Journal of Interdisciplinary Mathematics
ISSN: 0972-0502 (Print), ISSN: 2169-012X (Online)
Vol. 28 (2025), No. 4, pp. 1315–1334
DOI : 10.47974/JIM-1780

Intuitionistic fuzzy (bifuzzy) BCK-algebras under norms

Rasul Rasuli
Department of Mathematics
Payame Noor University (PNU)
P. O. Box 19395-3697
Tehran
Iran

Abstract

As this study, with respect to norms, we introduce the intuitionistic fuzzification of the concept of subalgebras, ideals, implicative ideals and positive implicative ideals in BCK-algebras and investigate many of their conditions. Next, we investigate intersection, cartesian product and homomorphism(image and pre image) of them.

Subject Classification (2010): *11S45, 03E72, 15A60, 55N45, 51A10.*

Keywords: *Algebra and orders, Theory of fuzzy sets, Bifuzzy sets, Norms, Products and intersections, Homomorphisms.*

1. Introduction

The notion of **BCK** -Algebra was introduced by Imai and Iseki[5]. Zadeh[26] defined the concept of fuzzy sets. Atanassov[2, 3] introduced and investigated the concept of bifuzzy set (IFS). Many authors considered fuzzy **BCK** -Algebras[6, 7, 8, 9, 10, 13, 25]. Jun and Kim obtained some results about bifuzzy subAlgebras and ideals in *BCK* -Algebras[11]. Satyanarayana and Prasad investigated nifuzzy (implicative, positive implicative, commutative) ideals in **BCK** -Algebras[22]. In previous works, the author obtained many results of fuzzy Algebraic structures with respect to norms[14-21]. In Section 2, we remember the basic concepts. In Section 3, by using norms (T,C) we define bifuzzy subAlgebras $(IFSN(\Upsilon))$, bifuzzy ideals $(IFIN(\Upsilon))$, bifuzzy implicative ideals $(IFIIN(\Upsilon))$ and bifuzzy positive implicative ideals $(IFPIIN(\Upsilon))$ of **BCK** -Algebra Υ.

E-mail: `Rasuli@pnu.ac.ir`

Next, we characterize $IFSN(\Upsilon)$ with subalgebras of **BCK**-Algebra Υ and $IFIN(\Upsilon)$ with ideals of **BCK**-Algebra Υ and $IFIIN(\Upsilon)$ with implicative ideals of **BCK**-Algebra Υ and $IFPIIN(\Upsilon)$ with positive implicative ideals of **BCK**-Algebra Υ. Also we investigate and consider relainship between $IFSN(\Upsilon)$, $IFIN(\Upsilon)$, $IFIIN(\Upsilon)$ and $IFPIIN(\Upsilon)$ under some conditions. In Section 4, we investigate $IFSN(\Upsilon)$, $IFIN(\Upsilon)$, $IFIIN(\Upsilon)$ and $IFPIIN(\Upsilon)$ under intersection, cartesian product and homomorphism(image and pre image) of **BCK**-Algebras Υ and Y.

2. Preliminaries

In this section we give the main concepts and readers can refere to [1, 2, 3, 4, 12, 13, 14, 23, 24].

Definition 2.1: We say $(\varnothing \neq \Upsilon, \star, 0)$ is **BCK**-Algebra with a binary operation $\star$ and a constant 0 if:

(1) $((p_1 \star q_1) \star (p \star z_1)) \preceq (z_1 \star q_1)$,

(2) $(p_1 \star (p_1 \star q_1)) \preceq q_1$,

(3) $p_1 \preceq p_1$,

(4) $p_1 \preceq q_1$ and $q_1 \preceq p_1$ imply that $p_1 = q_1$,

(5) $0 \preceq p_1$

for all $p_1, q_1, z_1 \in \Upsilon$.

A partial ordering $\preceq$ on Υ can be defined by $p_1 \preceq q_1$ if and only if $p_1 \star q_1 = 0$. Also we have:

(6) $p_1 \star 0 = p_1$,

(7) $p_1 \star q_1 \preceq p_1$,

(8) $(p_1 \star q_1) \star z_1 = (p_1 \star z_1) \star q_1$,

(9) $(p_1 \star z_1) \star (q_1 \star z_1) \preceq p_1 \star q_1$,

(10) $p_1 \star (p \star (p_1 \star q_1)) = p_1 \star q_1$,

(11) if $p_1 \preceq q_1$, then $p_1 \star z_1 \preceq q_1 \star z_1$ and $z_1 \star q_1 \preceq z_1 \star p_1$

in any **BCK**-Algebra Υ and every $p_1, q_1, z_1 \in \Upsilon$.

Definition 2.2: Assume $E \neq \varnothing$ be a subset of a **BCK**-Algebra Υ. We say E is subalgebra of Υ if $p \star q \in E$ for each $p, q \in E$.

Definition 2.3: Define a **BCK**-Algebra Υ as implicative if $p = p \star (q \star p)$, for every $p, q \in \Upsilon$.

Definition 2.4: We say that a **BCK**-Algebra Υ will be positive implicative if $(p \star q) \star z = (p \star z) \star (q \star z)$ which $p, q, z \in \Upsilon$.

Definition 2.5: $E \neq \varnothing$ be a subset of a **BCK**-Algebra Υ. Define E as an ideal of Υ if

(1) $0 \in E$,

(2) if $m \star n \in E$ and $n \in E$, then $m \in E$ that $m, n \in \Upsilon$.

Definition 2.6: A non-empty subset E of a **BCK**-Algebra Υ is defined an implicative ideal of Υ if

(1) $0 \in E$,

(2) as $(p \star (q \star p)) \star z \in E$ and $z \in E$ so $p \in E$ such that $p, q, z \in \Upsilon$.

Definition 2.7: A non-empty subset E of a **BCK**-Algebra Υ is called a positive implicative ideal of Υ if

(1) $0 \in E$,

(2) $(p \star q) \star z \in E$ and $q \star z \in E$ imply that $p \star z \in E$ which $p, q, z \in \Upsilon$.

Definition 2.8: We call the mapping $g : \Upsilon \to Y$ of **BCK**-Algebras is a homomorphism if $g(p \star q) = g(p) \star g(q)$, for each $p, q \in \Upsilon$.

Definition 2.9: Assume $\Upsilon \neq \varnothing$ be an arbitrary set. We say that the function $\chi : \Upsilon \to [0,1]$ is A fuzzy subset of Υ and the set of all fuzzy subsets of Υ is called the $[0,1]$-power set of Υ and is denoted $[0,1]^{\Upsilon}$. Assume $s \in [0,1]$, the set $\chi_s = \{p \in \Upsilon : \chi(p) \geq s\}$ will be an upper level of χ and the set $\chi_t = \{p \in \Upsilon : \chi(p) \leq t\}$ will be a lower level of χ.

Definition 2.10: The complex mapping $A = (\chi_A, \lambda_A) : \Upsilon \to [0,1] \times [0,1]$ will be defined an bifuzzy set (IFS) in Υ if $\chi_A + \lambda_A \leq 1$ such that $\chi_A : \Upsilon \to [0,1]$ and $\lambda_A : \Upsilon \to [0,1]$. Also define $\varnothing_{\Upsilon}(p) = (0,1) \sim 0$ and $U_{\Upsilon}(p) = (1,0) \sim 1$, as an bifuzzy empty set and an bifuzzy whole set in Υ respectively. Denote $IFS(\Upsilon)$ for the sets of all $IFSs$ in Υ.

