j : \mathbb{R} \to \mathbb{R}$, $g_j : \mathbb{R} \to \mathbb{R}$ and $H_j : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$, $j = 1,2$, are continuous functions.

In the present paper, we shall introduce a new differential problem of the Langevin fractional differential equation type that involves sequential form derivative of Caputo. The proposed problem is complemented with initial conditions. So, we consider the following problem:

$$\begin{cases} {}^{C}D^{\varepsilon_1}\ldots{}^{C}D^{\varepsilon_{\mathbf{n}}}[{}^{C}D^{\delta}+\lambda]\mathfrak{Z}(t)=f(t,\mathfrak{Z}(t)) & ,0<\varepsilon_\iota\le 1,\iota=1,\ldots,\mathbf{n},\mathbf{m}-1<\delta\le\mathbf{m} \\ {}^{C}D^{\sum_{j=\iota+1}^{\mathbf{n}}\varepsilon_j+\delta}\mathfrak{Z}(0)+\lambda\,{}^{C}D^{\sum_{j=\iota+1}^{\mathbf{n}}\varepsilon_j}\mathfrak{Z}(0)=\Delta_\iota & \text{for } \iota=1,\ldots,\mathbf{n}-1 \\ {}^{C}D^{\delta}\mathfrak{Z}(0)+\lambda\mathfrak{Z}(0)=\Lambda_\delta & \\ \mathfrak{Z}^{l}(0)=\Lambda_l & l=0,\ldots,\mathbf{m}-1, \end{cases} \tag{1}$$

where, ${}^{C}D^{(.)}$ is The operator of the Caputo derivative of non-integer order, $t\in 0,\mathbb{T}]$, $\lambda\in\mathbb{R}_{+}^{\not 0}$, $f:[0,\mathbb{T}]\times\mathbb{R}\to\mathbb{R}$ is a given function.

The goal of this work lies in the following two points:

- Formulate a new class of problems concerning Langevin type fractional differential equation.
- Use the properties of Mittag-Leffler function (see for instance [21], [47], [52]) to investigate the proposed results.

2. Preliminaries

We recall the following definitions that are in [21], [26], [32], [33], [43], [52].

- The notation that represents the Banach space of all continuous functions $\mathfrak{Z}$ defined from $[0,\mathbb{T}]$ to the set of real numbers is $C([0,\mathbb{T}],\mathbb{R})$, with

$$\|\mathfrak{Z}\|_\infty = \underset{t\in 0,\mathbb{T}]}{Sup}|\mathfrak{Z}(t)|.$$

- Let

$$AC^{n}([0,\mathbb{T}],\mathbb{R}):=\{\mathfrak{Z}:[0,\mathbb{T}]\to\mathbb{R}:\tfrac{d^{n}\mathfrak{Z}(t)}{dt^{n}}\in AC([0,\mathbb{T}],\mathbb{R})\},$$

such that $AC([0,\mathbb{T}],\mathbb{R})$ is space of the absolutely continuous functions.

Definition 2.1: [33] [43] The integral in the sense of R-L of order $\theta>0$ of $\mathfrak{Z}:[0,+\infty)\to\mathbb{R}$ which is continuous is given by

$$I^{\theta}\mathfrak{Z}(t)=\frac{1}{\Gamma(\theta)}\int_0^t (t-s)^{\theta-1}\mathfrak{Z}(s)ds.$$

Definition 2.2: [33] [32] For any function $\mathfrak{z}$ given on the interval $[0,+\infty)$, the derivative of Caputo of order θ is defined by

$$^{C}D^{\theta}\mathfrak{z}(t)=\frac{1}{\Gamma(\mathbf{n}-\theta)}\int_0^t (t-s)^{\mathbf{n}-\theta-1}\mathfrak{z}^{(\mathbf{n})}(s)ds,$$

such that $[\theta]$ represents the integer part of θ and $\mathbf{n}=[\theta]+1$.

Lemma 2.3: [33][43] *For* $\theta>0$, *the homogenous equation*

$$^{C}D^{\theta}\mathfrak{z}(t)=0,$$

has solutions

$$\mathfrak{z}(t)=\sum_{\iota=0}^{n-1}c_{\iota}t^{\iota},$$

under the fact that $c_{\iota}\in\mathbb{R},\iota=\overline{0,n-1}$, *and* $n=[\theta]+1$.

Lemma 2.4: [33][43] *For* $\theta>0$, *and* $\mathfrak{z}\in AC^{n}([0,\mathbb{T}],\mathbb{R})$, *we have*

$$I^{\theta C}D^{\theta}\mathfrak{z}(t)=\sum_{\iota=0}^{n-1}c_{\iota}t^{\iota}+\mathfrak{z}(t),$$

with $c_{\iota}\in\mathbb{R},\iota=0,\dots,n-1,n-1=[\theta]$.

Definition 2.5: [47] The canonical expression for the Mittag-Leffler function with two parameter is given as

$$E_{\varepsilon,\beta}(z)=\sum_{\iota=0}^{\infty}\frac{z^{\iota}}{\Gamma(\beta+\alpha\iota)},\varepsilon>0,\beta>0,$$

for all $z\in\mathbb{R}$.

Lemma 2.6: [26] *For* $\delta,\gamma\in\mathbb{R}_{+}$, *and* $m\in\mathbb{N}$, *we have*

$$t^{m}E_{\delta,\varepsilon+m\delta}(t)=E_{\delta,\varepsilon}(t)-\sum_{\iota=0}^{m-1}\frac{t^{\iota}}{\Gamma(\varepsilon+\iota\delta)}.$$

In particular

$$E_{\delta,\varepsilon}(t)=tE_{\delta,\delta+\varepsilon}(t)+\frac{1}{\Gamma(\varepsilon)}. \tag{2}$$

Lemma 2.7: [52], [21] *Let* $\alpha,\beta>0$. *The functions* $E_{\alpha},E_{\alpha,\alpha}$ *and* $E_{\alpha,\alpha+\beta}$ *are nonnegative with:*

(i) If $\lambda>0$ and $t\in[0,\mathbb{T}]$,

$$E_{\alpha,\alpha}(-\lambda t^{\alpha}) \le \frac{1}{\Gamma(\alpha)}, E_{\alpha,\beta}(-\lambda t^{\alpha}) \le \frac{1}{\Gamma(\alpha+\beta)} \tag{3}$$

$$E_{\alpha,\alpha}(t) = \alpha E'_{\alpha}(t) \tag{4}$$

$$E_{\alpha,\alpha+\beta}(t) = \int_0^1 \frac{u^{\alpha-1}(1-u)^{\beta-1}}{\Gamma(\beta)} E_{\alpha,\alpha}(u^{\alpha} t)du \tag{5}$$

