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Geometry of hypersurfaces in Euclidean 4-space with Caputo fractional derivatives

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Abstract

The aim of this paper is to generalize the paper in [7] in 4-dimensional Euclidean space. We define and calculate the geometries of hypersurface in an Euclidean four space using the Caputo fractional derivative. Using the fractional calculus we give the geometrics of the considered hypersurface. We give some example using the graphs of some functions in $\mathbb{R}^3$.

Subject Classification: *53A10, 53C42.*

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1. Introduction

The applications and theory of fractional calculus are important in diverse areas of mathematical physics and engineering sciences. Standard differentiation or integration into a random non-integer hierarchy is a generalization. For detail see [2, 4, 5]. The ϕ –Caputo fractional derivative

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in Banach spaces is used by the authors in [5] to examine a particular type of starting value problem for nonlinear implicit fractional differential equations.

In the other hand the differential geometry via fractional calculus is very interesting and has been studied by several authors. In [7], the authors give a method to construct fractional differential geometry of curves. They use the fractional operator to define the tangent vector of a plane curve. They provide the fractional Frenet-Serret formulas and the expression of the Frenet frame of curve using a simplification for the fractional derivative of the composite function. .

In [3] the authors introduce and define the geometric operators of plan curves and space curves using the fractional calculus and provide some relations between them. They give the Frenet-Serret formulas of the curve in $\mathbb{R}^3$ and then construct the ways of determining a curve in $\mathbb{R}^3$ in terms of the new invariants.

In [8] the authors study the geometric structures of surfaces with Caputo fractional derivatives. They introduce the Gauss frame with fractional order. They define the two curvatures of the surface using fractional calculus.

The study of hypersurfaces on a Euclidean space is a vast and very interesting subject which allows us to better understand the geometry of ambient space, see for example [1]. And this paper try to give the same geometry with the Caputo fractional derivative.

Motivated by the above works, we investigate the geometries of hypersurfaces in Euclidean four space via fractional derivative. We define and calculate the geometries of hypersurface in an Euclidean four space using the Caputo fractionnal derivative. The two curvatures of this hypersurface are studied and calculated with the Capito fractional derivative.

The paper is organised as follow: in section 2, we recall some preliminaries results about the geometries of hypersurface in Euclidean for space and some basic tool in Caputo fractional derivative. In the section 3, we study the differential geometries of hypersurface in $\mathbb{R}^4$ using the Caputo fractional derivative operator.

2. Preliminaries

2.1 *Hypersurfaces in Euclidean 4-space*

A hypersurface in an Euclidean 4-space can be defined by the differential map

$$\varphi : D \subset \mathbb{R}^3 \to \mathbb{R}^4$$
$$(x,y,z) \mapsto \varphi(x,y,z) = (\varphi_1(x,y,z), \varphi_2(x,y,z), \varphi_3(x,y,z), \varphi_4(x,y,z)) \tag{1}$$

were D is an open domain of $\mathbb{R}^3$.

Let $\vec{u} = (u_1, u_2, u_3, u_4)$, $\vec{v} = (v_1, v_2, v_3, v_4)$ and $\vec{w} = (w_1, w_2, w_3, w_4)$ be three vectors in $\mathbb{R}^4$.

Then the inner product and the vector product are defined respectively by:

$$\langle \vec{u}, \vec{v} \rangle = u_1 v_1 + u_2 v_2 + u_3 v_3 + u_4 v_4 \tag{2}$$

and

$$\vec{u} \times \vec{v} \times \vec{w} = \det \begin{bmatrix} e_1 & e_2 & e_3 & e_4 \\ u_1 & u_2 & u_3 & u_4 \\ v_1 & v_2 & v_3 & v_4 \\ w_1 & w_2 & w_3 & w_4 \end{bmatrix}, \tag{3}$$

where (e_1, e_2, e_3, e_4) is the canonical basis in $\mathbb{R}^4$.

The norm of $\vec{u}$ is expressed by $\| \vec{u} \| = \sqrt{\langle \vec{u}, \vec{u} \rangle}$.

The unit normal vector field, the first and second fundamental forms of φ

$$n_\alpha = \frac{\varphi_x \times \varphi_y \times \varphi_z}{\| \varphi_x \times \varphi_y \times \varphi_z \|}, \tag{4}$$

$$[g_{ij}] = \begin{bmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{bmatrix} \tag{5}$$

and

$$[h_{ij}] = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix}, \tag{6}$$

respectively, where the coefficients $g_{ij} = \langle \varphi_{x_i}, \varphi_{x_j} \rangle$, $h_{ij} = \langle \varphi_{x_i x_j}, n_\alpha \rangle$, $\varphi_{x_i} = \frac{\partial \varphi(x_0, x_1, x_2)}{\partial x_i}$, $\varphi_{x_i x_i} = \frac{\partial^2 \varphi(x_0, x_1, x_2)}{\partial x_i x_j}$, $i, j \in \{0,1,2\}$.

The formula of the shape operator of φ is given by

$$S = [s_{ij}] = [g^{ij}] \cdot [h_{ij}], \tag{7}$$

where $[g^{ij}]$ is the inverse matrix of $[g_{ij}]$.

By using the equations in (5)-(7), we get the Gaussian curvature and mean curvature of φ

$$K = \frac{\det[h_{ij}]}{\det[g_{ij}]} \tag{8}$$

and

$$3H = trace(S), \tag{9}$$

respectively.

2.2 *Caputo fractionnal derivative*

In this section, we recall some definitions and preliminaries that will be used in this work. For more details see [6].

Definition 2.1: The Riemann-Liouville integral of order $\alpha > 0$ for a continuous function $\varphi \in L^1((0,1],\mathbb{R})$ is given by:

$$I^{\alpha}\varphi(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\varphi(\tau)d\tau,\ \forall t \in (0,1] \tag{10}$$

with $\Gamma(\alpha) := \int_0^{\infty} e^{-u}u^{\alpha-1}du.$

Definition 2.2: If $\varphi \in C^n([0,1],\mathbb{R})$ and $n-1 < \alpha \leq n,$ then, the Caputo fractional derivative is given by:

$$\begin{aligned} D^{\alpha}\varphi(t) &= I^{n-\alpha}\frac{d^n}{dt^n}(\varphi(t)) \\ &= \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{n-\alpha-1}\varphi^{(n)}(s)ds. \end{aligned} \tag{11}$$

For example the Caputo fractional derivative of order $0 < \alpha \leq 1$ for φ is given by

$$\begin{aligned} D^{\alpha}\varphi(t) &= I^{1-\alpha}\frac{d}{dt}(\varphi(t)) \\ &= \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}\varphi'(s)ds. \end{aligned} \tag{12}$$

The Leibniz formula via the Caputo fractional derivative is given by

$$(D^{\alpha} fg)(t) = \sum_{k=0}^{\infty} C_{\alpha}^{k} \frac{d^k f}{dt^k}(D^{\alpha-k} g)(t) - \frac{t^{-\alpha}}{\Gamma(1-\alpha)} f(0)g(0). \tag{13}$$

By using relation (13), the fractional derivative of composite function $f(t(s))$ is given by

$$(D^{\alpha} f)(t(s)) = \frac{f(t(s)) - f(t(0))}{\Gamma(1-\alpha)} s^{-\alpha} + \sum_{k=1}^{\infty} C_{\alpha}^{k} \frac{s^{k-\alpha}}{\Gamma(k-\alpha+1)} \frac{d^k f(t(s))}{ds^k}. \tag{14}$$