Definition 2.11: Assume $\phi : \Upsilon \to Y$ be a function of sets which $A = (\chi_A, \lambda_A) \in IFS(\Upsilon)$ and $B = (\chi_B, \lambda_B) \in IFS(Y)$. we define

$$\rho(A)(q) = (\rho(\chi_A)(q),\ \rho(\lambda_A)(q))$$

$$= \begin{cases} (Sup\{\chi_A(p) \mid p \in \Upsilon, \rho(p) = q\}, \inf\{\lambda_A(p) \mid p \in \Upsilon, \rho(p) = q\}) & \text{as } \rho^{-1}(q) \neq \phi \\ (0,1) & \text{as } \rho^{-1}(q) = \phi \end{cases}$$

and $\rho^{-1}(B)\ (p) = (\rho^{-1}(\chi_B)\ (p),\ \rho^{-1}(\lambda_B)\ (p)) = (\chi_B(\rho(p)),\ \lambda_B(\rho(p)))$ for every $p \in \Upsilon,\ q \in Y$.

Definition 2.12: Assume $p_2, q_2, z_2 \in \mathcal{I} = [0,1]$. Then

(i) A triangular norm is a map $T^{nor} : \mathcal{I} \times \mathcal{I} \to \mathcal{I}$, by $T^{nor}(p_2, 1) = p_2$, $T^{nor}(p_2,\ q_2) \leq T^{nor}(p_2,\ q_2) \leq T^{nor}(p_2,\ z_2)$ if $q_2 \leq z_2$, $T^{nor}(p_2,\ q_2) = T^{nor}(q_2,\ p_2)$ and $T^{nor}(p_2,\ T^{nor}(q_2,\ z_2)) = T^{nor}\ (T^{nor}(p_2,\ q), z_2)$.

(ii) A triangular conorm is a function $C^{con} : \mathcal{I} \times \mathcal{I} \to \mathcal{I}$, by $C^{con}(p_2,\ 0) = p_2$, $C^{con}(p_2,\ q_2) \leq C^{con}(p_2,\ z_2)$ if $q_2 \leq z_2$, $C^{con}(p_2,\ q_2) = C^{con}(q_2,\ p_2)$ and $C^{con}(p_2,\ C^{con}(q_2,\ z_2)) = C^{con}(C^{con}\ (p_2,\ q_2), z_2)$.

We say that T^{nor} and C^{con} be idempotent if for all $p_2 \in \mathcal{I}$ we have $T^{nor}(p_2,\ p_2) = p_2$ and $C^{con}(p_2,\ p_2) = p_2$.

Definition 2.13: Suppose $A = (\chi_A,\ \lambda_A) \in IFS(\Upsilon)$ and $B = (\chi_B,\ \lambda_B) \in IFS(\Upsilon)$. Define

$$A \cap B = (\chi_{A \cap B}, \lambda_{A \cap B}) : \Upsilon \to \mathcal{I}$$

as $\chi_{A \cap B}(p) = T^{nor}(\chi_A(p),\ \chi_B(p))$ and $\lambda_{A \cap B}(p) = C^{con}(\lambda_A(p),\ \lambda_B(p))$ for all $p \in \Upsilon$.

Definition 2.14: Assume $A = (\chi_A,\ \lambda_A) \in IFS(\Upsilon)$ and $B = (\chi_B,\ \lambda_B) \in IFS(Y)$. Define the cartesian product of A and B as $A \times B : \Upsilon \times Y \to \mathcal{I}$ with

$$\begin{aligned} (A \times B)(p,q) &= ((\chi_A, \lambda_A) \times (\chi_B, \lambda_B))(p,q) \\ &= (\chi_{A \times B}, \lambda_{A \times B})(p,q) \\ &= (\chi_{A \times B}(p,q), \lambda_{A \times B}(p,q)) \\ &= (T^{nor}(\chi_A(p), \chi_B(q)), C^{con}(\lambda_A(p), \lambda_B(q))) \end{aligned}$$

$\forall (p,q) \in \Upsilon \times Y$.

Lemma 2.15: *We have*

$$C^{con}(C^{con}(p,q), C^{con}(w,z)) = C^{con}(C^{con}(p,w), C^{con}(q,z)),$$

and

$$T^{nor}(T^{nor}(p,q),T^{nor}(w,z))=T^{nor}(T^{nor}(p,w),T^{nor}(q,z)),$$

$\forall p,q,w,z\in\mathcal{I}.$

3. $IFSN(\Upsilon), IFIN(\Upsilon), IFIIN(\Upsilon), IFPIIN(\Upsilon)$

In this paper we assume that Υ, Y will be two **BCK** -Algebras.

Definition 3.1: Assume $A=(\chi_A,\lambda_A)\in IFS(\Upsilon)$. Thus A will be called as an bifuzzy subalgebra of Υ with respect to norms if $\forall p,q\in\Upsilon$,

(1) $\chi_A(p\star q)\geq T^{nor}(\chi_A(p),\chi_A(q))$,

(2) $\lambda_A(p\star q)\leq C^{con}(\lambda_A(p),\lambda_A(q))$.

The set of all bifuzzy subalgebras of Υ with respect to norms will be denoted by $IFSN(\Upsilon)$.

Example 3.2: Suppose $\Upsilon=\{0,10,20,30\}$ be a set which:

$\star$	0	10	20	0
0	0	0	0	0
10	10	0	0	10
20	20	a	0	20
30	30	30	30	0

Hence $(\Upsilon,\star,0)$ is a **BCK** -Algebra.

Assume $\chi_A:(\Upsilon,\star,0)\to\mathcal{I}$ which

$$\chi_A(p)=\begin{cases}0.35 & \text{if } p=0,10,30\\ 0.25 & \text{if } p=20\end{cases}$$

and

$$\lambda_A(p)=\begin{cases}0.45 & \text{if } p=0,10,30\\ 0.55 & \text{if } p=20.\end{cases}$$

Assume $T^{nor}(a_1, b_1) = T_p^{nor}(a_1, b_1) = a_1b_1$ and $C^{con}(a_1, b_1) = C_p^{con}(a_1, b_1) = a_1 + b_1 - a_1b_1$ for every $a_1,b_1\in\mathcal{I}$ thus $A=(\chi_A,\lambda_A)\in IFSN(\Upsilon)$.

Proposition 3.3: Assume $A=(\chi_A,\lambda_A)\in IFS(\Upsilon)$ that T^{nor},C^{con} be idempotent. Hence the following items are equivalent.

(1) $A=(\chi_A,\lambda_A)\in IFSN(\Upsilon)$.

(2) $A_{m,n}=\{p\in\Upsilon : A(p)\supseteq(m,n)\}\neq\emptyset$ is a subalgebra of Υ with $m,n\in[0,1]$.

Proof: Assume $A=(\chi_A,\lambda_A)\in IFSN(\Upsilon)$ with $p,q\in A_{m,n}$. Then

$$\chi_A(p\star q)\geq T^{nor}(\chi_A(p),\chi_A(q))\geq T^{nor}(m,m)=m$$

and

$$\lambda_A(p\star q)\leq C^{con}(\lambda_A(p),\lambda_A(q))\leq C^{con}(n,n)=n$$

hence $p\star q\in A_{m,n}$ and so $A_{m,n}$ is a subalgebra of Υ for every $m,n\in[0,1]$.

Conversely, Assume $A_{m,n}\neq\emptyset$ be a subalgebra of Υ for every $m,n\in[0,1]$. Suppose $m=T^{nor}(\chi_A(p),\chi_A(q))$ and $n=C^{con}(\lambda_A(p),\lambda_A(q))$ that $p,q\in A_{m,n}$. Since $A_{m,n}$ is a subalgebra of Υ so $p\star q\in A_{m,n}$ and thus

$$\chi_A(p\star q)\geq m=T^{nor}(\chi_A(p),\chi_A(q))$$

and

$$\lambda_A(p\star q)\leq n=C^{con}(\lambda_A(p),\lambda_A(q))$$

then $A=(\chi_A,\lambda_A)\in IFSN(\Upsilon)$.

Proposition 3.4: If $A=(\chi_A,\ \lambda_A)\in IFSN(\Upsilon)$ and T^{nor},C^{con} be idempotent, then $A(0)\supseteq A(p)$ that $p\in\Upsilon$.