(ii) For any $\lambda > 0$ and $t_2, t_1 \in [0,\mathbb{T}]$:

$$|E_{\alpha,\alpha}(-\lambda t_2^{\alpha}) - E_{\alpha,\alpha}(-\lambda t_1^{\alpha})| := O(|t_2^{\alpha} - t_1^{\alpha}|) \text{ as } t_2 \to t_1 \tag{6}$$

$$|E_{\alpha,\alpha+\beta}(-\lambda t_2^{\alpha}) - E_{\alpha,\alpha+\beta}(-\lambda t_1^{\alpha})| := O(|t_2^{\alpha} - t_1^{\alpha}|) \text{ as } t_2 \to t_1 \tag{7}$$

3. Main Results

In this section, we begin by presenting the integral form of solutions of the problem (1). Then, we can pass to investigate our results. Before doing that, we give the definition:

Definition 3.1: A function $\mathfrak{Z} \in C([0,\mathbb{T}],\mathbb{R})$, such that its Caputo derivative is well defined on $[0,\mathbb{T}]$, is considered as solution of (1) if: (1).

For the existence results of solutions for (1), we have to work with:

$$\begin{cases} {}^{C}D^{\varepsilon_1} \dots {}^{C}D^{\varepsilon_{\mathbf{n}}}[{}^{C}D^{\delta} + \lambda]\mathfrak{Z}(t) = h(t) & ,0 < \varepsilon_{\iota} \le 1, \iota = 1,\dots,\mathbf{n}, \mathbf{m}-1 < \delta \le \mathbf{m} \\ {}^{C}D^{\sum_{j=\iota+1}^{\mathbf{n}} \varepsilon_j + \delta}\mathfrak{Z}(0) + \lambda {}^{C}D^{\sum_{j=\iota+1}^{\mathbf{n}} \varepsilon_j}\mathfrak{Z}(0) = \Delta_{\iota} & \iota = 1,\dots,\mathbf{n}-1 \\ {}^{C}D^{\delta}\mathfrak{Z}(0) + \lambda\mathfrak{Z}(0) = \Lambda_{\delta} & \\ \mathfrak{Z}^{l}(0) = \Lambda_l & l = 0,\dots,\mathbf{m}-1 \end{cases} \tag{8}$$

where $t \in [0,\mathbb{T}]$, λ is a strictly positive real number, and $h \in C([0,\mathbb{T}],\mathbb{R})$.

Lemma 3.2: *The set of solutions of (8)is given by:*

$$\mathfrak{Z}(t) = \sum_{l=0}^{\mathbf{m}-1} \Lambda_l t^l E_{\delta,l+1}(-\lambda t^{\delta}) + \Lambda_{\delta} t^{\delta} E_{\delta,\delta}(-\lambda t^{\delta}) + \frac{1}{\lambda}\sum_{\iota=1}^{\mathbf{n}-1} \Delta_{\iota} t^{\sum_{j=\iota+1}^{\mathbf{n}} \varepsilon_j} \left(\frac{1}{\Gamma\left(\sum_{j=\iota+1}^{\mathbf{n}} \varepsilon_j + 1\right)} - E_{\delta}(-\lambda t^{\delta}) \right)$$

$$+ \int_0^t (t-\rho)^{\sum_{\iota=1}^{\mathbf{n}} \varepsilon_{\iota} + \delta - 1} E_{\delta, \sum_{\iota=1}^{\mathbf{n}} \varepsilon_{\iota} + \delta}(-\lambda(t-\rho)^{\delta}) h(\rho) d\rho.$$

Proof: We take $\mathfrak{Z}(t)$ as a solution of (8). The fact that $h \in C([0,\mathbb{T}],\mathbb{R})$ gives

$$^{C}D^{\varepsilon_1} \dots {}^{C}D^{\varepsilon_n}[{}^{C}D^{\delta} + \lambda]\mathfrak{Z}(t) = h(t).$$

By using lemma 2.4, we get

$$[{}^{C}D^{\delta} + \lambda]\mathfrak{Z}(t) = I^{\sum_{\iota=1}^{\mathbf{n}} \varepsilon_\iota} h(t) + \sum_{\iota=1}^{\mathbf{n}-1} \Delta_\iota \frac{t^{\sum_{j=\iota+1}^{\mathbf{n}} \varepsilon_j}}{\Gamma\left(\sum_{j=\iota+1}^{\mathbf{n}} \varepsilon_j + 1\right)} + \Lambda_\delta.$$

As a result of Theorem 7.2 in [14], the solution of (8) is:

$$\mathfrak{Z}(t) = \sum_{l=1}^{\mathbf{m}-1} \Lambda_l t^l E_{\delta,l+1}(-\lambda t^\delta) - \frac{1}{\lambda}\int_0^t e'_\delta(z) w(t-z)dz,$$

where $w(z) = I^{\sum_{\iota=1}^{\mathbf{n}} \varepsilon_\iota} h(z) + \sum_{\iota=1}^{\mathbf{n}-1} \Delta_\iota \frac{z^{\sum_{j=\iota+1}^{\mathbf{n}} \varepsilon_j}}{\Gamma\left(\sum_{j=\iota+1}^{\mathbf{n}} \varepsilon_j + 1\right)} + \Lambda_\delta$, $e_\delta(z) = E_\delta(-\lambda z^\delta)$, and $e'_\delta(z) = -\lambda z^{\delta-1} E_{\delta,\delta}(-\lambda z^\delta)$.