3. Geometries of hypersurfaces in $\mathbb{R}^4$ via Caputo fractional derivative

3.1 *First and second fondamental forms*

Let us consider a smooth hypersurface $\varphi = \varphi(x, y, z)$ in a Euclidean 4-space $\mathbb{R}^4$:

$$\varphi(x, y, z) = (\varphi^1(x, y, z), \varphi^2(x, y, z), \varphi^3(x, y, z), \varphi^4(x, y, z)), \tag{15}$$

where x y and z are the coordinates on $D \subseteq \mathbb{R}^3$. In the fractional case, we consider the case where only the Gaussian basis of the hypersurface S is defined using the fractional derivative of Caputo and the other derivatives are the classical ones. In this case, the vectors of this basis are defined as:

$$\{{}^{C}D_{x}^{\alpha}\varphi, {}^{C}D_{y}^{\alpha}\varphi, {}^{C}D_{z}^{\alpha}\varphi, n_{\alpha}\}, \tag{16}$$

where n_{α} is the normal vector of the hypersurface $\varphi,$ given by

$$n_{\alpha} = \frac{{}^{C}D_{x}^{\alpha}\varphi \times {}^{C}D_{y}^{\alpha}\varphi \times {}^{C}D_{z}^{\alpha}\varphi}{\| {}^{C}D_{x}^{\alpha}\varphi \times {}^{C}D_{y}^{\alpha}\varphi \times {}^{C}D_{z}^{\alpha}\varphi \|}; \quad \alpha \in]0,1] \tag{17}$$

Now we put $(x, y, z) \equiv (x^1, x^2, x^3)$ Using the basis vectors in (16), the induce metric of the hypersurface $\varphi = \varphi(x^i)$ is given by

$$g_{ij}^{(\alpha)} = \langle {}^{C}D_{x^i}^{\alpha}\varphi, {}^{C}D_{x^j}^{\alpha}\varphi \rangle, \tag{18}$$

where $\alpha \in]0,1]$ is the order of the Caputo fractionnal derivative of the componentsof φ.

The classical derivative of the first fundamental form is given by:

$$\frac{\partial g_{ij}^{(\alpha)}}{\partial x^k} + \frac{\partial g_{jk}^{(\alpha)}}{\partial x^i} - \frac{\partial g_{ki}^{(\alpha)}}{\partial x^j} = \frac{\partial}{\partial x^k}\langle {}^{C}D_{x^i}^{\alpha}\varphi, {}^{C}D_{x^j}^{\alpha}\varphi \rangle + \frac{\partial}{\partial x^i}\langle {}^{C}D_{x^j}^{\alpha}\varphi, {}^{C}D_{x^k}^{\alpha}\varphi \rangle - \frac{\partial}{\partial x^j}\langle {}^{C}D_{x^k}^{\alpha}\varphi, {}^{C}D_{x^i}^{\alpha}\varphi \rangle \tag{19}$$

Using the fractionnal case, the Christoffel symbols of the first type is defined by:

$$\Gamma^{(\alpha)}_{ikj} = \left\langle {}^{C}D^{\alpha}_{x^k}\varphi, \frac{\partial}{\partial x^i}({}^{C}D^{\alpha}_{x^j}\varphi)\right\rangle. \tag{20}$$

And the Christoffel symbols of the second type is defined by:

$$\Gamma^{(\alpha)k}_{i\,j} = g^{(\alpha)kl}\Gamma^{(\alpha)}_{il\,j}, \tag{21}$$

where $g^{(\alpha)kl}$ is the matrix inverse of $g^{(\alpha)}_{kl}$.

The torsion is given by $T^{(\alpha)}_{ikj} \equiv \Gamma^{(\alpha)}_{ikj} - \Gamma^{(\alpha)}_{jki}$. Then, the equation (19) becomes

$$\frac{\partial g^{(\alpha)}_{ij}}{\partial x^k} + \frac{\partial g^{(\alpha)}_{jk}}{\partial x^i} - \frac{\partial g^{(\alpha)}_{ki}}{\partial x^j} + T^{(\alpha)}_{jki} + T^{(\alpha)}_{jik} = \Gamma^{(\alpha)}_{kji} + \Gamma^{(\alpha)}_{ijk}. \tag{22}$$

Because, since (19) we have

$$\begin{aligned}\frac{\partial g^{(\alpha)}_{ij}}{\partial x^k} + \frac{\partial g^{(\alpha)}_{jk}}{\partial x^i} - \frac{\partial g^{(\alpha)}_{ki}}{\partial x^j} &= \Gamma^{(\alpha)}_{kij} + \Gamma^{(\alpha)}_{kji} + \Gamma^{(\alpha)}_{ikj} + \Gamma^{(\alpha)}_{ijk} - \Gamma^{(\alpha)}_{jki} - \Gamma^{(\alpha)}_{jik}\\ &= -(\Gamma^{(\alpha)}_{jki} - \Gamma^{(\alpha)}_{ikj}) - (\Gamma^{(\alpha)}_{jik} - \Gamma^{(\alpha)}_{kij}) + \Gamma^{(\alpha)}_{ijk} + \Gamma^{(\alpha)}_{kij}\\ &= -T^{(\alpha)}_{jki} - T^{(\alpha)}_{jik} + \Gamma^{(\alpha)}_{ijk} + \Gamma^{(\alpha)}_{kij}\end{aligned} \tag{23}$$

and that give the result. Now we introduce the following expression

$$G_{i\,j} = \frac{\partial}{\partial x^i}({}^{C}D^{\alpha}_{x^j}\varphi) - \Gamma^{(\alpha)h}_{i\,j}\,{}^{C}D^{\alpha}_{x^h}\varphi. \tag{24}$$

Proposition 3.1: The vector $G_{i\,j}$ is orthogonal to $D^{\alpha}_{x^k}\varphi$.
Proof: With an easy computation, we have

$$\begin{aligned}\langle G_{i\,j}, {}^{C}D^{\alpha}_{x^k}\varphi\rangle &= \left\langle \frac{\partial}{\partial x^i}({}^{C}D^{\alpha}_{x^j}\varphi) - \Gamma^{(\alpha)h}_{i\,j}\,{}^{C}D^{\alpha}_{x^h}\varphi, {}^{C}D^{\alpha}_{k}\varphi\right\rangle\\ &= \left\langle \frac{\partial}{\partial x^i}({}^{C}D^{\alpha}_{x^j}\varphi), {}^{C}D^{\alpha}_{x^k}\varphi\right\rangle - \Gamma^{(\alpha)h}_{ij}\langle {}^{C}D^{\alpha}_{x^h}\varphi, {}^{C}D^{\alpha}_{x^k}\varphi\rangle\\ &= \left\langle {}^{C}D^{\alpha}_{x^k}\varphi, \frac{\partial}{\partial x^i}({}^{C}D^{\alpha}_{x^j}\varphi)\right\rangle - \Gamma^{(\alpha)h}_{ij} g^{(\alpha)}_{hk}\\ &= \Gamma^{(\alpha)}_{ikj} - \Gamma^{(\alpha)}_{ikj}\\ &= 0\end{aligned}$$

Theorem 3.2: *The fractional Gauss equation of the hypersurface* φ *is given by*

$$\frac{\partial}{\partial x^i}({}^C D^{\alpha}_{x^j}\varphi) = \Gamma^{(\alpha)h}_{i\,j}\,{}^C D^{\alpha}_{x^h}\varphi + h^{(\alpha)}_{ij} n_{\alpha}, \tag{25}$$

where $h^{(\alpha)}_{ij} n_{\alpha}$ *is called the fractional second fundamental form of* φ.