Definition 3.5: Define $A=(\chi_A,\lambda_A)\in IFS(\Upsilon)$ is an intuitionistic fuzzy ideal of Υ with respect to norms (t-norm T^{nor} and t-conorm C^{con}) if :

(1) $\chi_A(0)\geq\chi_A(p)$,
(2) $\chi_A(p)\geq T^{nor}(\chi_A(p\star q),\chi_A(q))$,
(3) $\lambda_A(0)\leq\lambda_A(p)$,
(4) $\lambda_A(p)\leq C^{con}(\lambda_A(p\star q),\lambda_A(q))$,
for all $p,q\in\Upsilon$.

$IFIN(\Upsilon)$, will be the set of all bifuzzy ideals of Υ with respect to norms (T^{nor} and C^{con}).

Example 3.6: Assume $\Upsilon=\{0,1,2,3,4\}$ be a set as:

$\star$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	0	0
3	3	3	3	0	0
4	4	3	4	1	0

Hence $(\Upsilon,\star,0)$ is a **BCK**-Algebra. Suppose $A=(\chi_A,\lambda_A)\in IFS(\Upsilon)$ which

$$\chi_A(p)=\begin{cases}1 & \text{if } p=0,2\\ t & \text{if } p=1,3,4\end{cases}$$

and

$$\lambda_A(p)=\begin{cases}0 & \text{if } p=0,2\\ s & \text{if } p\neq 1,3,4\end{cases}$$

such that $s,t\in\mathcal{I}$ and $s+t\leq 1$.

Assume $T^{nor}(a',b')=T_p^{nor}(a',b')=a'b'$ and $C^{con}(a',b')=C_p^{con}(a',b')=a'+b'-a'b'$ for every $a',b'\in\mathcal{I}$ so $A=(\chi_A,\lambda_A)\in IFIN(\Upsilon)$

Proposition 3.7: Assume $A=(\chi_A,\lambda_A)\in IFS(\Upsilon)$ with T^{nor},C^{con} be idempotent. Then $A=(\chi_A,\lambda_A)\in IFIN(\Upsilon)$ if and only if the

$$A_{p,q}=\{p\in\Upsilon:\chi_A(p)\geq p,\lambda_A(p)\leq q\}$$

is either empty or an ideal of Υ for every $p,q\in[0,1]$.

Proposition 3.8: Assume $A=(\chi_A,\lambda_A)\in IFIN(\Upsilon)$ and $a'\star q\preceq z$. Then $\forall a',q,z\in\Upsilon$ we will have that $\chi_A(a')\geq T^{nor}(\chi_A(q),\chi_A(z))$ and $\lambda_A(a')\leq C^{con}(\lambda_A(q),\lambda_A(z))$.

Proof: As $a'\star q\preceq z$ so $(a'\star q)\star z=0$ which $a',q,z\in\Upsilon$ so

$$\begin{aligned}\chi_A(a')&\geq T^{nor}(\chi_A(a'\star q),\chi_A(q))\\&\geq T^{nor}(T^{nor}(\chi_A((a'\star q)\star z),\chi_A(z)),\chi_A(q))\\&=T^{nor}(T^{nor}(\chi_A(0),\chi_A(z)),\chi_A(q))\\&=T^{nor}(\chi_A(z),\chi_A(q))\\&=T^{nor}(\chi_A(q),\chi_A(z))\end{aligned}$$

then $\chi_A(a')\geq T^{nor}(\chi_A(q),\chi_A(z))$. Moreover

$$\begin{aligned}\lambda_A(a') &\leq C^{con}(\lambda_A(a'\star q),\lambda_A(q))\\ &\leq C^{con}(C^{con}(\lambda_A((a'\star q)\star z),\lambda_A(z)),\lambda_A(q))\\ &= C^{con}(C^{con}(\lambda_A(0),\lambda_A(z)),\lambda_A(q))\\ &= C^{con}(\lambda_A(z),\lambda_A(q))\\ &= C^{con}(\lambda_A(q),\lambda_A(z))\end{aligned}$$

thus $\lambda_A(a') \leq C^{con}(\lambda_A(q),\lambda_A(z))$.

Proposition 3.9: If $A=(\chi_A,\lambda_A) \in IFIN(\Upsilon)$ and $p' \preceq q$, then $\forall p', q \in \Upsilon$ we get $A(p') \supseteq A(q)$.

Now we prove that every $IFIN(\Upsilon)$ is $IFSN(\Upsilon)$.

Proposition 3.10: Assume $A=(\chi_A,\lambda_A) \in IFIN(\Upsilon)$. Hence $A=(\chi_A,\lambda_A) \in IFSN(\Upsilon)$.

Proof: By using $p\star q \preceq p$ and Proposition 3.9 we will have $A(p\star q) \supseteq A(p)$.

$$\chi_A(p\star q) \geq \chi_A(p) \geq T^{nor}(\chi_A(p\star q),\chi_A(q)) \geq T^{nor}(\chi_A(p),\chi_A(q))$$

and

$$\lambda_A(p\star q) \leq \lambda_A(p) \leq C^{con}(\lambda_A(p\star q),\lambda_A(q)) \leq C^{con}(\lambda_A(p),\lambda_A(q))$$

thus $A=(\chi_A,\lambda_A) \in BFSN(\Upsilon)$.

Remark 3.11: Note that the converse of Proposition 3.10 is not true. Example 3.2 give us $A=(\chi_A,\lambda_A) \in IFSN(\Upsilon)$ but $\chi_A(b) = 0.25 \ngeq T^{nor}(\chi_A(b\star a), \chi_A(a)) = T^{nor}(\chi_A(a), \chi_A(a)) = \chi_A(a) = 0.35$ hence $A=(\chi_A, \lambda_A) \notin IFIN(\Upsilon)$.

Now we show that under a condition every $IFSN(\Upsilon)$ is $IFIN(\Upsilon)$.

Proposition 3.12: Assume $A=(\chi_A, \lambda_A) \in IFSN(\Upsilon)$ such that $\chi_A(p) \geq T^{nor}(\chi_A(y), \chi_A(z))$ and $\lambda_A(p) \leq C^{con}(\lambda_A(y), \lambda_A(z))$ and $p\star y \preceq z$ hold $\forall p,y,z \in \Upsilon$. Hence $A=(\chi_A, \lambda_A) \in IFIN(\Upsilon)$.

Proof: Using Proposition 3.4 we have $A(0) \supseteq A(p)$. Since $p\star(p\star y) \preceq y$ then

$$A(p) \supseteq (T^{nor}(\chi_A(p\star y),\chi_A(y)), C^{con}(\lambda_A(p\star y),\lambda_A(y))).$$

Thus $A=(\chi_A,\lambda_A) \in IFIN(\Upsilon)$.

Proposition 3.13: Assume $A=(\chi_A,\lambda_A) \in IFS(\Upsilon)$. Then

$$A=(\chi_A,\lambda_A)\in IFIN(\Upsilon)\Leftrightarrow \sqsubset A=(\chi_A,\overline{\chi}_A)\in IFIN(\Upsilon)$$

and $\sqsupset A=(\overline{\lambda}_A,\lambda_A)\in IFIN(\Upsilon)$
with $\overline{\chi}_A=1-\chi_A$ and $\overline{\lambda}_A=1-\lambda_A$.

Definition 3.14: $A=(\chi_A,\lambda_A)\in IFS(\Upsilon)$ is called an bifuzzy implicative ideal of Υ according norms (*t*-norm T^{nor} and *t*-conorm C^{con}) if $\forall p,y,z\in\Upsilon$:

(1) $A(0)\,Supseteq\,A(p)$,

(2) $A(p)\,Supseteq(T^{nor}(\chi_A(p\star(y\star p)),\chi_A(z)),C^{con}(\lambda_A(p\star(y\star p)),\lambda_A(z)))$.