So,

$$\begin{aligned}
\mathfrak{Z}(t) &= \sum_{l=0}^{\mathbf{m}-1} \Lambda_l t^l E_{\delta,l+1}(-\lambda t^\delta) - \frac{1}{\lambda}\int_0^t e'_\delta(z) w(t-z)dz \\
&= \sum_{l=0}^{\mathbf{m}-1} \Lambda_l t^l E_{\delta,l+1}(-\lambda t^\delta) + \int_0^t z^{\delta-1} E_{\delta,\delta}(-\lambda z^\delta) w(t-z)dz \\
&= \sum_{l=0}^{\mathbf{m}-1} \Lambda_l t^l E_{\delta,l+1}(-\lambda t^\delta) + \int_0^t (t-\rho)^{\delta-1} E_{\delta,\delta}(-\lambda(t-\rho)^\delta) w(\rho)d\rho \\
&= \sum_{l=0}^{\mathbf{m}-1} \Lambda_l t^l E_{\delta,l+1}(-\lambda t^\delta) + \Lambda_\delta \int_0^t (t-\rho)^{\delta-1} E_{\delta,\delta}(-\lambda(t-\rho)^\delta)d\rho \\
&\quad + \int_0^t (t-\rho)^{\delta-1} E_{\delta,\delta}(-\lambda(t-\rho)^\delta) \sum_{\iota=1}^{\mathbf{n}-1} \Delta_\iota \frac{\rho^{\sum_{j=\iota+1}^{\mathbf{n}} \varepsilon_j}}{\Gamma(\sum_{j=\iota+1}^{\mathbf{n}} \varepsilon_j + 1)} d\rho \\
&\quad + \int_0^t (t-\rho)^{\delta-1} E_{\delta,\delta}(-\lambda(t-\rho)^\delta) I^{\sum_{\iota=1}^{\mathbf{n}} \varepsilon_\iota} h(\rho)d\rho.
\end{aligned}$$

By using the result (see [21]), for $u > 0$:

$$\int_0^t (t-\rho)^{\mu-1} E_{\mu,\mu}(-\lambda(t-\rho)^\mu)\rho^u d\rho = \frac{1}{\lambda}(t^u - \Gamma(u+1)t^u E_\mu(-\lambda t^\mu)),$$

we conclud that

$$\begin{aligned}
\mathfrak{Z}(t) &= \sum_{l=0}^{\mathbf{m}-1} \Lambda_l t^l E_{\delta,l+1}(-\lambda t^\delta) + \Lambda_\delta t^\delta E_{\delta,\delta}(-\lambda t^\delta) \\
&+ \sum_{\iota=1}^{\mathbf{n}-1} \frac{\Delta_\iota}{\Gamma\left(\sum_{j=\iota+1}^{\mathbf{n}} \varepsilon_j + 1\right)} \left(\frac{1}{\lambda} \left(t^{\sum_{j=\iota+1}^{\mathbf{n}} \varepsilon_j} - \Gamma\left(\sum_{j=\iota+1}^{\mathbf{n}} \varepsilon_j + 1\right) t^{\sum_{j=\iota+1}^{\mathbf{n}} \varepsilon_j} E_\delta(-\lambda t^\delta) \right)\right) \\
&+ \int_0^t (t-\rho)^{\delta-1} E_{\delta,\delta}(-\lambda (t-\rho)^\delta) I^{\sum_{\iota=1}^{\mathbf{n}} \varepsilon_\iota} h(\rho) d\rho \\
&= \sum_{l=0}^{\mathbf{m}-1} \Lambda_l t^l E_{\delta,l+1}(-\lambda t^\delta) + \Lambda_\delta t^\delta E_{\delta,\delta}(-\lambda t^\delta) \\
&+ \frac{1}{\lambda} \sum_{\iota=1}^{\mathbf{n}-1} \Delta_\iota t^{\sum_{j=\iota+1}^{\mathbf{n}} \varepsilon_j} \left(\frac{1}{\Gamma\left(\sum_{j=\iota+1}^{\mathbf{n}} \varepsilon_j + 1\right)} - E_\delta(-\lambda t^\delta) \right) \\
&+ \int_0^t (t-\rho)^{\delta + \sum_{j=\iota+1}^{\mathbf{n}} \varepsilon_\iota - 1} E_{\delta, \sum_{\iota=1}^{\mathbf{n}} \varepsilon_\iota + \delta} (-\lambda (t-\rho)^\delta) h(\rho) d\rho.
\end{aligned}$$

This completes the proof.

Notice that (1) has a solution provided that the operator defined from $C([0,\mathbb{T}],\mathbb{R})$ to $C([0,\mathbb{T}],\mathbb{R})$ by:

$$\begin{aligned}
(N\mathfrak{Z})(t) &= \sum_{l=0}^{\mathbf{m}-1} \Lambda_l t^l E_{\delta,l+1}(-\lambda t^\delta) + \Lambda_\delta t^\delta E_{\delta,\delta}(-\lambda t^\delta) \\
&+ \frac{1}{\lambda} \sum_{\iota=1}^{\mathbf{n}-1} \Delta_\iota t^{\sum_{j=\iota+1}^{\mathbf{n}} \varepsilon_j} \left(\frac{1}{\Gamma\left(\sum_{j=\iota+1}^{\mathbf{n}} \varepsilon_j + 1\right)} - E_\delta(-\lambda t^\delta) \right) \\
&+ \int_0^t (t-\rho)^{\delta + \sum_{\iota=1}^{\mathbf{n}} \varepsilon_\iota - 1} E_{\delta, \sum_{\iota=1}^{\mathbf{n}} \varepsilon_\iota + \delta} (-\lambda (t-\rho)^\delta) f(\rho, \mathfrak{Z}(\rho)) d\rho \qquad (9)
\end{aligned}$$

admits a fixed point.

We begin by proving the theorem:

Theorem 3.3: *Assume hold the conditions:*

(HP_1) One has $k > 0$ such that, for all $\overline{\mathfrak{Z}}, \overline{\mathfrak{Z}}' \in \mathbb{R}$, it yields that

$$|f(t,\overline{\mathfrak{Z}}) - f(t,\overline{\mathfrak{Z}}')| \le k |\overline{\mathfrak{Z}} - \overline{\mathfrak{Z}}'|, \forall t \in [0,\mathbb{T}].$$

If

$$\frac{k\mathbb{T}^{\sum_{\iota=1}^{\mathbf{n}}\varepsilon_\iota+\delta}}{\Gamma\left(\sum_{\iota=1}^{\mathbf{n}}\varepsilon_\iota+1+\delta\right)}<1,$$

then (1) has one and only one solution on $[0,\mathbb{T}]$.

Proof: We aim to demonstrate that the mapping N, as specified in equation (9), satisfies the conditions of contraction.