Proof: Using the proposition 3.1, we have $\langle G_{i\,j}, {}^C D^{\alpha}_{x^k}\varphi\rangle = 0;$ so there exists a coefficient $h^{(\alpha)}_{ij}$ such that

$$G_{i\,j} = h^{(\alpha)}_{ij} n_{\alpha} = \frac{\partial}{\partial x^i}({}^C D^{\alpha}_{x^j}\varphi) - \Gamma^{(\alpha)h}_{i\,j}\,{}^C D^{\alpha}_{x^h}\varphi,$$

where $h^{(\alpha)}_{ij}$ is the fractional second fundamental form α defined by

$$h^{(\alpha)}_{ij} = \left\langle \frac{\partial}{\partial x^i}({}^C D^{\alpha}_{x^j}\varphi), n_{\alpha} \right\rangle. \tag{26}$$

We remark that the second fundamental form is also asymmetric because of the fractional non-local derivative $h^{(\alpha)}_{i\,j} \neq h^{(\alpha)}_{j\,i}$. Gauss's formula (25) and the vectors of the frame $\langle {}^C D^{\alpha}_{x^j}\varphi, n_{\alpha}\rangle = 0$ are orthogonal, expressing the second basic form as

$$h^{(\alpha)}_{i\,j} = -\left\langle {}^C D^{\alpha}_{x^j}\varphi, \frac{\partial n_{\alpha}}{\partial x^i} \right\rangle. \tag{27}$$

Because we have $\langle {}^C D^{\alpha}_{x^j}\varphi, n_{\alpha}\rangle = 0$ and differentiating the vector of the basis x^i we get

$$\begin{aligned}
\frac{\partial}{\partial x^i}\langle {}^C D^{\alpha}_{x^j}\varphi, n_{\alpha}\rangle &= \frac{\partial}{\partial x^i}\langle {}^C D^{\alpha}_{x^j}\varphi, n_{\alpha}\rangle + \left\langle {}^C D^{\alpha}_{x^j}\varphi, \frac{\partial n_{\alpha}}{\partial x^i} \right\rangle \\
&= h^{(\alpha)}_{ij} + \left\langle {}^C D^{\alpha}_{x^j}\varphi, \frac{\partial n_{\alpha}}{\partial x^i} \right\rangle \\
&= h^{(\alpha)}_{ij} - h^{(\alpha)}_{ij} = 0
\end{aligned}$$

and then the result.The classical derivative of $\langle n_{\alpha}, n_{\alpha}\rangle = 1$ gives $\left\langle \frac{\partial n_{\alpha}}{\partial x^i}, n_{\alpha} \right\rangle = 0.$ So there exists a coefficient L^k_i such that the derivative of the normal vector is expressed as

$$\frac{\partial n_{\alpha}}{\partial x^i} = L^k_i\,{}^C D^{\alpha}_{x^k}\varphi. \tag{28}$$

If replace (28) in (27), the coefficient L^k_i is given by :

$$h_{ij}^{(\alpha)} = -L_i^k \langle {}^C D_{x^j}^{\alpha} \varphi, {}^C D_{x^k}^{\alpha} \varphi \rangle = -L_i^k g_{jk}^{(\alpha)}. \tag{29}$$

That is $h_i^{(\alpha)k} = g^{(\alpha)kl} h_{il}^{(\alpha)} = -L_i^k$. Thus the fractional Weingarten is given by:

$$\frac{\partial n_\alpha}{\partial x^i} = -h_i^{(\alpha)k}\, {}^C D_{x^k}^{\alpha} \varphi. \tag{30}$$

3.2 *Condition of Integrability and curvatures*

Here we study conditions for integrating the hypersurface into the fractional basis. First, the non-commutativity of the classical derivative of the basis vectors is related to the asymmetry of Christoffel symbols of type 2 and to the second fundamental form. And then we have

$$\frac{\partial}{\partial x^i}({}^C D_{x^j}^{\alpha} \varphi) - \frac{\partial}{\partial x^j}({}^C D_{x^i}^{\alpha} \varphi) = (\Gamma_{ij}^{(\alpha)k} - \Gamma_{ji}^{(\alpha)k})\, {}^C D_{x^k}^{\alpha} \varphi + (h_{ij}^{(\alpha)} - h_{ji}^{(\alpha)}) n_\alpha. \tag{31}$$

Moreover, we consider the commutativity of the following integer derivatives:

$$\frac{\partial}{\partial x^k}\left[\frac{\partial}{\partial x^i}({}^C D_{x^j}^{\alpha} \varphi)\right] = \frac{\partial}{\partial x^i}\left[\frac{\partial}{\partial x^k}({}^C D_{x^j}^{\alpha} \varphi)\right] \tag{32}$$

By (25) and (30), the left term in (32) is given by

$$\begin{aligned}
&\frac{\partial}{\partial x^k}\left[\frac{\partial}{\partial x^i}({}^C D_{x^j}^{\alpha} \varphi)\right] \\
&\quad = \frac{\partial}{\partial x^k}\left[\Gamma_{ij}^{(\alpha)l}\, {}^C D_{x^l}^{\alpha} \varphi + h_{ij}^{(\alpha)} n_\alpha\right] \\
&\quad = \frac{\partial}{\partial x^k}\Gamma_{ij}^{(\alpha)l}\, {}^C D_{x^l}^{\alpha} \varphi + \Gamma_{ij}^{(\alpha)l} \frac{\partial}{\partial x^k}({}^C D_{x^l}^{\alpha} \varphi) + \frac{\partial}{\partial x^k} h_{ij}^{(\alpha)} n_\alpha + h_{ij}^{(\alpha)} \frac{\partial n_\alpha}{\partial x^k} \\
&\quad = \frac{\partial}{\partial x^k}\Gamma_{ij}^{(\alpha)l}\, {}^C D_{x^l}^{\alpha} \varphi + \Gamma_{ij}^{(\alpha)l} \frac{\partial}{\partial x^k}({}^C D_{x^l}^{\alpha} \varphi) + \frac{\partial}{\partial x^k} h_{ij}^{(\alpha)} n_\alpha + h_{ij}^{(\alpha)} (-h_k^{(\alpha)l}\, {}^C D_{x^l}^{\alpha} \varphi) \\
&\quad = \frac{\partial}{\partial x^k}\Gamma_{ij}^{(\alpha)l}\, {}^C D_{x^l}^{\alpha} \varphi + \Gamma_{ij}^{(\alpha)m} (\Gamma_{km}^{(\alpha)l}\, {}^C D_{x^l}^{\alpha} \varphi + h_{kl}^{(\alpha)} n_\alpha) + \frac{\partial}{\partial x^k} h_{ij}^{(\alpha)} n_\alpha - h_{ij}^{(\alpha)} h_k^{(\alpha)l}\, {}^C D_{x^k}^{\alpha} \varphi \\
&\quad = \left(\frac{\partial}{\partial x^k}\Gamma_{ij}^{(\alpha)l} + \Gamma_{ij}^{(\alpha)m} \Gamma_{km}^{(\alpha)l} - h_{ij}^{(\alpha)} h_k^{(\alpha)l}\right)^C D_{x^l}^{\alpha} \varphi + \left(\frac{\partial}{\partial x^k} h_{ij}^{(\alpha)} + \Gamma_{ij}^{(\alpha)} h_{kl}^{(\alpha)}\right) n_\alpha.
\end{aligned} \tag{33}$$