Denoting $IFIIN(\Upsilon)$ for the set of all bifuzzy implicative ideals of Υ according norms (T^{nor} and C^{con}).

Proposition 3.15: Suppose $A=(\chi_A,\lambda_A)\in IFS(\Upsilon)$ with T^{nor},C^{con} be idempotent. Then

$$A=(\chi_A,\lambda_A)\in IFIIN(\Upsilon)\Leftrightarrow A_{m,n}=\{p\in\Upsilon:\chi_A(p)\geq m,\lambda_A(p)\leq n\}\neq\varnothing$$

and $A_{m,n}$ be an implicative ideal of Υ for every $m,n\in[0,1]$.

Definition 3.16: $A=(\chi_A,\lambda_A)\in IFS(\Upsilon)$ will be as an bifuzzy positive implicative ideal of Υ according norms (*t*-norm T^{nor} and *t*-conorm C^{con}) if $\forall p,y,z\in\Upsilon$:

(1) $A(0)$ *Supseteq* $A(p)$,

(2) $A(p\star z)\supseteq(T^{nor}(\chi_A((p\star y)\star z),\chi_A(y\star z)),C^{con}(\lambda_A((p\star y)\star z),$
$\lambda_A(y\star z)))$.

We display $IFPIIN(\Upsilon)$ as the set of all bifuzzy positive implicative ideals of Υ according norms (T^{nor} and C^{con}).

Proposition 3.17: Assume $A=(\chi_A,\lambda_A)\in IFS(\Upsilon)$ that T^{nor},C^{con} be idempotent. Thus

$$A=(\chi_A,\lambda_A)\in IFPIIN(\Upsilon)\Leftrightarrow A_{m,n}=\{p\in\Upsilon:A(p)\supseteq(m,n)\}\neq\varnothing$$

and $A_{m,n}$ will be a positive implicative ideal of Υ which $m,n\in[0,1]$.

Proposition 3.18: Assume $A=(\chi_A,\lambda_A)\in IFIN(\Upsilon)$ with

$$\chi_A(p_{10}\star y_{10})\geq T^{nor}(\chi_A(((p_{10}\star y_{10})\star y_{10})\star z_{10}),\chi_A(z_{10}))$$

and

$$\lambda_A(p_{10} \star y_{10}) \leq C^{con}(\lambda_A(((p_{10} \star y_{10}) \star y_{10}) \star z_{10}), \lambda_A(z_{10}))$$

$\forall p_{10}, y_{10}, z_{10} \in \Upsilon$. Hence $A = (\chi_A, \lambda_A) \in IFPIIN(\Upsilon)$.

Proof: Assume $p_{10}, y_{10}, z_{10} \in \Upsilon$. Then

$$((p_{10} \star z_{10}) \star z_{10}) \star (y_{10} \star z_{10}) \preceq (p_{10} \star z_{10}) \star y_{10} = (p_{10} \star y_{10}) \star z_{10}$$

and

$$\chi_A(((p_{10} \star z_{10}) \star z_{10}) \star (y_{10} \star z_{10})) \geq \chi_A((p_{10} \star y_{10}) \star z_{10})$$

and

$$\lambda_A(((p_{10} \star z_{10}) \star z_{10}) \star (y_{10} \star z_{10})) \leq \lambda_A((p_{10} \star y_{10}) \star z_{10}).$$

As we get $y_{10} = z_{10}$ and $z_{10} = y_{10} \star z_{10}$ hence

$$\chi_A(p_{10} \star z_{10}) \geq T^{nor}(\chi_A(((p_{10} \star z_{10}) \star z_{10}) \star (y_{10} \star z_{10})), \chi_A(y_{10} \star z_{10}))$$

and

$$\lambda_A(p_{10} \star z_{10}) \leq C^{con}(\lambda_A(((p_{10} \star z_{10}) \star z_{10}) \star (y_{10} \star z_{10})), \lambda_A(y_{10} \star z_{10})).$$

Then

$$\chi_A(p_{10} \star z_{10}) \geq T^{nor}(\chi_A(((p_{10} \star z_{10}) \star z_{10}) \star (y_{10} \star z_{10})), \chi_A(y_{10} \star z_{10})) \geq T^{nor}(\chi_A((p_{10} \star y_{10}) \star z_{10}), \chi_A(y_{10} \star z_{10})))$$

and

$$\lambda_A(p_{10} \star z_{10}) \leq C^{con}(\lambda_A(((p_{10} \star z_{10}) \star z_{10}) \star (y_{10} \star z_{10})), \lambda_A(y_{10} \star z_{10})) \leq C^{con}(\lambda_A((p_{10} \star y_{10}) \star z_{10}), \lambda_A(y_{10} \star z_{10}))).$$

Thus $A = (\chi_A, \lambda_A) \in IFPIIN(\Upsilon)$.

Proposition 3.19: Assume $A = (\chi_A, \lambda_A) \in IFIN(\Upsilon)$. Then

$$A = (\chi_A, \lambda_A) \in IFPIIN(\Upsilon) \Leftrightarrow A((p_{11} \star z_{11}) \star (y_{11} \star z_{11})) \supseteq A((p_{11} \star y_{11}) \star z_{11})$$

$$\forall p_{11}, y_{11}, z_{11} \in \Upsilon.$$

Proof: Suppose

$$\chi_A((p_{11} \star z_{11}) \star (y_{11} \star z_{11})) \geq \chi_A((p_{11} \star y_{11}) \star z_{11})$$

and

$$\lambda_A((p_{11}\star z_{11})\star(y_{11}\star z_{11}))\leq\lambda_A((p_{11}\star y_{11})\star z_{11})$$

$\forall p_{11}, y_{11}, z_{11}\in\Upsilon$. Hence $(p_{11}\star z_{11})\star(y_{11}\star z_{11})\preceq p_{11}\star y_{11}$ and

$$\begin{aligned}\chi_A(p_{11}\star z_{11})&\geq T^{nor}(\chi_A(y_{11}\star z_{11}),\chi_A(p_{11}\star y_{11}))\\&\geq T^{nor}(\chi_A(y_{11}\star z_{11}),\chi_A((p_{11}\star z_{11})\star(y_{11}\star z_{11})))\\&=T^{nor}(\chi_A((p_{11}\star z_{11})\star(y_{11}\star z_{11})),\chi_A(y_{11}\star z_{11}))\\&\geq T^{nor}(\chi_A((p_{11}\star y_{11})\star z_{11}),\chi_A(y_{11}\star z_{11}))\end{aligned}$$

and

$$\begin{aligned}\lambda_A(p_{11}\star z_{11})&\leq C^{con}(\lambda_A(y_{11}\star z_{11}),\lambda_A(p_{11}\star y_{11}))\\&\leq C^{con}(\lambda_A(y_{11}\star z_{11}),\lambda_A((p_{11}\star z_{11})\star(y_{11}\star z_{11})))\\&=C^{con}(\lambda_A((p_{11}\star z_{11})\star(y_{11}\star z_{11})),\lambda_A(y_{11}\star z_{11}))\\&\leq C^{con}(\lambda_A((p_{11}\star y_{11})\star z_{11}),\lambda_A(y_{11}\star z_{11})).\end{aligned}$$

Thus

$$\chi_A(p_{11}\star z_{11})\geq T^{nor}(\chi_A((p_{11}\star y_{11})\star z_{11}),\chi_A(y_{11}\star z_{11}))$$

and

$$\lambda_A(p_{11}\star z_{11})\leq C^{con}(\lambda_A((p_{11}\star y_{11})\star z_{11}),\lambda_A(y_{11}\star z_{11}))$$

thus $A=(\chi_A,\lambda_A)\in IFPIIN(\Upsilon)$.