For $\overline{\mathfrak{z}},\overline{\mathfrak{z}}'\in C([0,\mathbb{T}],\mathbb{R})$, and $\forall t\in[0,\mathbb{T}]$, we have

$$\left|N\overline{\mathfrak{z}}(t)-N\overline{\mathfrak{z}}'(t)\right|\le\int_0^t(t-\rho)^{\delta+\sum_{\iota=1}^{\mathbf{n}}\varepsilon_\iota-1}E_{\delta,\sum_{\iota=1}^{\mathbf{n}}\varepsilon_\iota+\delta}(-\lambda(t-\rho)^\delta)\,|f(\rho,\overline{\mathfrak{z}}(\rho))-f(\rho,\overline{\mathfrak{z}}'(\rho))|\,d\rho$$

$$\le\frac{k}{\Gamma\left(\delta+\sum_{\iota=1}^{\mathbf{n}}\varepsilon_\iota\right)}\int_0^t(t-\rho)^{\delta+\sum_{\iota=1}^{\mathbf{n}}\varepsilon_\iota-1}|\overline{\mathfrak{z}}(\rho)-\overline{\mathfrak{z}}'(\rho)|\,d\rho$$

$$\le\frac{k\|\overline{\mathfrak{z}}-\overline{\mathfrak{z}}'\|_\infty}{\Gamma\left(\delta+\sum_{\iota=1}^{\mathbf{n}}\varepsilon_\iota\right)}\int_0^t(t-\rho)^{\delta+\sum_{\iota=1}^{\mathbf{n}}\varepsilon_\iota-1}d\rho$$

$$\le\frac{k\mathbb{T}^{\delta+\sum_{\iota=1}^{\mathbf{n}}\varepsilon_\iota}}{\Gamma\left(\delta+\sum_{\iota=1}^{\mathbf{n}}\varepsilon_\iota+1\right)}\|\overline{\mathfrak{z}}-\overline{\mathfrak{z}}'\|_\infty .$$

Thus,

$$\|N\overline{\mathfrak{z}}-N\overline{\mathfrak{z}}'\|_\infty\le\frac{k\mathbb{T}^{\delta+\sum_{\iota=1}^{\mathbf{n}}\varepsilon_\iota}}{\Gamma\left(\delta+\sum_{\iota=1}^{\mathbf{n}}\varepsilon_\iota+1\right)}\|\overline{\mathfrak{z}}-\overline{\mathfrak{z}}'\|_\infty .$$

The theorem is proved.

Next, by applying the Schaefer fixed point theorem, we prove an existence result of solutions of our problem.

Theorem 3.4: *Assume the validation of:*

(HP_2) Our function *f* can be continuous.

(HP_3) One has a strictly positive number $M>0$, such that

$$|f(t,\mathfrak{z}(t))|\le M,\text{for each } t\in[0,\mathbb{T}],\forall\mathfrak{z}\in C([0,\mathbb{T}],\mathbb{R})$$

Then, (1) has one or more solutions in $C([0,\mathbb{T}],\mathbb{R})$.

Proof: We utilise Schaefer theorem.

Firstly: Let $\{\mathfrak{z}_{\mathbf{n}}\}$ with $\mathfrak{z}_{\mathbf{n}} \to \mathfrak{z}$ in $C([0,\mathbb{T}],\mathbb{R})$. So, taking $t \in [0,\mathbb{T}]$

$$|N\mathfrak{z}_{\mathbf{n}}(t) - N\mathfrak{z}(t)| \le \int_0^t (t-\rho)^{\delta+\sum_{\iota=1}^{\mathbf{n}}\varepsilon_j - 1} E_{\delta,\sum_{\iota=1}^{\mathbf{n}}\varepsilon_\iota+\delta}(-\lambda(t-\rho)^\delta)\,|f(\rho,\mathfrak{z}_{\mathbf{n}}(\rho)) - f(\rho,\mathfrak{z}(\rho))|\,d\rho$$

$$\le \frac{\|f(.,\mathfrak{z}_{\mathbf{n}}(.)) - f(.,\mathfrak{z}(.))\|_\infty}{\Gamma\left(\delta+\sum_{\iota=1}^{\mathbf{n}}\varepsilon_\iota\right)} \int_0^t (t-\rho)^{\delta+\sum_{\iota=1}^{\mathbf{n}}\varepsilon_\iota - 1} d\rho$$

$$\le \frac{\mathbb{T}^{\delta+\sum_{\iota=1}^{\mathbf{n}}\varepsilon_\iota}}{\Gamma\left(\delta+\sum_{\iota=1}^{\mathbf{n}}\varepsilon_\iota+1\right)} \| f(.,\mathfrak{z}_{\mathbf{n}}(.)) - f(.,\mathfrak{z}(.))\|_\infty$$

We have *f* continuous. Hence,

$$\| N\mathfrak{z}_{\mathbf{n}} - N\mathfrak{z}\|_\infty \to 0 \text{ as } n \to \infty.$$

Hence, *N* is continuous too.

Secondly: We need to prove, $\forall \mu > 0, \exists r > 0,$ what follows:

$$\forall \mathfrak{z} \in B_\mu = \{\mathfrak{z} \in C([0,\mathbb{T}],\mathbb{R}), \|\mathfrak{z}\|_\infty \le \mu\} \Rightarrow \| N\mathfrak{z}\|_\infty \le r.$$

$\forall t \in [0,\mathbb{T}],$ we have

$$|N\mathfrak{z}(t)| \le \left|\sum_{l=0}^{\mathbf{m}-1} \Lambda_l t^l E_{\delta,l+1}(-\lambda t^\delta)\right| + |\Lambda_\delta t^\delta E_{\delta,\delta}(-\lambda t^\delta)|$$
$$+\left|\frac{1}{\lambda}\sum_{\iota=1}^{\mathbf{n}-1} \Delta_\iota t^{\sum_{j=\iota+1}^{\mathbf{n}}\varepsilon_j}\left(\frac{1}{\Gamma\left(\sum_{j=\iota+1}^{\mathbf{n}}\varepsilon_j+1\right)} - E_\delta(-\lambda t^\delta)\right)\right|$$
$$+\int_0^t (t-\rho)^{\delta+\sum_{\iota=1}^{\mathbf{n}}\varepsilon_\iota - 1} E_{\delta,\sum_{\iota=1}^{\mathbf{n}}\varepsilon_\iota+\delta}(-\lambda(t-\rho)^\delta)\,|f(\rho,\mathfrak{z}(\rho))|\,d\rho$$
$$\le \sum_{l=0}^{\mathbf{m}-1}\frac{|\Lambda_l| t^l}{\Gamma(l+1)} + |\Lambda_\delta|\, t^\delta E_{\delta,\delta}(-\lambda t^\delta) + \frac{1}{\lambda}\sum_{\iota=1}^{\mathbf{n}-1}|\Delta_\iota|\, t^{\sum_{j=\iota+1}^{\mathbf{n}}\varepsilon_j}\left|\frac{1}{\Gamma\left(\sum_{j=\iota+1}^{\mathbf{n}}\varepsilon_j+1\right)} - E_\delta(-\lambda t^\delta)\right|$$
$$+\frac{M}{\Gamma\left(\delta+\sum_{\iota=1}^{\mathbf{n}}\varepsilon_\iota\right)}\int_0^t (t-\rho)^{\delta+\sum_{\iota=1}^{\mathbf{n}}\varepsilon_\iota - 1} d\rho$$