With the same methode we obtain the second fundamental form on the right term of (32):

$$\frac{\partial}{\partial x^i}\left[\frac{\partial}{\partial x^k}({}^C D_{x^j}^{\alpha}\varphi)\right]=\left(\frac{\partial}{\partial x^i}\Gamma_{kj}^{(\alpha)l}+\Gamma_{kj}^{(\alpha)m}\Gamma_{im}^{(\alpha)l}-h_{kj}^{(\alpha)}h_i^{(\alpha)l}\right)^C D_{x^l}^{\alpha}\varphi+\left(\frac{\partial}{\partial x^i}h_{kj}^{(\alpha)}+\Gamma_{kj}^{(\alpha)}h_{il}^{(\alpha)}\right)n_\alpha. \tag{34}$$

Thus we obtain the fractional differential equations of Gauss-Mainardi-Codazzi by the following expression:

$$\frac{\partial\Gamma_{i\,j}^{(\alpha)l}}{\partial x^k}-\frac{\partial\Gamma_{k\,j}^{(\alpha)l}}{\partial x^i}+\Gamma_{i\,j}^{(\alpha)m}\Gamma_{k\,m}^{(\alpha)l}-\Gamma_{k\,j}^{(\alpha)m}\Gamma_{i\,m}^{(\alpha)l}-(h_{i\,j}^{(\alpha)}h_k^{(\alpha)l}-h_{k\,j}^{(\alpha)}h_i^{(\alpha)l})=0, \tag{35}$$

$$\frac{\partial h_{i\,j}^{(\alpha)}}{\partial x^k}-\frac{\partial h_{k\,j}^{(\alpha)}}{\partial x^i}+\Gamma_{i\,j}^{(\alpha)l}h_{kl}^{(\alpha)l}-\Gamma_{k\,j}^{(\alpha)l}h_{il}^{(\alpha)l}=0. \tag{36}$$

With the equation (35), and using the Christoffel of second type, we get

$$R_{jik}^{(\alpha)l}=\frac{\partial\Gamma_{i\,j}^{(\alpha)l}}{\partial x^k}-\frac{\partial\Gamma_{k\,j}^{(\alpha)l}}{\partial x^i}+\Gamma_{i\,j}^{(\alpha)m}\Gamma_{k\,m}^{(\alpha)l}-\Gamma_{k\,j}^{(\alpha)m}\Gamma_{i\,m}^{(\alpha)l}. \tag{37}$$

Because, since (35) we get

$$\frac{\partial\Gamma_{i\,j}^{(\alpha)l}}{\partial x^k}-\frac{\partial\Gamma_{k\,j}^{(\alpha)l}}{\partial x^i}+\Gamma_{i\,j}^{(\alpha)m}\Gamma_{k\,m}^{(\alpha)l}-\Gamma_{k\,j}^{(\alpha)m}\Gamma_{i\,m}^{(\alpha)l}=h_{i\,j}^{(\alpha)}h_k^{(\alpha)l}-h_{k\,j}^{(\alpha)}h_i^{(\alpha)l}.$$

If we put $R_{jik}^{(\alpha)l}=h_{i\,j}^{(\alpha)}\,h_k^{(\alpha)l}-h_{k\,j}^{(\alpha)}\,h_i^{(\alpha)l}$, we get the result. And the relation $R_{jlik}^{(\alpha)}=g_{ml}^{(\alpha)}\,R_{jik}^{(\alpha)m}$ become, by using the non-symmetry of the second form fundamental:

$$\begin{aligned}R_{jik}^{(\alpha)m}&=h_{i\,j}^{(\alpha)}h_k^{(\alpha)m}-h_{k\,j}^{(\alpha)}h_i^{(\alpha)m}\\&=h_{ij}^{(\alpha)}g^{(\alpha)lm}h_{kl}^{(\alpha)}-h_{kj}^{(\alpha)}g^{(\alpha)lm}h_{il}^{(\alpha)}\\&=(h_{ij}^{(\alpha)}h_{kl}^{(\alpha)}-h_{kj}^{(\alpha)}h_{il}^{(\alpha)})g^{(\alpha)lm}.\end{aligned}$$

And by putting $R_{jlik}^{(\alpha)}=h_{ij}^{(\alpha)}h_{kl}^{(\alpha)}-h_{kj}^{(\alpha)}h_{il}^{(\alpha)}$, we get: $R_{jlik}^{(\alpha)}=g_{ml}^{(\alpha)}R_{jik}^{(\alpha)m}$.

Theorem 3.3: *The fractional Gauss curvature of the hypersurface φ is given by*

$$K^{(\alpha)}=\frac{\det h_{ij}^{(\alpha)}}{\det g_{ij}^{(\alpha)}}, \tag{38}$$

where

$$\det[g_{ij}^{(\alpha)}] = -g_{13}^{(\alpha)}g_{22}^{(\alpha)}g_{31}^{(\alpha)} + g_{12}^{(\alpha)}g_{23}^{(\alpha)}g_{31}^{(\alpha)} + g_{13}^{(\alpha)}g_{21}^{(\alpha)}g_{32}^{(\alpha)} - g_{11}^{(\alpha)}g_{23}^{(\alpha)}g_{32}^{(\alpha)} - g_{12}^{(\alpha)}g_{21}^{(\alpha)}g_{33}^{(\alpha)} + g_{11}^{(\alpha)}g_{22}^{(\alpha)}g_{33}^{(\alpha)}.$$

and

$$\det[h_{ij}^{(\alpha)}] = -h_{13}^{(\alpha)}h_{22}^{(\alpha)}h_{31}^{(\alpha)} + h_{12}^{(\alpha)}h_{23}^{(\alpha)}h_{31}^{(\alpha)} + h_{13}^{(\alpha)}h_{21}^{(\alpha)}h_{32}^{(\alpha)} - h_{11}^{(\alpha)}h_{23}^{(\alpha)}h_{32}^{(\alpha)} - h_{12}^{(\alpha)}h_{21}^{(\alpha)}h_{33}^{(\alpha)} + h_{11}^{(\alpha)}h_{22}^{(\alpha)}h_{33}^{(\alpha)}.$$

Using the relation in (7) the expression of the inverse of $[g_{ij}^{\alpha}]$ is given by:

$$[g^{(\alpha)ij}] = \frac{1}{\det[g_{ij}^{(\alpha)}]}\begin{bmatrix} g_{22}^{(\alpha)}g_{33}^{(\alpha)} - g_{23}^{(\alpha)}g_{32}^{(\alpha)} & g_{13}^{(\alpha)}g_{32}^{(\alpha)} - g_{12}^{(\alpha)}g_{33}^{(\alpha)} & g_{12}^{(\alpha)}g_{23}^{(\alpha)} - g_{13}^{(\alpha)}g_{22}^{(\alpha)} \\ g_{23}^{(\alpha)}g_{31}^{(\alpha)} - g_{21}^{(\alpha)}g_{33}^{(\alpha)} & g_{11}^{(\alpha)}g_{33}^{(\alpha)} - g_{13}^{(\alpha)}g_{31}^{(\alpha)} & g_{13}^{(\alpha)}g_{21}^{(\alpha)} - g_{11}^{(\alpha)}g_{23}^{(\alpha)} \\ g_{21}^{(\alpha)}g_{32}^{(\alpha)} - g_{22}^{(\alpha)}g_{31}^{(\alpha)} & g_{12}^{(\alpha)}g_{31}^{(\alpha)} - g_{11}^{(\alpha)}g_{32}^{(\alpha)} & g_{11}^{(\alpha)}g_{22}^{(\alpha)} - g_{12}^{(\alpha)}g_{21}^{(\alpha)} \end{bmatrix}. \tag{39}$$