Conversely, Suppose $A=(\chi_A,\ \lambda_A)\in IFPIIN(\Upsilon)$ and $p_{11},\ y_{11},\ z_{11}\in\Upsilon$ with $a_{11}=p_{11}\star(y_{11}\star z_{11})$ and $b_{11}=p_{11}\star y_{11}$. Thus $((p_{11}\star(y_{11}\star z_{11}))\star(p_{11}\star y_{11}))\preceq y_{11}\star(y_{11}\star z_{11})$ and thus $((p_{11}\star(y_{11}\star z_{11}))\star(p_{11}\star y_{11}))\star z_{11}\preceq y_{11}\star(y_{11}\star z_{11})\star z_{11}=0$ and $\chi_A(((p_{11}\star(y_{11}\star z_{11}))\star(p_{11}\star y_{11}))\star z_{11})\geq\chi_A(0)$ and $\lambda_A(((p_{11}\star(y_{11}\star z_{11}))\star(p_{11}\star y_{11}))\star z_{11})\leq\lambda_A(0)$. Then $\chi_A((a_{11}\star b_{11})\star z_{11})=\chi_A((p_{11}\star(y_{11}\star z_{11})\star p_{11}\star y_{11})\star z_{11})\geq\chi_A(0)$ and $\lambda_A((a_{11}\star b_{11})\star z_{11})=\star(y_{11}\star z_{11})\star p_{11}\star y_{11})\star z_{11})\leq\lambda_A(0)$. Now

$$\begin{aligned}\chi_A((p_{11}\star z_{11})\star(y_{11}\star z_{11}))&=\chi_A(p_{11}\star(y_{11}\star z_{11})\star z_{11})\\&=\chi_A(a_{11}\star z_{11})\\&\geq T^{nor}(\chi_A((a_{11}\star b_{11})\star z_{11}),\chi_A(b_{11}\star z_{11}))\\&\geq T^{nor}(\chi_A(0),\chi_A(b_{11}\star z_{11}))\\&=\chi_A(b_{11}\star z_{11})\\&=\chi_A((p_{11}\star y_{11})\star z_{11})\end{aligned}$$

and

$$\begin{aligned}\lambda_A((p_{11}\star z_{11})\star(y_{11}\star z_{11})) &= \lambda_A(p_{11}\star(y_{11}\star z_{11})\star z)\\ &= \lambda_A(a_{11}\star z_{11})\\ &\leq C^{con}(\lambda_A((a_{11}\star b_{11})\star z_{11}),\lambda_A(b_{11}\star z_{11}))\\ &\leq C^{con}(\lambda_A(0),\lambda_A(b_{11}\star z_{11}))\\ &= \lambda_A(b_{11}\star z_{11})\\ &= \lambda_A((p_{11}\star y_{11})\star z_{11}).\end{aligned}$$

Thus $\chi_A((p_{11}\star z_{11})\star(y_{11}\star z_{11}))\geq \chi_A((p_{11}\star y_{11})\star z_{11})$ and $\lambda_A((p_{11}\star z_{11})\star(y_{11}\star z_{11}))\leq \lambda_A((p_{11}\star y_{11})\star z_{11})$ for every $p_{11},y_{11},z_{11}\in\Upsilon$.

Proposition 3.20: Assume $A=(\chi_A,\lambda_A)\in IFPIIN(\Upsilon)$ and $p_3,y_3,z_3,a_3,b_3\in\Upsilon$.

(1) As $((p_3\star y_3)\star y_3)\star a_3\preceq b_3$, so

$$\chi_A(p_3\star y_3)\geq T^{nor}(\chi_A(a_3),\chi_A(b_3))$$

and

$$\lambda_A(p_3\star y_3)\leq C^{con}(\lambda_A(a_3),\lambda_A(b_3)).$$

(2) If $((p_3\star y_3)\star z_3)\star a_3\preceq b_3$, then

$$\chi_A((p_3\star z_3)\star(y_3\star z_3))\geq T^{nor}(\chi_A(a_3),\chi_A(b_3))$$

and

$$\lambda_A((p_3\star z_3)\star(y_3\star z_3))\leq C^{con}(\lambda_A(a_3),\lambda_A(b_3)).$$

Proof: Assume $A=(\chi_A,\lambda_A)\in IFPIIN(\Upsilon)$ and $p_3,y_3,z_3,a_3,b_3\in\Upsilon$.

(1) Let $((p_3\star y_3)\star y_3)\star a_3\preceq b_3$ thus $\chi_A((p_3\star y_3)\star y_3)\geq T^{nor}(\chi_A(a_3),\chi_A(b_3))$ and $\lambda_A((p_3\star y_3)\star y_3)\leq C^{con}(\lambda_A(a_3),\lambda_A(b_3))$. Hence

$$\begin{aligned}\chi_A(p_3\star y_3) &\geq T^{nor}(\chi_A((p_3\star y_3)\star y_3),\chi_A(y_3\star y_3))\\ &= T^{nor}(\chi_A((p_3\star y_3)\star y_3),\chi_A(0))\\ &= \chi_A((p_3\star y_3)\star y_3)\\ &\geq T^{nor}(\chi_A(a_3),\chi_A(b_3))\end{aligned}$$

and

$$\begin{aligned}\lambda_A(p_3 \star y_3) &\leq C^{con}(\lambda_A((p_3 \star y_3) \star y_3), \lambda_A(y_3 \star y_3)) \\ &= C^{con}(\lambda_A((p_3 \star y_3) \star y_3), \lambda_A(0)) \\ &= \lambda_A((p_3 \star y_3) \star y_3) \\ &\leq C^{con}(\lambda_A(a_3), \lambda_A(b_3))\end{aligned}$$

then

$$\chi_A(p_3 \star y_3) \geq T^{nor}(\chi_A(a_3), \chi_A(b_3))$$

and

$$\lambda_A(p_3 \star y_3) \leq C^{con}(\lambda_A(a_3), \lambda_A(b_3)).$$

(2) Let $((p_3 \star y_3) \star z_3) \star a_3 \preceq b_3$, so from Proposition 3.8 we get that

$$\chi_A((p_3 \star z_3) \star (y_3 \star z_3)) \geq \chi_A((p_3 \star y_3) \star z_3) \geq T^{nor}(\chi_A(a_3), \chi_A(b_3))$$

and

$$\lambda_A((p_3 \star z_3) \star (y_3 \star z_3)) \leq \lambda_A((p_3 \star y_3) \star z_3) \leq C^{con}(\lambda_A(a_3), \lambda_A(b_3)).$$

Proposition 3.21: Suppose $A = (\chi_A, \lambda_A) \in IFS(\Upsilon)$ with $((p_4 \star y_4) \star y_4) \star a_4 \preceq b_4$ that $p_4, y_4, a_4, b_4 \in \Upsilon$. If $A(p_4 \star y_4) \subseteq (T^{nor}(\chi_A(a_4), \chi_A(b_4)), C^{con}(\lambda_A(a), \lambda_A(b_4)))$, then $A = (\chi_A, \lambda_A) \in IFPIIN(\Upsilon)$.