$$\leq \sum_{l=0}^{\mathbf{m}-1}\frac{|\Lambda_l|\mathbb{T}^l}{\Gamma(l+1)}+\frac{|\Lambda_\delta|\mathbb{T}^\delta}{\Gamma(\delta)}+\frac{1}{\lambda}\sum_{\iota=1}^{\mathbf{n}-1}|\Delta_\iota|\,\mathbb{T}^{\sum_{j=\iota+1}^{\mathbf{n}}\varepsilon_j}\left(\frac{1}{\Gamma\left(\sum_{j=\iota+1}^{\mathbf{n}}\varepsilon_j+1\right)}+1\right)$$

$$+\frac{M\mathbb{T}^{\delta+\sum_{\iota=1}^{\mathbf{n}}\varepsilon_\iota}}{\Gamma\left(\delta+\sum_{\iota=1}^{\mathbf{n}}\varepsilon_\iota+1\right)}=r.$$

So,

$$\|N\mathfrak{Z}\|_\infty\leq \sum_{l=0}^{\mathbf{m}-1}\frac{|\Lambda_l|\mathbb{T}^l}{\Gamma(l+1)}+\frac{|\Lambda_\delta|\mathbb{T}^\delta}{\Gamma(\delta)}+\frac{1}{\lambda}\sum_{\iota=1}^{\mathbf{n}-1}|\Delta_\iota|\,\mathbb{T}^{\sum_{j=\iota+1}^{\mathbf{n}}\varepsilon_j}\left(\frac{1}{\Gamma\left(\sum_{j=\iota+1}^{\mathbf{n}}\varepsilon_j+1\right)}+1\right)+\frac{M\mathbb{T}^{\delta+\sum_{\iota=1}^{\mathbf{n}}\varepsilon_\iota}}{\Gamma\left(\delta+\sum_{\iota=1}^{\mathbf{n}}\varepsilon_\iota+1\right)}=r.$$

Thus, N is bounded operator.

Thirdly: Let $t_2,t_1\in[0,\mathbb{T}]$, $t_1<t_2$, and B_μ be a set that is bounded of $C([0,\mathbb{T}],\mathbb{R})$ as above. For $\mathfrak{Z}\in B_\mu$, we get

$$\begin{aligned}|N\mathfrak{Z}(t_2)-N\mathfrak{Z}(t_1)|\leq&\left|\sum_{l=0}^{\mathbf{m}-1}\Lambda_l t_2^l E_{\delta,l+1}(-\lambda t_2^\delta)-\sum_{l=0}^{\mathbf{m}-1}\Lambda_l t_1^l E_{\delta,l+1}(-\lambda t_1^\delta)\right|\\
&+|\Lambda_\delta t_2^\delta E_{\delta,\delta}(-\lambda t_2^\delta)-\Lambda_\delta t_1^\delta E_{\delta,\delta}(-\lambda t_1^\delta)|\\
&+\frac{1}{\lambda}\left|\sum_{\iota=1}^{\mathbf{n}-1}\frac{\Delta_\iota t_2^{\sum_{j=\iota+1}^{\mathbf{n}}\varepsilon_j}}{\Gamma\left(\sum_{j=\iota+1}^{\mathbf{n}}\varepsilon_j+1\right)}-\sum_{\iota=1}^{\mathbf{n}-1}\frac{\Delta_\iota t_1^{\sum_{j=\iota+1}^{\mathbf{n}}\varepsilon_j}}{\Gamma\left(\sum_{j=\iota+1}^{\mathbf{n}}\varepsilon_j+1\right)}\right|\\
&+\frac{1}{\lambda}\left|\sum_{\iota=1}^{\mathbf{n}-1}\Delta_\iota t_2^{\sum_{j=\iota+1}^{\mathbf{n}}\varepsilon_j}E_\delta(-\lambda t_2^\delta)-\sum_{\iota=1}^{\mathbf{n}-1}\Delta_\iota t_1^{\sum_{j=\iota+1}^{\mathbf{n}}\varepsilon_j}E_\delta(-\lambda t_1^\delta)\right|\\
&+\left|\int_0^{t_2}(t_2-s)^{\delta+\sum_{\iota=1}^{\mathbf{n}}\varepsilon_\iota-1}E_{\delta,\sum_{\iota=1}^{\mathbf{n}}\varepsilon_\iota+\delta}(-\lambda(t_2-\rho)^\delta)f(\rho,\mathfrak{Z}(\rho))ds\right.\\
&\left.-\int_0^{t_1}(t_1-\rho)^{\delta+\sum_{\iota=1}^{\mathbf{n}}\varepsilon_\iota-1}E_{\delta,\sum_{\iota=1}^{\mathbf{n}}\varepsilon_\iota+\delta}(-\lambda(t_1-\rho)^\delta)f(\rho,\mathfrak{Z}(\rho))d\rho\right|\end{aligned}$$