The mean curvature of S via the fractional way is given by the relation

$$3H^{(\alpha)} = trace([g^{(\alpha)ij}]\cdot[h_{ij}^{(\alpha)}]). \tag{40}$$

Since (3.2) we have

$$H^{(\alpha)} = \frac{trace\left(\begin{bmatrix} g_{22}^{(\alpha)}g_{33}^{(\alpha)} - g_{23}^{(\alpha)}g_{32}^{(\alpha)} & g_{13}^{(\alpha)}g_{32}^{(\alpha)} - g_{12}^{(\alpha)}g_{33}^{(\alpha)} & g_{12}^{(\alpha)}g_{23}^{(\alpha)} - g_{13}^{(\alpha)}g_{22}^{(\alpha)} \\ g_{23}^{(\alpha)}g_{31}^{(\alpha)} - g_{21}^{(\alpha)}g_{33}^{(\alpha)} & g_{11}^{(\alpha)}g_{33}^{(\alpha)} - g_{13}^{(\alpha)}g_{31}^{(\alpha)} & g_{13}^{(\alpha)}g_{21}^{(\alpha)} - g_{11}^{(\alpha)}g_{23}^{(\alpha)} \\ g_{21}^{(\alpha)}g_{32}^{(\alpha)} - g_{22}^{(\alpha)}g_{31}^{(\alpha)} & g_{12}^{(\alpha)}g_{31}^{(\alpha)} - g_{11}^{(\alpha)}g_{32}^{(\alpha)} & g_{11}^{(\alpha)}g_{22}^{(\alpha)} - g_{12}^{(\alpha)}g_{21}^{(\alpha)} \end{bmatrix}\cdot\begin{bmatrix} h_{11}^{(\alpha)} & h_{12}^{(\alpha)} & h_{13}^{(\alpha)} \\ h_{21}^{(\alpha)} & h_{22}^{(\alpha)} & h_{23}^{(\alpha)} \\ h_{31}^{(\alpha)} & h_{32}^{(\alpha)} & h_{33}^{(\alpha)} \end{bmatrix}\right)}{3\det[g_{ij}^{(\alpha)}]}. \tag{41}$$

Theorem 3.4: *The fractional mean curvature of the hypersurface φ is given by:*

$$H^{(\alpha)} = \frac{\begin{bmatrix} h_{11}^{(\alpha)}(g_{22}^{(\alpha)}g_{33}^{(\alpha)} - g_{23}^{(\alpha)}g_{32}^{(\alpha)}) + h_{21}^{(\alpha)}(g_{13}^{(\alpha)}g_{32}^{(\alpha)} - g_{12}^{(\alpha)}g_{33}^{(\alpha)}) + h_{13}^{(\alpha)}(g_{12}^{(\alpha)}g_{23}^{(\alpha)} - g_{13}^{(\alpha)}g_{22}^{(\alpha)}) \\ +h_{12}^{(\alpha)}(g_{23}^{(\alpha)}g_{31}^{(\alpha)} - g_{21}^{(\alpha)}g_{33}^{(\alpha)}) + h_{22}^{(\alpha)}(g_{11}^{(\alpha)}g_{33}^{(\alpha)} - g_{13}^{(\alpha)}g_{31}^{(\alpha)}) + h_{32}^{(\alpha)}(g_{13}^{(\alpha)}g_{21}^{(\alpha)} - g_{11}^{(\alpha)}g_{23}^{(\alpha)}) \\ +h_{13}^{(\alpha)}(g_{21}^{(\alpha)}g_{32}^{(\alpha)} - g_{22}^{(\alpha)}g_{31}^{(\alpha)}) + h_{23}^{(\alpha)}(g_{12}^{(\alpha)}g_{31}^{(\alpha)} - g_{11}^{(\alpha)}g_{32}^{(\alpha)}) + h_{33}^{(\alpha)}(g_{11}^{(\alpha)}g_{22}^{(\alpha)} - g_{12}^{(\alpha)}g_{21}^{(\alpha)}) \end{bmatrix}}{3\begin{bmatrix} -g_{13}^{(\alpha)}g_{22}^{(\alpha)}g_{31}^{(\alpha)} + g_{12}^{(\alpha)}g_{23}^{(\alpha)}g_{31}^{(\alpha)} + g_{13}^{(\alpha)}g_{21}^{(\alpha)}g_{32}^{(\alpha)} \\ -g_{11}^{(\alpha)}g_{23}^{(\alpha)}g_{32}^{(\alpha)} - g_{12}^{(\alpha)}g_{21}^{(\alpha)}g_{33}^{(\alpha)} + g_{11}^{(\alpha)}g_{22}^{(\alpha)}g_{33}^{(\alpha)} \end{bmatrix}}. \tag{42}$$

Example 3.5: Let the function *f* given by

$$\begin{array}{rcl} f:U\subset\mathbb{R}^3 & \to & \mathbb{R} \\ (x,y,z) & \mapsto & (x,y,z)=x^2+y^2+z^2 \qquad \text{where } U\subset\mathbb{R}^3. \end{array} \tag{43}$$

Now let M be the graph parametrized by $\varphi = (\varphi^1, \varphi^2, \varphi^3, \varphi^4) = (x, y, z, f(x, y, z))$. The fractional vectors of the frame of the tangent space TM are:

$$\begin{aligned} {}^{C}D_x^{\alpha}\varphi^1 &= \frac{1}{\Gamma(1-\alpha)}\int_0^x \frac{1}{(x-t)^{\alpha}}dt \\ &= \frac{1}{\Gamma(1-\alpha)}\left[-\frac{(x-t)^{1-\alpha}}{1-\alpha}\right]_0^x \\ &= \frac{x^{1-\alpha}}{(1-\alpha)\Gamma(1-\alpha)} = \frac{x^{1-\alpha}}{\Gamma(2-\alpha)}; \end{aligned}$$

$${}^{C}D_x^{\alpha}\varphi^2 = 0,$$

$${}^{C}D_x^{\alpha}\varphi^3 = 0$$

and

$${}^{C}D_x^{\alpha}\varphi^4 = \frac{1}{\Gamma(1-\alpha)}\int_0^x \frac{2t}{\Gamma(x-t)^{\alpha}}dt.$$

With an easy computation we get

$${}^{C}D_x^{\alpha}\varphi^4 = \frac{2x^{2-\alpha}}{\Gamma(3-\alpha)}.$$

Then we get

$${}^{C}D_x^{\alpha}\varphi = \frac{x^{1-\alpha}}{\Gamma(2-\alpha)}\left(1,0,0,\frac{2x}{3-\alpha}\right). \tag{44}$$

The same computation gives

$${}^{C}D_y^{\alpha}\varphi = \frac{y^{1-\alpha}}{\Gamma(2-\alpha)}\left(0,1,0,\frac{2y}{3-\alpha}\right) \tag{45}$$

and

$${}^{C}D_z^{\alpha}\varphi = \frac{z^{1-\alpha}}{\Gamma(2-\alpha)}\left(0,0,1,\frac{2z}{3-\alpha}\right). \tag{46}$$