Proof: Firstly, we prove that $A = (\chi_A, \lambda_A) \in IFIN(\Upsilon)$. Assume $p_4, y_4, z_4 \in \Upsilon$ which $p_4 \star y_4 \preceq z_4$. Hence $((p_4 \star 0) \star 0) \star y_4 = (p_4 \star y_4) \star z_4 = 0$ thus $((p \star 0) \star 0) \star y_4 \preceq z_4$. Put $y_4 = 0, a_4 = y_4, b_4 = z_4$ in hypothesis then $\chi_A(p_4) = \chi_A(p_4 \star 0) \geq T^{nor}(\chi_A(y_4), \chi_A(z_4))$ and $\lambda_A(p_4) = \lambda_A(p_4 \star 0) \leq C^{con}(\lambda_A(y_4), \lambda_A(z_4))$. Thus from Proposition 3.12 we get that $A = (\chi_A, \lambda_A) \in IFIN(\Upsilon)$. As $(((p_4 \star y_4) \star y_4) \star ((p_4 \star y_4) \star y_4)) \star 0 = 0$ so $(((p_4 \star y_4) \star y_4) \star ((p_4 \star y_4) \star y_4)) \preceq 0$ for all $p_4, y_4 \in \Upsilon$. As hypothesis $\chi_A(p_4 \star y_4) \geq T^{nor}(\chi_A((p_4 \star y_4) \star y_4), \chi_A(0)) = \chi_A((p_4 \star y_4) \star y_4)$ and $\lambda_A(p_4 \star y_4) \leq C^{con}(\lambda_A((p_4 \star y_4) \star y_4), \lambda_A(0)) = \lambda_A((p_4 \star y_4) \star y_4)$. Then $A = (\chi_A, \lambda_A)$ $IFPIIN(\Upsilon)$.

Proposition 3.22: Assume $A = (\chi_A, \lambda_A) \in IFS(\Upsilon)$ and $((p_5 \star y_5) \star z_5) \star a_5 \preceq b_5$ that $p_5, y_5, z_5, a_5, b_5 \in \Upsilon$. If $A((p_5 \star y_5) \star (y_5 \star z_5)) \supseteq (T^{nor}(\chi_A(a_5), \chi_A(b_5)), C^{con}(\lambda_A(a_5), \lambda_A(b_5)))$, then $A = (\chi_A, \lambda_A) \in IFPIIN(\Upsilon)$.

Proof: Let $((p_5 \star y_5) \star z_5) \star a_5 \preceq b_5$ such that $p_5, y_5, z_5, a_5, b_5 \in \Upsilon$. Then $(((p_5 \star y_5) \star z_5) \star a_5) \star b_5 = 0$. Now

$\chi_A(p_5 \star y_5) = \chi_A((p_5 \star y_5) \star 0) = \chi_A((p_5 \star y_5) \star (y_5 \star y_5)) \geq T^{nor}(\chi_A(a_5), \chi_A(b_5))$

and

$\lambda_A(p_5 \star y_5) = \lambda_A((p_5 \star y_5) \star 0) = \lambda_A((p_5 \star y_5) \star (y_5 \star y_5)) \leq C^{con}(\lambda_A(a_5), \lambda_A(b_5))$

hence $A = (\chi_A, \lambda_A) \in IFPIIN(\Upsilon)$.

Proposition3.23:*If* $A = (\chi_A, \lambda_A) \in IFPIIN(\Upsilon)$, *then* $\Delta A = (\chi_A, \overline{\chi}_A) \in IFPIIN(\Upsilon)$.

Proof: Suppose $p_6, y_6, z_6 \in \Upsilon$. As $A = (\chi_A, \lambda_A) \in IFPIIN(\Upsilon)$ so

$$\chi_A(0) \geq \chi_A(p_6) \tag{1}$$

and then $1 - \chi_A(0) \leq 1 - \chi_A(p_6)$ thus

$$\overline{\chi}_A(0) \leq \overline{\chi}_A(p_6). \tag{2}$$

Moreover

$$\chi_A(p_6 \star z_6) \geq T^{nor}(\chi_A((p_6 \star y_6) \star z_6), \chi_A(y_6 \star z_6)) \tag{3}$$

thus

$$1 - \chi_A(p_6 \star z_6) \leq 1 - T^{nor}(\chi_A((p_6 \star y_6) \star z_6), \chi_A(y_6 \star z_6))$$

then

$$\overline{\chi}_A(p_6 \star z_6) \leq C^{con}(1 - \chi_A((p_6 \star y_6) \star z_6), 1 - \chi_A(y_6 \star z_6)))$$

then

$$\overline{\chi}_A(p_6 \star z_6) \leq C^{con}(\overline{\chi}_A((p_6 \star y_6) \star z_6), \overline{\chi}_A(y_6 \star z_6)). \tag{4}$$

Now (1)-(4) give us that $\Delta A = (\chi_A, \overline{\chi}_A) \in IFPIIN(\Upsilon)$.

Proposition 3.24: If $A = (\chi_A, \lambda_A) \in IFPIIN(\Upsilon)$, then $\nabla A = (\overline{\lambda}_A, \lambda_A) \in IFPIIN(\Upsilon)$.

Proof: Let $p_{15}, y_{15}, z_{15} \in \Upsilon$. As $A = (\chi_A, \lambda_A) \in IFPIIN(\Upsilon)$ so

$$\lambda_A(0) \leq \lambda_A(p_{15}) \tag{1}$$

and then $1 - \lambda_A(0) \geq 1 - \lambda_A(p_{15})$ then

$$\overline{\lambda}_A(0) \geq \overline{\lambda}_A(p_{15}). \tag{2}$$

Also

$$\lambda_A(p_{15} \star z_{15}) \leq C^{con}(\lambda_A((p_{15} \star y_{15}) \star z_{15}), \lambda_A(y_{15} \star z_{15})) \tag{3}$$

thus

$$1 - \lambda_A(p_{15} \star z_{15}) \geq 1 - C^{con}(\lambda_A((p_{15} \star y_{15}) \star z_{15}), \lambda_A(y_{15} \star z_{15}))$$

then

$$\overline{\lambda}_A(p_{15} \star z_{15}) \geq T^{nor}(1 - \lambda_A((p_{15} \star y_{15}) \star z_{15}), 1 - \lambda_A(y_{15} \star z_{15})))$$

then

$$\overline{\lambda}_A(p_{15} \star z_{15}) \geq T^{nor}(\overline{\lambda}_A((p_{15} \star y_{15}) \star z_{15}), \overline{\lambda}_A(y_{15} \star z_{15})). \tag{4}$$

As (1)-(4) so $\nabla A = (\chi_A, \overline{\chi}_A) \in IFPIIN(\Upsilon)$.

Proposition 3.5: $A = (\chi_A, \lambda_A) \in IFPIIN(\Upsilon)$ if and only if $\Delta A = (\chi_A, \overline{\chi}_A) \in IFPIIN(\Upsilon)$ and $\nabla A = (\overline{\lambda}_A, \lambda_A) \in IFPIIN(\Upsilon)$.

4. $IFSN(\Upsilon), IFIN(\Upsilon), IFIIN(\Upsilon), IFPIIN(\Upsilon)$ under intersection, cartesian product and homomorphism

Proposition 4.1: Assume $A = (\chi_A, \lambda_A) \in IFSN(\Upsilon)$ and $B = (\chi_B, \lambda_B) \in IFSN(\Upsilon)$. Then $A \cap B \in IFSN(\Upsilon)$.

Proof: Let $p_7, y_7 \in \Upsilon$. Then

$$\begin{aligned}
\chi_{A\cap B}(p_7 \star y_7) &= T^{nor}(\chi_A(p_7 \star y_7), \chi_B(p_7 \star y_7)) \\
&\geq T^{nor}(T^{nor}(\chi_A(p_7), \chi_A(y_7)), T^{nor}(\chi_B(p_7), \chi_B(y_7))) \\
&= T^{nor}(T^{nor}(\chi_A(p_7), \chi_B(p_7)), T^{nor}(\chi_A(y_7), \chi_B(y_7))) \\
&= T^{nor}(\chi_{A\cap B}(p_7), \chi_{A\cap B}(y_7))
\end{aligned}$$

thus

$$\chi_{A\cap B}(p_7 \star y_7) \geq T^{nor}(\chi_{A\cap B}(p_7), \chi_{A\cap B}(y_7)).$$

Also

$$\begin{aligned}\lambda_{A\cap B}(p_7 \star y_7) &= C^{con}(\lambda_A(p_7 \star y_7), \lambda_B(p_7 \star y_7)) \\ &\leq C^{con}(C^{con}(\lambda_A(p_7), \lambda_A(y_7)), C^{con}(\lambda_B(p_7), \lambda_B(y_7))) \\ &= C^{con}(C^{con}(\lambda_A(p_7), \lambda_B(p_7)), C^{con}(\lambda_A(y_7), \lambda_B(y_7))) \\ &= C^{con}(\lambda_{A\cap B}(p_7), \lambda_{A\cap B}(y_7))\end{aligned}$$

then

$$\lambda_{A\cap B}(p_7 \star y_7) \leq C^{con}(\lambda_{A\cap B}(p_7), \lambda_{A\cap B}(y_7)).$$

Thus $A \cap B = (\chi_{A\cap B}, \lambda_{A\cap B}) \in IFSN(\Upsilon)$.