Thanks to lemma 2.6 and lemma 2.7, we get

$$\begin{aligned}&|N\mathfrak{Z}(t_2)-N\mathfrak{Z}(t_1)|\leq\\
&\sum_{l=0}^{\mathbf{m}-1}|\Lambda_l|\left|t_2^l\left(-\lambda t_2^\delta E_{\delta,\delta+l+1}(-\lambda t_2^\delta)+\frac{1}{\Gamma(l+1)}\right)-t_1^l\left(-\lambda t_1^\delta E_{\delta,\delta+l+1}(-\lambda t_1^\delta)+\frac{1}{\Gamma(l+1)}\right)\right|\\
&+|\Lambda_\delta|\,|t_2^\delta E_{\delta,\delta}(-\lambda t_2^\delta)-t_1^\delta E_{\delta,\delta}(-\lambda t_2^\delta)+t_1^\delta E_{\delta,\delta}(-\lambda t_2^\delta)-t_1^\delta E_{\delta,\delta}(-\lambda t_1^\delta)|\end{aligned}$$

$$+\frac{1}{\lambda}\sum_{\iota=1}^{\mathbf{n}-1}\frac{|\Delta_\iota|}{\Gamma\left(\sum_{j=\iota+1}^{\mathbf{n}}\varepsilon_j+1\right)}\left|t_2^{\sum_{j=\iota+1}^{\mathbf{n}}\varepsilon_j}-t_1^{\sum_{j=\iota+1}^{\mathbf{n}}\varepsilon_j}\right|$$

$$+\frac{1}{\lambda}\sum_{\iota=1}^{\mathbf{n}-1}|\Delta_\iota|\,|t_2^{\sum_{j=\iota+1}^{\mathbf{n}}\varepsilon_j}E_\delta(-\lambda t_2^\delta)-t_1^{\sum_{j=\iota+1}^{\mathbf{n}}\varepsilon_j}E_\delta(-\lambda t_2^\delta)+t_1^{\sum_{j=\iota+1}^{\mathbf{n}}\varepsilon_j}E_\delta(-\lambda t_2^\delta)-t_1^{\sum_{j=\iota+1}^{\mathbf{n}}\varepsilon_j}E_\delta(-\lambda t_1^\delta)|$$

$$+|\int_0^{t_2}(t_2-\rho)^{\sum_{\iota=1}^{\mathbf{n}}\varepsilon_\iota+\delta-1}E_{\delta,\sum_{\iota=1}^{\mathbf{n}}\varepsilon_\iota+\delta}(-\lambda(t_2-\rho)^\delta)f(\rho,\mathfrak{Z}(\rho))d\rho$$

$$-\int_0^{t_2}(t_1-\rho)^{\sum_{\iota=1}^{\mathbf{n}}\varepsilon_\iota+\delta-1}E_{\delta,\sum_{\iota=1}^{\mathbf{n}}\varepsilon_\iota+\delta}(-\lambda(t_2-\rho)^\delta)f(\rho,\mathfrak{Z}(\rho))d\rho$$

$$+\int_0^{t_2}(t_1-\rho)^{\sum_{\iota=1}^{\mathbf{n}}\varepsilon_\iota+\delta-1}E_{\delta,\sum_{\iota=1}^{\mathbf{n}}\varepsilon_\iota+\delta}(-\lambda(t_2-\rho)^\delta)f(\rho,\mathfrak{Z}(\rho))d\rho$$

$$-\int_0^{t_1}(t_1-\rho)^{\sum_{\iota=1}^{\mathbf{n}}\varepsilon_\iota+\delta-1}E_{\delta,\sum_{\iota=1}^{\mathbf{n}}\varepsilon_\iota+\delta}(-\lambda(t_1-\rho)^\delta)f(\rho,\mathfrak{Z}(\rho))d\rho|$$

$$\leq\sum_{l=0}^{\mathbf{m}-1}\left|\Lambda_l\right|\left(\lambda\,|t_2^{\delta+l}E_{\delta,\delta+l+1}(-\lambda t_2^\delta)-t_1^{\delta+l}E_{\delta,\delta+l+1}(-\lambda t_1^\delta)|+\frac{|t_2^l-t_1^l|}{\Gamma(l+1)}\right)$$

$$+|\Lambda_\delta|\left(\frac{|t_2^\delta-t_1^\delta|}{\Gamma(\delta)}+t_1^\delta\,|E_{\delta,\delta}(-\lambda t_2^\delta)-E_{\delta,\delta}(-\lambda t_1^\delta)|\right)$$

$$+\frac{1}{\lambda}\sum_{\iota=1}^{\mathbf{n}-1}\frac{\left|\Delta_\iota\right|}{\Gamma\left(\sum_{j=\iota+1}^{\mathbf{n}}\varepsilon_j+1\right)}\left|t_2^{\sum_{j=\iota+1}^{\mathbf{n}}\varepsilon_j}-t_1^{\sum_{j=\iota+1}^{\mathbf{n}}\varepsilon_j}\right|$$

$$+\frac{1}{\lambda}\sum_{\iota=1}^{\mathbf{n}-1}|\Delta_\iota|\left(\left|t_2^{\sum_{j=\iota+1}^{\mathbf{n}}\varepsilon_j}-t_1^{\sum_{j=\iota+1}^{\mathbf{n}}\varepsilon_j}\right|+t_1^{\sum_{j=\iota+1}^{\mathbf{n}}\varepsilon_j}\,|E_\delta(-\lambda t_2^\delta)-E_\delta(-\lambda t_1^\delta)|\right)$$

$$+\frac{M}{\Gamma\left(\delta+\sum_{\iota=1}^{\mathbf{n}}\varepsilon_\iota\right)}\int_0^{t_2}\left|(t_2-\rho)^{\delta+\sum_{\iota=1}^{\mathbf{n}}\varepsilon_\iota-1}-(t_1-\rho)^{\delta+\sum_{\iota=1}^{\mathbf{n}}\varepsilon_\iota-1}\right|d\rho$$

$$+Mt_1^{\delta+\sum_{\iota=1}^{\mathbf{n}}\varepsilon_\iota-1}\int_0^{t_1}\left|E_{\delta,\delta+\sum_{\iota=1}^{\mathbf{n}}\varepsilon_\iota}(-\lambda(t_2-\rho)^\delta)-E_{\delta,\delta+\sum_{\iota=1}^{\mathbf{n}}\varepsilon_\iota}(-\lambda(t_1-\rho)^\delta)\right|d\rho$$

$$+\frac{M}{\Gamma\left(\delta+\sum_{\iota=1}^{\mathbf{n}}\varepsilon_\iota\right)}\int_{t_1}^{t_2}|t_1-\rho|^{\delta+\sum_{\iota=1}^{\mathbf{n}}\varepsilon_\iota-1}\,d\rho$$

$$\leq\sum_{l=0}^{\mathbf{m}-1}|\Lambda_l|\left(\lambda\left(\frac{|t_2^{\delta+l}-t_1^{\delta+l}|}{\Gamma(\delta+l+1)}+t_1^{\delta+l}\left|E_{\delta,\delta+l+1}(-\lambda t_2^\delta)-E_{\delta,\delta+l+1}(-\lambda t_1^\delta)\right|\right)+\frac{|t_2^l-t_1^l|}{\Gamma(l+1)}\right)$$