Using the vector product formula, we get

$$ {}^{C}D_x^{\alpha}\varphi \times {}^{C}D_y^{\alpha}\varphi \times {}^{C}D_z^{\alpha}\varphi = \frac{x^{1-\alpha}y^{1-\alpha}z^{1-\alpha}}{\Gamma^3(2-\alpha)}\left(\frac{2x}{3-\alpha},\frac{2y}{3-\alpha},\frac{2z}{3-\alpha},-1\right). \tag{47} $$

The norm is

$$ \begin{aligned} \|{}^{C}D_x^{\alpha}\varphi \times {}^{C}D_y^{\alpha}\varphi \times {}^{C}D_z^{\alpha}\varphi\| &= \frac{x^{1-\alpha}y^{1-\alpha}z^{1-\alpha}}{\Gamma^3(2-\alpha)}\left[\frac{4x^2}{(x-\alpha)^2}+\frac{4y^2}{(y-\alpha)^2}+\frac{4z^2}{(z-\alpha)^2}+1\right]^{\frac{1}{2}} \\ &= \frac{2x^{1-\alpha}y^{1-\alpha}z^{1-\alpha}}{(3-\alpha)\Gamma^3(2-\alpha)}\left[x^2+y^2+z^2+\left(\frac{3-\alpha}{2}\right)^2\right]^{\frac{1}{2}}. \end{aligned} \tag{48} $$

Finaly the fractional normal vector is

$$ n_\alpha = \left[x^2+y^2+z^2+\left(\frac{3-\alpha}{2}\right)^2\right]^{-\frac{1}{2}}\left(x,y,z,\frac{\alpha-3}{2}\right). \tag{49} $$

the coefficients of the first fundamental form via fractional calculus are

$$ g_{11}^{(\alpha)} = \langle {}^{C}D_x^{\alpha}\varphi, {}^{C}D_x^{\alpha}\varphi\rangle = \frac{x^{2-2\alpha}}{\Gamma^2(2-\alpha)}\left(1+\frac{4x^2}{(3-\alpha)^2}\right) \tag{50} $$

$$ g_{22}^{(\alpha)} = \langle {}^{C}D_y^{\alpha}\varphi, {}^{C}D_y^{\alpha}\varphi\rangle = \frac{y^{2-2\alpha}}{\Gamma^2(2-\alpha)}\left(1+\frac{4y^2}{(3-\alpha)^2}\right) \tag{51} $$

$$ g_{33}^{(\alpha)} = \langle {}^{C}D_z^{\alpha}\varphi, {}^{C}D_z^{\alpha}\varphi\rangle = \frac{z^{2-2\alpha}}{\Gamma^2(2-\alpha)}\left(1+\frac{4z^2}{(3-\alpha)^2}\right) \tag{52} $$

$$ g_{12}^{(\alpha)} = g_{21}^{\alpha} = \langle {}^{C}D_x^{\alpha}\varphi, {}^{C}D_y^{\alpha}\varphi\rangle = \frac{4x^{2-\alpha}y^{2-\alpha}}{\Gamma^2(3-\alpha)} \tag{53} $$

$$ g_{13}^{(\alpha)} = g_{31}^{\alpha} = \langle {}^{C}D_x^{\alpha}\varphi, {}^{C}D_z^{\alpha}\varphi\rangle = \frac{4x^{2-\alpha}z^{2-\alpha}}{\Gamma^2(3-\alpha)} \tag{54} $$

$$ g_{23}^{(\alpha)} = g_{32}^{\alpha} = \langle {}^{C}D_y^{\alpha}\varphi, {}^{C}D_z^{\alpha}\varphi\rangle = \frac{4y^{2-\alpha}z^{2-\alpha}}{\Gamma^2(3-\alpha)}. \tag{55} $$

By using the normal n_α, the coefficients of the second fundamental form via fractional way, are

$$h_{11}^{(\alpha)} = \left\langle \frac{\partial}{\partial x}({}^{C}\mathrm{D}_{x}^{\alpha}\varphi), n_{\alpha} \right\rangle = \left\langle \frac{(1-\alpha)x^{-\alpha}}{\Gamma(2-\alpha)}\left(1,0,0,\frac{2(2-\alpha)x}{(1-\alpha)(3-\alpha)}\right), \right.$$

$$\left. \left[x^2+y^2+z^2+\left(\frac{3-\alpha}{2}\right)^2\right]^{-\frac{1}{2}}\left(x,y,z,\frac{\alpha-3}{2}\right)\right\rangle$$

$$h_{11}^{(\alpha)} = -\frac{x^{1-\alpha}}{\Gamma(2-\alpha)}\left[x^2+y^2+z^2+\left(\frac{3-\alpha}{2}\right)^2\right]^{-\frac{1}{2}} \tag{56}$$

$$h_{22}^{(\alpha)} = \left\langle \frac{\partial}{\partial y}({}^{C}D_{y}^{\alpha}\varphi), n_{\alpha} \right\rangle = \left\langle \frac{(1-\alpha)y^{-\alpha}}{\Gamma(2-\alpha)}\left(1,0,0,\frac{2(2-\alpha)y}{(1-\alpha)(3-\alpha)}\right), \right.$$

$$\left. \left[x^2+y^2+z^2+\left(\frac{3-\alpha}{2}\right)^2\right]^{-\frac{1}{2}}\left(x,y,z,\frac{\alpha-3}{2}\right)\right\rangle$$

$$h_{22}^{(\alpha)} = -\frac{y^{1-\alpha}}{\Gamma(2-\alpha)}\left[x^2+y^2+z^2+\left(\frac{3-\alpha}{2}\right)^2\right]^{-\frac{1}{2}} \tag{57}$$

$$h_{33}^{(\alpha)} = \left\langle \frac{\partial}{\partial z}({}^{C}D_{z}^{\alpha}\varphi), n_{\alpha} \right\rangle = \left\langle \frac{(1-\alpha)z^{-\alpha}}{\Gamma(2-\alpha)}\left(1,0,0,\frac{2(2-\alpha)z}{(1-\alpha)(3-\alpha)}\right), \right.$$

$$\left. \left[x^2+y^2+z^2+\left(\frac{3-\alpha}{2}\right)^2\right]^{-\frac{1}{2}}\left(x,y,z,\frac{\alpha-3}{2}\right)\right\rangle h_{33}^{(\alpha)}$$

$$= -\frac{z^{1-\alpha}}{\Gamma(2-\alpha)}\left[x^2+y^2+z^2+\left(\frac{3-\alpha}{2}\right)^2\right]^{-\frac{1}{2}} \tag{58}$$

$$h_{12}^{(\alpha)} = 0\,; h_{13}^{(\alpha)} = 0\,; h_{21}^{(\alpha)} = 0\,; h_{23}^{(\alpha)} = 0\,; h_{31}^{(\alpha)} = 0 \, et \, h_{32}^{(\alpha)} = 0. \tag{59}$$