Proposition 4.2: Assume $A = (\chi_A, \lambda_A) \in IFIN(\Upsilon)$ and $B = (\chi_B, \lambda_B) \in IFIN(\Upsilon)$. Hence $A \cap B \in IFIN(\Upsilon)$.

Proposition 4.3: Suppose $A = (\chi_A, \lambda_A) \in IFIIN(\Upsilon)$ and $B = (\chi_B, \lambda_B) \in IFIIN(\Upsilon)$. Therefore $A \cap B \in IFIIN(\Upsilon)$.

Proposition 4.4: If $A = (\chi_A, \lambda_A) \in IFPIIN(\Upsilon)$ and $B = (\chi_B, \lambda_B) \in IFPIIN(\Upsilon)$, then $A \cap B \in IFPIIN(\Upsilon)$.

Proposition 4.5: As $A = (\chi_A, \lambda_A) \in IFSN(\Upsilon)$ and $B = (\chi_B, \lambda_B) \in IFSN(Y)$ so $A \times B \in IFSN(\Upsilon \times Y)$.

Proof: Let $(a_1, y_1), (a_2, y_2) \in \Upsilon \times Y$. Then

$$\begin{aligned}(\chi_{A\times B})((a_1, y_1) \star (a_2, y_2)) &= (\chi_{A\times B})(a_1 \star a_2, y_1 \star y_2) \\ &= T^{nor}(\chi_A(a_1 \star a_2), \chi_B(y_1 \star y_2)) \\ &\geq T^{nor}(T^{nor}(\chi_A(a_1), \chi_A(a_2)), T^{nor}(\chi_B(y_1), \chi_B(y_2))) \\ &= T^{nor}(T^{nor}(\chi_A(a_1), \chi_B(y_1)), T^{nor}(\chi_A(a_2), \chi_B(y_2))) \\ &= T^{nor}(\chi_{A\times B}(a_1, y_1), \chi_{A\times B}(a_2, y_2))\end{aligned}$$

thus

$$\chi_{A\times B}((a_1, y_1) \star (a_2, y_2)) \geq T^{nor}(\chi_{A\times B}(a_1, y_1), \chi_{A\times B}(a_2, y_2)).$$

Also

$$\begin{aligned}(\lambda_{A\times B})((a_1,y_1)\star(a_2,y_2)) &= (\lambda_{A\times B})(a_1\star a_2, y_1\star y_2)\\ &= C^{con}(\lambda_A(a_1\star a_2),\lambda_B(y_1\star y_2))\\ &\le C^{con}(C^{con}(\lambda_A(a_1),\lambda_A(a_2)),C^{con}(\lambda_B(y_1),\lambda_B(y_2)))\\ &= C^{con}(C^{con}(\lambda_A(a_1),\lambda_B(y_1)),C^{con}(\lambda_A(a_2),\lambda_B(y_2)))\\ &= C^{con}(\lambda_{A\times B}(a_1,y_1),\lambda_{A\times B}(a_2,y_2))\end{aligned}$$

then

$$\lambda_{A\times B}((a_1,y_1)\star(a_2,y_2))\le C^{con}(\lambda_{A\times B}(a_1,y_1),\lambda_{A\times B}(a_2,y_2)).$$

Therefore

$$A\times B=(\chi_{A\times B},\lambda_{A\times B})\in IFSN(\Upsilon\times Y).$$

Proposition 4.6: Assume $A=(\chi_A,\ \lambda_A)\in IFIN(\Upsilon)$ and $B=(\chi_B,\ \lambda_B)\in IFIN(Y)$. Then $A\times B\in IFIN(\Upsilon\times Y)$.

Proposition 4.7: Suppose $A=(\chi_A,\ \lambda_A)\in IFIIN(\Upsilon)$ and $B=(\chi_B,\ \lambda_B)\in IFIIN(Y)$. Then $A\times B\in IFIIN(\Upsilon\times Y)$.

Proposition 4.8: Assume $A=(\chi_A,\ \lambda_A)\in IFPIIN(\Upsilon)$ and $B=(\chi_B,\ \lambda_B)\in IFPIIN(Y)$. Then $A\times B\in IFPIIN(\Upsilon\times Y)$.

Proposition 4.9: If $A=(\chi_A,\ \lambda_A)\in IFSN(\Upsilon)$ and $\rho:\Upsilon\to Y$ be a homomorphism of **BCK**-Algebras, then $\rho(A)\in IFSN(Y)$.

Proof: Let $y_1,y_2\in Y$ and $a_1,a_2\in\Upsilon$ such that $\rho(a_1)=y_1$ and $\rho(a_2)=y_2$. Then

$$\rho(\chi_A)(y_1\star y_2)=\sup\{\chi_A(a_1\star a_2)\,|\,a_1,a_2\in\Upsilon,\rho(a_1)=y_1,\rho(a_2)=y_2\}$$

$$\ge\sup\{T^{nor}(\chi_A(a_1),\chi_A(a_2)\,|\,a_1,a_2\in\Upsilon,\rho(a_1)=y_1,\rho(a_2)=y_2\}$$

$$=T^{nor}(\sup\{\chi_A(a_1)\,|\,a_1\in\Upsilon,\rho(a_1)=y_1\},\sup\{\chi_A(a_2)\,|\,a_2\in\Upsilon,\rho(a_2)=y_2\})$$

$$=T^{nor}(\rho(\chi_A)(y_1),\phi(\chi_A)(y_2))$$

thus

$$\rho(\chi_A)(y_1\star y_2)\ge T^{nor}(\rho(\chi_A)(y_1),\phi(\chi_A)(y_2)).$$

Also

$$\rho(\lambda_A)(y_1\star y_2)=\inf\{\lambda_A(a_1\star a_2)\,|\,a_1,a_2\in\Upsilon,\rho(a_1)=y_1,\rho(a_2)=y_2\}$$

$$\leq \inf\{C^{con}(\lambda_A(a_1), \lambda_A(a_2) \mid a_1, a_2 \in \Upsilon, \rho(a_1) = y_1, \rho(a_2) = y_2\}$$

$$= C^{con}(\inf\{\chi_A(a_1) \mid a_1 \in \Upsilon, \rho(a_1) = y_1\}, \inf\{\lambda_A(a_2) \mid a_2 \in \Upsilon, \rho(a_2) = y_2\})$$

$$= C^{con}(\rho(\lambda_A)(y_1), \phi(\chi_A)(y_2))$$

so

$$\rho(\lambda_A)(y_1 \star y_2) \leq C^{con}(\rho(\lambda_A)(y_1), \phi(\lambda_A)(y_2)).$$

Then $\rho(A) = (\rho(\chi_A), \phi(\lambda_A)) \in IFSN(Y)$.

Proposition 4.10: If $B = (\chi_B, \lambda_B) \in IFSN(Y)$ and $\eta : \Upsilon \to Y$ be a homomorphism of **BCK** -Algebras, then $\eta^{-1}(B) \in IFSN(\Upsilon)$.