$$+|\Lambda_\delta|\left(\frac{|t_2^\delta-t_1^\delta|}{\Gamma(\delta)}+t_1^\delta|E_{\delta,\delta}(-\lambda t_2^\delta)-E_{\delta,\delta}(-\lambda t_1^\delta)|\right)+\frac{1}{\lambda}\sum_{\iota=1}^{\mathbf{n}-1}\frac{|\Delta_\iota|}{\Gamma\left(\sum_{j=\iota+1}^{\mathbf{n}}\varepsilon_j+1\right)}\left|t_2^{\sum_{j=\iota+1}^{\mathbf{n}}\varepsilon_j}-t_1^{\sum_{j=\iota+1}^{\mathbf{n}}\varepsilon_j}\right|$$

$$+\frac{1}{\lambda}\sum_{\iota=1}^{\mathbf{n}-1}|\Delta_\iota|\left(\left|t_2^{\sum_{j=\iota+1}^{\mathbf{n}}\varepsilon_j}-t_1^{\sum_{j=\iota+1}^{\mathbf{n}}\varepsilon_j}\right|+t_1^{\sum_{j=\iota+1}^{\mathbf{n}}\varepsilon_j}|E_\delta(-\lambda t_2^\delta)-E_\delta(\mathrm{v})|\right)$$

$$+\frac{M}{\Gamma\left(\delta+\sum_{\iota=1}^{\mathbf{n}}\varepsilon_\iota\right)}\int_0^{t_2}\left|(t_2-\rho)^{\sum_{\iota=1}^{\mathbf{n}}\varepsilon_\iota+\delta-1}-(t_1-\rho)^{\sum_{\iota=1}^{\mathbf{n}}\varepsilon_\iota+\delta-1}\right|d\rho$$

$$+Mt_1^{\delta+\sum_{\iota=1}^{\mathbf{n}}\varepsilon_\iota-1}\int_0^{t_1}\left|E_{\delta,\sum_{\iota=1}^{\mathbf{n}}\varepsilon_\iota+\delta}(-\lambda(t_2-\rho)^\delta)-E_{\delta,\sum_{\iota=1}^{\mathbf{n}}\varepsilon_\iota+\delta}(-\lambda(t_1-\rho)^\delta)\right|d\rho$$

$$+\frac{M}{\Gamma\left(\delta+\sum_{\iota=1}^{\mathbf{n}}\varepsilon_\iota\right)}\int_{t_1}^{t_2}|t_1-\rho|^{\sum_{\iota=1}^{\mathbf{n}}\varepsilon_\iota+\delta-1}\,d\rho$$

$$\leq\sum_{l=0}^{\mathbf{m}-1}|\Lambda_l|\left(\frac{|t_2^l-t_1^l|}{\Gamma(l+1)}+\lambda\left(\frac{|t_2^{\delta+l}-t_1^{\delta+l}|}{\Gamma(\delta+l+1)}+t_1^{\delta+l}O(|t_2^\delta-t_1^\delta|)\right)\right)+\left|\Lambda_\delta\right|\left(\frac{|t_2^\delta-t_1^\delta|}{\Gamma(\delta)}+t_1^\delta O(|t_2^\delta-t_1^\delta|)\right)$$

$$+\frac{1}{\lambda}\sum_{\iota=1}^{\mathbf{n}-1}\frac{|\Delta_\iota|}{\Gamma\left(\sum_{j=\iota+1}^{\mathbf{n}}\varepsilon_j+1\right)}\left|t_2^{\sum_{j=\iota+1}^{\mathbf{n}}\varepsilon_j}-t_1^{\sum_{j=\iota+1}^{\mathbf{n}}\varepsilon_j}\right|$$

$$+\frac{1}{\lambda}\sum_{\iota=1}^{\mathbf{n}-1}|\Delta_\iota|\left(\left|t_2^{\sum_{j=\iota+1}^{\mathbf{n}}\varepsilon_j}-t_1^{\sum_{j=\iota+1}^{\mathbf{n}}\varepsilon_j}\right|+t_1^{\sum_{j=\iota+1}^{\mathbf{n}}\varepsilon_j}O(|t_2^\delta-t_1^\delta|)\right)+\frac{2M|t_2-t_1|^{\delta+\sum_{\iota=1}^{\mathbf{n}}\varepsilon_\iota}}{\Gamma\left(\delta+\sum_{\iota=1}^{\mathbf{n}}\varepsilon_\iota+1\right)}$$

$$+\frac{M}{\Gamma\left(\delta+\sum_{i=1}^{\mathbf{n}}\varepsilon_\iota+1\right)}\left(t_2^{\delta+\sum_{\iota=1}^{\mathbf{n}}\varepsilon_\iota}-t_1^{\delta+\sum_{\iota=1}^{\mathbf{n}}\varepsilon_\iota}\right)+\frac{Mt_1^{\delta+\sum_{\iota=1}^{\mathbf{n}}\varepsilon_i}}{\delta+\sum_{i=1}^{n}\varepsilon_i}O(|t_2^\delta-t_1^\delta|),$$

we can observe that N is completely continuous.

Finally: The last step is to prove the boundedness of

$$\varepsilon=\{\mathfrak{Z}\in C([0,\mathbb{T}],\mathbb{R}):\mathfrak{Z}=\zeta N(\mathfrak{Z}),0<\zeta<1\}.$$

So, we consider

$$\mathfrak{Z}(t)=\zeta[\sum_{l=0}^{\mathbf{m}-1}\Lambda_l t^l E_{\delta,l+1}(-\lambda t^\delta)+\Lambda_\delta t^\delta E_{\delta,\delta}(-\lambda t^\delta)$$

$$+\frac{1}{\lambda}\sum_{i=1}^{\mathbf{n}-1}\Delta_i t^{\sum_{j=i+1}^{\mathbf{n}}\varepsilon_j}\left(\frac{1}{\Gamma\left(\sum_{j=i+1}^{\mathbf{n}}\varepsilon_j+1\right)}-E_\delta(-\lambda t^\delta)\right)$$

$$+\int_0^t (t-\rho)^{\delta+\sum_{i=1}^{\mathbf{n}}\varepsilon_i-1} E_{\delta,\delta+\sum_{i=1}^{\mathbf{n}}\varepsilon_i}(-\lambda(t-\rho)^{\delta}) f(\rho,\mathfrak{Z}(\rho))d\rho]$$