Thus, for the fractional Gauss curvature, we have

$$K^{(\alpha)} = \frac{\det\begin{bmatrix} h_{11}^{(\alpha)} & 0 & 0 \\ 0 & h_{22}^{(\alpha)} & 0 \\ 0 & 0 & h_{33}^{(\alpha)} \end{bmatrix}}{\det\begin{bmatrix} g_{11}^{(\alpha)} & g_{12}^{(\alpha)} & g_{13}^{(\alpha)} \\ g_{21}^{(\alpha)} & g_{22}^{(\alpha)} & g_{23}^{(\alpha)} \\ g_{31}^{(\alpha)} & g_{32}^{(\alpha)} & g_{33}^{(\alpha)} \end{bmatrix}} \tag{60}$$

where

$$\det[h_{ij}^{(\alpha)}] = h_{11}^{(\alpha)}h_{22}^{(\alpha)}h_{33}^{(\alpha)} = \frac{-x^{1-\alpha}y^{1-\alpha}z^{1-\alpha}}{\Gamma^3(2-\alpha)}\left[x^2+y^2+z^2+\left(\frac{3-\alpha}{2}\right)^2\right]^{-\frac{3}{2}}$$

and

$$\begin{aligned}\det[g_{ij}^{(\alpha)}] &= -g_{13}^{(\alpha)}g_{22}^{(\alpha)}g_{31}^{(\alpha)} + g_{12}^{(\alpha)}g_{23}^{(\alpha)}g_{31}^{(\alpha)} + g_{13}^{(\alpha)}g_{21}^{(\alpha)}g_{32}^{(\alpha)} - g_{11}^{(\alpha)}g_{23}^{(\alpha)}g_{32}^{(\alpha)} - \\ &\quad g_{12}^{(\alpha)}g_{21}^{(\alpha)}g_{33}^{(\alpha)} + g_{11}^{(\alpha)}g_{22}^{(\alpha)}g_{33}^{(\alpha)} \\ &= \frac{x^{2-2\alpha}y^{2-2\alpha}z^{2-2\alpha}}{\Gamma^6(3-\alpha)}\left[\begin{array}{l}(3-\alpha)^4\left(1+\frac{4x^2}{(3-\alpha)^2}\right)\left(1+\frac{4y^2}{(3-\alpha)^2}\right)\left(1+\frac{4z^2}{(3-\alpha)^2}\right)+ \\ 128x^2y^2z^2-16(3-\alpha)^2x^2z^2\left(1+\frac{4y^2}{(3-\alpha)^2}\right)- \\ \left(16(3-\alpha)^2(x^2y^2\left(1+\frac{4z^2}{(3-\alpha)^2}\right)+y^2z^2\left(1+\frac{4x^2}{(3-\alpha)^2}\right)\right)\end{array}\right].\end{aligned}$$

So we obtain

$$K^{(\alpha)} = \frac{-x^{\alpha-1}y^{\alpha-1}z^{\alpha-1}\left[x^2+y^2+z^2+\left(\frac{3-\alpha}{2}\right)^2\right]^{-\frac{3}{2}}}{\left[\begin{array}{l}\frac{(3-\alpha)}{\Gamma^3(3-\alpha)}\left(1+\frac{4x^2}{(3-\alpha)^2}\right)\left(1+\frac{4y^2}{(3-\alpha)^2}\right)\left(1+\frac{4z^2}{(3-\alpha)^2}\right)+ \\ \frac{128}{\Gamma^3(4-\alpha)}x^2y^2z^2-\frac{16}{\Gamma(4-\alpha)\Gamma^2(3-\alpha)}x^2z^2\left(1+\frac{4y^2}{(3-\alpha)^2}\right)- \\ \frac{16}{\Gamma(4-\alpha)\Gamma^2(3-\alpha)}\left(x^2y^2\left(1+\frac{4z^2}{(3-\alpha)^2}\right)+y^2z^2\left(1+\frac{4x^2}{(3-\alpha)^2}\right)\right)\end{array}\right]}. \tag{61}$$

For the fractional mean curvature, we have

$$H^{(\alpha)} = \frac{h_{11}^{(\alpha)}(g_{22}^{(\alpha)}g_{33}^{(\alpha)}-g_{23}^{(\alpha)}g_{32}^{(\alpha)})+h_{22}^{(\alpha)}(g_{11}^{(\alpha)}g_{33}^{(\alpha)}-g_{13}^{(\alpha)}g_{31}^{(\alpha)})+h_{33}^{(\alpha)}(g_{11}^{(\alpha)}g_{22}^{(\alpha)}-g_{12}^{(\alpha)}g_{21}^{(\alpha)})}{3\det g_{ij}^{(\alpha)}}$$

$$h_{11}^{(\alpha)}(g_{22}^{(\alpha)}g_{33}^{(\alpha)}-g_{23}^{(\alpha)}g_{32}^{(\alpha)}) = \frac{-x^{1-\alpha}y^{2-\alpha}z^{2-\alpha}}{\Gamma(2-\alpha)\sqrt{x^2+y^2+z^2}}\left[\begin{array}{l}\frac{y^{-\alpha}z^{-\alpha}}{\Gamma^4(2-\alpha)}\left(1+\frac{4y^2}{(3-\alpha)^2}\right)\left(1+\frac{4z^2}{(3-\alpha)^2}\right) \\ -\frac{8}{\Gamma^2(3-\alpha)}\end{array}\right]$$

$$h_{22}^{(\alpha)}(g_{11}^{(\alpha)}g_{33}^{(\alpha)}-g_{13}^{(\alpha)}g_{31}^{(\alpha)}) = \frac{-y^{1-\alpha}x^{2-\alpha}z^{2-\alpha}}{\Gamma(2-\alpha)\sqrt{x^2+y^2+z^2}}\left[\begin{array}{l}\frac{y^{-\alpha}z^{-\alpha}}{\Gamma^4(2-\alpha)}\left(1+\frac{4x^2}{(3-\alpha)^2}\right)\left(1+\frac{4z^2}{(3-\alpha)^2}\right) \\ -\frac{8}{\Gamma^2(3-\alpha)}\end{array}\right]$$

$$h_{33}^{(\alpha)}(g_{11}^{(\alpha)}g_{22}^{(\alpha)}-g_{12}^{(\alpha)}g_{21}^{(\alpha)})=\frac{-z^{1-\alpha}x^{2-\alpha}y^{2-\alpha}}{\Gamma(2-\alpha)\sqrt{x^2+y^2+z^2}}\left[\begin{array}{l}\frac{x^{-\alpha}y^{-\alpha}}{\Gamma^4(2-\alpha)}\left(1+\frac{4x^2}{(3-\alpha)^2}\right)\left(1+\frac{4y^2}{(3-\alpha)^2}\right)\\-\frac{8}{\Gamma^2(3-\alpha)}\end{array}\right].$$