Proof: Let $a_1, a_2 \in \Upsilon$. Then

$$\begin{aligned}\eta^{-1}(\chi_B)(a_1 \star a_2) &= \chi_B(\eta(a_1 \star a_2)) \\ &= \chi_B(\eta(a_1) \star \eta(a_2)) \\ &\geq T^{nor}(\chi_B(\eta(a_1)), \chi_B(\eta(a_2))) \\ &= T^{nor}(\eta^{-1}(\chi_B)(a_1), \eta^{-1}(\chi_B)(a_2))\end{aligned}$$

thus

$$\eta^{-1}(\chi_B)(a_1 \star a_2) \geq T^{nor}(\eta^{-1}(\chi_B)(a_1), \eta^{-1}(\chi_B)(a_2)).$$

Also

$$\begin{aligned}\eta^{-1}(\lambda_B)(a_1 \star a_2) &= \lambda_B(\eta(a_1 \star a_2)) \\ &= \lambda_B(\eta(a_1) \star \eta(a_2)) \\ &\leq C^{con}(\lambda_B(\eta(a_1)), \lambda_B(\eta(a_2))) \\ &= C^{con}(\eta^{-1}(\lambda_B)(a_1), \eta^{-1}(\lambda_B)(a_2))\end{aligned}$$

then

$$\eta^{-1}(\lambda_B)(a_1 \star a_2) \leq C^{con}(\eta^{-1}(\lambda_B)(a_1), \eta^{-1}(\lambda_B)(a_2)).$$

Therefore $\eta^{-1}(B) = (\eta^{-1}(\chi_B), \eta^{-1}(\lambda_B)) \in IFSN(\Upsilon)$.

Proposition 4.11: Let $\varphi : \Upsilon \to Y$ be a homomorphism of **BCK** -Algebras such that $A = (\chi_A, \lambda_A) \in IFS(\Upsilon)$ and $B = (\chi_B, \lambda_B) \in IFS(Y)$.

(1) If $A \in IFIN(\Upsilon)$, then $\varphi(A) \in IFIN(Y)$.

(2) If $B \in IFIN(Y)$, then $\varphi^{-1}(B) \in IFIN(\Upsilon)$.

(3) If $A \in IFIIN(\Upsilon)$, then $\varphi(A) \in IFIIN(Y)$.

(4) If $B \in IFIIN(Y)$, then $\varphi^{-1}(B) \in IFIIN(\Upsilon)$.

(5) If $A \in IFPIIN(\Upsilon)$, then $\varphi(A) \in IFPIIN(Y)$.

(6) If $B \in IFPIIN(Y)$, then $\varphi^{-1}(B) \in IFPIIN(\Upsilon)$.

References

[1] Mohammad Tairab Abu Osman, On some products of fuzzy subgroups, *Fuzzy Sets and Systems*, vol. 24, pp. 79–86 (1987).

[2] Krassimir T. Atanassov, Intuitionistic fuzzy sets, *Fuzzy Sets and Systems*, vol. 20, pp. 87–96 (1986).

[3] Krassimir T. Atanassov, New operations defined over the intuitionistic fuzzy sets, *Fuzzy Sets and Systems*, vol. 61, pp. 137–142 (1994).

[4] James J. Buckley and Esfandiar Eslami, An Introduction to Fuzzy Logic and Fuzzy Sets, Berlin, Germany: Springer-Verlag (2002).

[5] Yosiyuki Imai and Katsumi Iseki, On axioms of proportional calculi XIV, Proceedings of the Japan Academy, vol. 42, pp. 19–22 (1966).

[6] Young Bae Jun, A note on fuzzy ideals in BCK-algebras, *Mathematica Japonica*, vol. 42, no. 2, pp. 233–235 (1995).

[7] Young Bae Jun, Finite valued fuzzy ideals in BCK-algebras, *Journal of Fuzzy Mathematics,* vol. 5, no. 1, pp. 111–114 (1997).

[8] Young Bae Jun, Characterizations of Noetherian BCK-algebras via fuzzy ideals, *Fuzzy Sets and Systems*, vol. 108, no. 2, pp. 231–234 (1997).

[9] Young Bae Jun, Seong Min Hong, Seong Joo Kim, and Sang Zi Song, Fuzzy ideals and fuzzy subalgebras of BCK-algebras, *Journal of Fuzzy Mathematics,* vol. 7, no. 2, pp. 411–418 (1999).

[10] Young Bae Jun and Eun Hee Roh, Fuzzy commutative ideals of BCK-algebras, *Fuzzy Sets and Systems*, vol. 64, no. 3, pp. 401–405 (1994).

[11] Young Bae Jun and Kyung Hoon Kim, Intuitionistic fuzzy ideals of a BCK-algebras, *International Journal of Mathematics and Mathematical Sciences*, vol. 24, no. 12, pp. 839–849 (2000).

[12] Dharam S. Malik and John N. Mordeson, Fuzzy Commutative Algebra, Singapore: World Scientific Publishing Co. Pte. Ltd. (1995).

[13] Jian Meng, Young Bae Jun, and Hyeon Sook Kim, Fuzzy implicative ideals of BCK-algebras, *Fuzzy Sets and Systems,* vol. 89, no. 2, pp. 243–248 (1997).

[14] Rahim Rasuli, Norms over intuitionistic fuzzy subrings and ideals of a ring, *Notes on Intuitionistic Fuzzy Sets*, vol. 22, no. 5, pp. 72–83 (2016).

[15] Rahim Rasuli, Characterization of Q-fuzzy subrings (Anti Q-fuzzy subrings) with respect to a T-norm (T-conorm), *Journal of Information and Optimization Sciences*, vol. 39, no. 4, pp. 1–12 (2018).

[16] Rahim Rasuli, Fuzzy subgroups over a T-norm, *Journal of Information and Optimization Sciences*, vol. 39, pp. 1757–1765 (2018).

[17] Rahim Rasuli, Fuzzy congruence on product lattices under T-norms, *Journal of Information and Optimization Sciences*, vol. 42, no. 2, pp. 333–343 (2021).

[18] Rahim Rasuli, Intuitionistic fuzzy congruences on product lattices under norms, *Journal of Interdisciplinary Mathematics*, vol. 24, no. 5, pp. 1281–1304 (2021).

[19] Rahim Rasuli, Normalization, commutativity and centralization of TFSM(G), *Journal of Discrete Mathematical Sciences and Cryptography*, vol. 26, no. 5, pp. 1027–1050 (2023).

[20] Rahim Rasuli, Intuitionistic fuzzy G-modules with respect to norms (T and S), *Notes on Intuitionistic Fuzzy Sets*, vol. 29, no. 3, pp. 277–291 (2023).

[21] Rahim Rasuli, Intuitionistic fuzzy complex subgroups with respect to norms (T and S), *Journal of Fuzzy Extension and Application,* vol. 4, no. 2, pp. 92–114 (2023).

[22] B. Satyanarayana and R. Durga Prasad, Some results on intuitionistic fuzzy ideals in BCK-algebras, *General Mathematics Notes*, vol. 4, no. 1, pp. 1–15 (2011).

[23] B. Satyanarayana, U. B. Madhavi, and R. Durga Prasad, On intuitionistic fuzzy H-ideals in BCK-algebras, *International Journal of Algebra*, vol. 4, no. 15, pp. 743–749 (2010).

[24] B. Satyanarayana, E. V. Koteswara Rao, and L. Krishna, On intuitionistic fuzzy BCK-algebras, *ANU Journal of Physical Sciences,* vol. 1, no. 2, pp. 21–32 (2009).

[25] Seyed Mohammad Seyedjoula, Ali Gilani, and Ebrahim Arfa, Exact BCK-sequence in BCK-algebra, *Journal of Information and Optimization Sciences,* vol. 41, no. 4, pp. 1153–1162 (2020).

[26] Lotfi A. Zadeh, *Fuzzy sets, Information and Control*, vol. 8, pp. 338–353 (1965).

Received May, 2023