For $\zeta\in[0,1]$, $\forall t\in[0,\mathbb{T}]$, we have

$$\|(N\mathfrak{Z})\|_{\infty} \le \sum_{l=0}^{\mathbf{m}-1}|\Lambda_l|\left(\frac{\lambda\mathbb{T}^{\delta+l}}{\Gamma(\delta+l+1)}+\frac{\mathbb{T}^{l}}{\Gamma(l+1)}\right)+\frac{|\Lambda_\delta|\mathbb{T}^{\delta}}{\Gamma(\delta)}$$

$$+\frac{1}{\lambda}\sum_{\iota=1}^{\mathbf{n}-1}|\Delta_\iota|\mathbb{T}^{\sum_{j=\iota+1}^{\mathbf{n}}\varepsilon_j}\left(\frac{1}{\Gamma\left(\sum_{j=\iota+1}^{\mathbf{n}}\varepsilon_j+1\right)}+1\right)+\frac{M\mathbb{T}^{\delta+\sum_{\iota=1}^{\mathbf{n}}\varepsilon_\iota}}{\Gamma\left(\delta+\sum_{\iota=1}^{\mathbf{n}}\varepsilon_\iota+1\right)}=R.$$

This gives us that ε is bounded. The above result is proved.

Applications: This section deals tow examples to illustrate the validity of the two main results.

Example 1: We take the following equation:

$\forall t\in\left[0,1\right]$

$$\begin{cases} {}^{C}D^{0.1C}D^{0.2C}D^{0.3C}D^{0.4}[{}^{C}D^{0.5}+\lambda]\mathfrak{Z}(t)=\frac{(t\sqrt{\pi}-3)}{2}\cos(\mathfrak{Z}(t))+\frac{\arctan(\mathfrak{Z}(t))}{t^2} \\ {}^{C}D^{0.5}\mathfrak{Z}(0)+\lambda\mathfrak{Z}(0)=0 \\ {}^{C}D^{0.4+0.5}\mathfrak{Z}(0)+\lambda{}^{C}D^{0.4}\mathfrak{Z}(0)=1 \\ {}^{C}D^{0.3+0.4+0.5}\mathfrak{Z}(0)+\lambda{}^{C}D^{0.3+0.4}\mathfrak{Z}(0)=-1 \\ {}^{C}D^{0.2+0.3+0.4+0.5}\mathfrak{Z}(0)+\lambda{}^{C}D^{0.2+0.3+0.4}\mathfrak{Z}(0)=0 \\ \mathfrak{Z}(0)=0. \end{cases} \tag{10}$$

Here, we have

$$f(t)=\frac{(t\sqrt{\pi}-3)}{2}\cos(\mathfrak{Z}(t))+\frac{\arctan(\mathfrak{Z}(t))}{t^2}.$$

So, $\forall\mathfrak{Z},\mathfrak{Z}'\in\mathbb{R}$,

$$|f(t,\mathfrak{Z})-f(t,\mathfrak{Z}')|\le\left(\frac{\sqrt{\pi}-3}{2}+1\right)|\mathfrak{Z}-\mathfrak{Z}'|,$$

thus, (HP_1) is satisfied, and

$$\frac{\frac{\sqrt{\pi}-3}{2}+1}{\Gamma(0.1+0.2+0.3+0.4+0.5+1)}=\frac{(\sqrt{\pi}-1)^2}{3\sqrt{\pi}}\le 1.$$

So, by Banach contraction principle, we can state that (10) has only one solution in $C([0,1],\mathbb{R})$.

Also, we have

$$\left|f(t,\mathfrak{z}(t))\right| \leq \frac{\sqrt{\pi}-3}{2}+\frac{\pi}{2}.$$

Hence,the condition (HP_3) holds. It follows by Theorem 3.4 that (10) has one or more solutions.

Example 2: We take the following equation: $\forall t \in [0,2]$

$$\begin{cases} {}^{C}D^{\frac{1}{2}}\,{}^{C}D^{\frac{1}{4}}\,{}^{C}D^{\frac{1}{8}}\,{}^{C}D^{\frac{1}{16}}\,{}^{C}D^{\frac{1}{32}}[{}^{C}D^{\frac{63}{32}}+\lambda]\mathfrak{z}(t)=\frac{\sin(\mathfrak{z}(t))}{t^2} \\ {}^{C}D^{\sum_{j=i+1}^{5}\left(\frac{1}{2}\right)^j+\frac{63}{32}}\mathfrak{z}(0)+\lambda\,{}^{C}D^{\sum_{j=i+1}^{5}\left(\frac{1}{2}\right)^j}\mathfrak{z}(0)=2^i \qquad ,i=1,\ldots,5 \\ \mathfrak{z}(0)=0,\mathfrak{z}'(0)=1. \end{cases} \tag{11}$$

We observe that $\varepsilon_i=\frac{1}{2^i}$, $i=1,\ldots,5$ and $f(t,\mathfrak{z}(t))=\frac{\sin(\mathfrak{z}(t))}{t^2}$. So, $\forall \mathfrak{z},\mathfrak{z}' \in \mathbb{R}$, we have

$$|f(t,\mathfrak{z})-f(t,\mathfrak{z}')| \leq \tfrac{1}{4}|\mathfrak{z}-\mathfrak{z}'|,$$

thus, (HP_1) is satisfied and

$$\frac{\frac{1}{4}\times 2^3}{\Gamma(4)}=\frac{1}{3}\leq 1.$$

Then Theorem 3.3 holds true, and therefore, problem (11) has only one solution on $[0,2]$.

In addition, we have

$$|f(t,\mathfrak{z}(t))| \leq \frac{1}{4}.$$

Hence, theorem 3.4 yields that there exists one or more solutions to problem (11) in $C([0,2]\mathbb{R})$.

5. Conclusion

In the above work, we have constructed a general form of sequential differential problem of Langevin type with under initial conditions. Also, we have proved a result of unique solutions by using the Banach fixed point. Later, by Schaefer fixed point theorem, we have established an existence result. Finally, we have discussed two examples to verify the main theorems.

In the future, looking forward in our research to study fractional differential inclusions of Langevin type. Is it possible to consider that?

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