Then we obtain

$$H^{(\alpha)}=\frac{\left[\begin{array}{l}\frac{-x^{1-\alpha}y^{2-\alpha}z^{2-\alpha}}{\Gamma(2-\alpha)\sqrt{x^2+y^2+z^2}}\left[\begin{array}{l}\frac{y^{-\alpha}z^{-\alpha}}{\Gamma^4(2-\alpha)}\left(1+\frac{4y^2}{(3-\alpha)^2}\right)\left(1+\frac{4z^2}{(3-\alpha)^2}\right)\\-\frac{8}{\Gamma^2(3-\alpha)}\end{array}\right]+\\\frac{-y^{1-\alpha}x^{2-\alpha}z^{2-\alpha}}{\Gamma(2-\alpha)\sqrt{x^2+y^2+z^2}}\left[\begin{array}{l}\frac{y^{-\alpha}z^{-\alpha}}{\Gamma^4(2-\alpha)}\left(1+\frac{4x^2}{(3-\alpha)^2}\right)\left(1+\frac{4z^2}{(3-\alpha)^2}\right)\\-\frac{8}{\Gamma^2(3-\alpha)}\end{array}\right]+\\\frac{-z^{1-\alpha}x^{2-\alpha}y^{2-\alpha}}{\Gamma(2-\alpha)\sqrt{x^2+y^2+z^2}}\left[\begin{array}{l}\frac{x^{-\alpha}y^{-\alpha}}{\Gamma^4(2-\alpha)}\left(1+\frac{4x^2}{(3-\alpha)^2}\right)\left(1+\frac{4y^2}{(3-\alpha)^2}\right)\\-\frac{8}{\Gamma^2(3-\alpha)}\end{array}\right]\end{array}\right]}{3\left[\begin{array}{l}\frac{(3-\alpha)}{\Gamma^3(3-\alpha)}\left(1+\frac{4x^2}{(3-\alpha)^2}\right)\left(1+\frac{4y^2}{(3-\alpha)^2}\right)\left(1+\frac{4z^2}{(3-\alpha)^2}\right)+\\\frac{128}{\Gamma^3(4-\alpha)}x^2y^2z^2-\frac{16}{\Gamma(4-\alpha)\Gamma^2(3-\alpha)}x^2z^2\left(1+\frac{4y^2}{(3-\alpha)^2}\right)-\\\frac{16}{\Gamma(4-\alpha)\Gamma^2(3-\alpha)}\left(x^2y^2\left(1+\frac{4z^2}{(3-\alpha)^2}\right)+y^2z^2\left(1+\frac{4x^2}{(3-\alpha)^2}\right)\right)\end{array}\right]}. \tag{62}$$

Example 3.6: We give another example of graph where the function *f* is a constant defined by

$$f:U\subset\mathbb{R}^3\to\mathbb{R}(x,y,z)\mapsto f(x,y,z)=K \qquad \textit{avec } U \text{ un ouvert de } \mathbb{R}^3 \tag{63}$$

where $k=constante$ non zero constant. The parametrization of the graph is: $\varphi=(\varphi^1,\ \varphi^2,\ \varphi^3,\ \varphi^4)=(x,\ y,\ z,\ f(x,\ y,\ z))$ Since the Caputo derivative of a constant is zero, we have ${}^{C}D_x^{\alpha}\varphi={}^{C}D_y^{\alpha}\varphi={}^{C}D_z^{\alpha}\varphi=0$.

Using (44), (45) et (46), we get

$${}^{C}D_x^{\alpha}\varphi=\frac{x^{1-\alpha}}{\Gamma(2-\alpha)}(1,0,0,0), \tag{64}$$

$${}^{C}D_y^{\alpha}\varphi=\frac{y^{1-\alpha}}{\Gamma(2-\alpha)}(0,1,0,0) \tag{65}$$

and

$$ {}^{C}D_{z}^{\alpha}\varphi = \frac{z^{1-\alpha}}{\Gamma(2-\alpha)}(0,0,1,0). \tag{66} $$

The vector product is

$$ {}^{C}D_{x}^{\alpha}\varphi \times {}^{C}D_{y}^{\alpha}\varphi \times {}^{C}D_{z}^{\alpha}\varphi = \frac{x^{1-\alpha}y^{1-\alpha}z^{1-\alpha}}{\Gamma^{3}(2-\alpha)}\vec{e_{z}} \tag{67} $$

and its norm is

$$ \| {}^{C}D_{x}^{\alpha}\varphi \times {}^{C}D_{y}^{\alpha}\varphi \times {}^{C}D_{z}^{\alpha}\varphi \| = \frac{x^{1-\alpha}y^{1-\alpha}z^{1-\alpha}}{\Gamma^{3}(2-\alpha)} \tag{68} $$

which gives the fractional normal vector of φ by

$$ n_{\alpha} = (0,0,0,1). \tag{69} $$

The fractional components of the first fundamental form are:

$$ \begin{aligned} g_{11}^{(\alpha)} &= \langle {}^{C}D_{x}^{\alpha}\varphi, {}^{C}D_{x}^{\alpha}\varphi \rangle \\ &= \frac{x^{2-2\alpha}}{\Gamma(2-\alpha)}; \end{aligned} \tag{70} $$

$$ \begin{aligned} g_{22}^{(\alpha)} &= \langle {}^{C}D_{x}^{\alpha}\varphi, {}^{C}D_{x}^{\alpha}\varphi \rangle \\ &= \frac{y^{2-2\alpha}}{\Gamma(2-\alpha)}; \end{aligned} \tag{71} $$

$$ \begin{aligned} g_{33}^{(\alpha)} &= \langle {}^{C}D_{x}^{\alpha}\varphi, {}^{C}D_{x}^{\alpha}\varphi \rangle \\ &= \frac{z^{2-2\alpha}}{\Gamma(2-\alpha)} \end{aligned} \tag{72} $$

$$ g_{12}^{(\alpha)} = g_{21}^{(\alpha)} = 0\,; g_{13}^{(\alpha)} = g_{31}^{(\alpha)} = 0\, et\, g_{23}^{(\alpha)} = g_{32}^{(\alpha)} = 0. \tag{73} $$

the coefficients of the second fundamental form via fractional way, are

$$ \begin{aligned} h_{11}^{(\alpha)} &= \left\langle \frac{\partial}{\partial x}({}^{C}D_{x}^{\alpha}\varphi, n_{\alpha}) \right\rangle \\ &= 0; \end{aligned} \tag{74} $$

$$h_{22}^{(\alpha)} = \left\langle \frac{\partial}{\partial y}({}^{C}D_{y}^{\alpha}\varphi, n_{\alpha}) \right\rangle$$
$$= 0; \tag{75}$$

$$h_{33}^{(\alpha)} = \left\langle \frac{\partial}{\partial z}({}^{C}D_{z}^{\alpha}\varphi, n_{\alpha}) \right\rangle$$
$$= 0; \tag{76}$$

$$h_{12}^{(\alpha)} = \left\langle \frac{\partial}{\partial x}({}^{C}D_{y}^{\alpha}\varphi, n_{\alpha}) \right\rangle$$
$$= 0; \tag{77}$$

$$h_{21}^{(\alpha)} = \left\langle \frac{\partial}{\partial y}({}^{C}D_{x}^{\alpha}\varphi, n_{\alpha}) \right\rangle$$
$$= 0; \tag{78}$$

$$h_{13}^{(\alpha)} = \left\langle \frac{\partial}{\partial x}({}^{C}D_{z}^{\alpha}\varphi, n_{\alpha}) \right\rangle$$
$$= 0; \tag{79}$$

$$h_{31}^{(\alpha)} = \left\langle \frac{\partial}{\partial z}({}^{C}D_{x}^{\alpha}\varphi, n_{\alpha}) \right\rangle$$
$$= 0; \tag{80}$$

$$h_{23}^{(\alpha)} = \left\langle \frac{\partial}{\partial y}({}^{C}D_{z}^{\alpha}\varphi, n_{\alpha}) \right\rangle$$
$$= 0 \tag{81}$$

and

$$h_{32}^{(\alpha)} = \left\langle \frac{\partial}{\partial z}({}^{C}D_{y}^{\alpha}\varphi, n_{\alpha}) \right\rangle$$
$$= 0. \tag{82}$$

Then $\det[h_{ij}^{(\alpha)}] = 0$ and so $K^{(\alpha)} = 0$ and $H^{(\alpha)} = 0.$

Corollary 3.7: *If the function f is a non zero constant then the hypersurface defined by the graph of f is flat and minimal.*

Conflict of interest

The authors declare that they have no conflict of interest.

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