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Complex Sadik integral transform of Bessel's function of first kind

Sonia Sharma †
Harish Nagar *
Department of Mathematics
Chandigarh University
Gharuan (Mohali) 140413
Punjab
India

Abstract

The Laplace equation, the heat equation, the wave equation, and the Helmholtz equation in cylindrical or spherical coordinates are just a few of the numerous equations that may be solved using Bessel's functions in modern engineering and natural science applications. In this paper, we present some important applications for assessing the integration as well as the intricate Sadik integral transform of Bessel's functions. The efficiency of the complex Sadik integral transform and its ability to provide an accurate solution with the fewest number of calculations are demonstrated using real-world numerical examples.

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1. Introduction

For engineers and physicists, Bessel's functions are crucial. Bessel's function appears to be the most commonly used function in physics and engineering science, after the trigonometric, hyperbolic, and exponential functions. The Helmholtz equation and the Schrodinger equation in cylindrical or spherical coordinated form are only two of the many problems that can be solved using Bessel's functions in the modern world of engineering and natural science.[1-2]

The Solution to the Bessel differential equation, which is named after the German astronomer Friedrich Bessel, are known as Bessel functions [3-4].

† *E-mail:* **skaushiksonia96@gmail.com**
* *E-mail:* **drharishngr@gmail.com (Corresponding Author)**

$$t^2\frac{d^2y}{d^2t}+t\frac{dy}{dt}+(t^2-n^2)y=0\,,\quad n>0$$

i.e. the second order differential equation with two linearly independent solutions. The general solution of Bessel differential equation is provided as:

$$Y=\mathrm{A}J_n(t)+\mathrm{B}\mathcal{Y}_n(t)$$

where the two linearly independent solutions, known as the Bessel functions of the first and second kind, respectively, of order n are denoted by $J_n(t)$ and $\mathcal{Y}_n(t)$. At t=0, $\mathcal{Y}_n(t)$is not finite, while $J_n(t)$ is. As usual, the known boundary conditions are used to determine the arbitrary constants A and B in the general solution.

Definition 1.1: Bessel function of first kind of order n

The Bessel's function of order $n \in N$, is represented as: [3-5]

$$J_n(t)=\frac{t^n}{2^n n!}\left[1-\frac{t^2}{2(2n+2)}+\frac{t^4}{2.4.(2n+2)(2n+4)}-\frac{t^6}{2.4.6.(2n+2)(2n+4)(2n+6)}+\cdots\right] \tag{1}$$

The infinite power series gives functions of Bessel function of zero order, one order, two order…. etc. if we take n=0, 1, 2, ..

Bessel function of zero order: If n = 0, the infinite power series gives Bessel function of zero order and it is represented by $J_0(t)$.

$$(t)=1-\frac{t^2}{4}+\frac{t^4}{4^3}-\frac{t^6}{4^3 6^2}+\cdots \tag{2}$$

Bessel function of one order: The infinite series for Bessel function of order one is obtained when n=1, and represented by $J_1(t)$.

$$(t)=\frac{t}{2}-\frac{t^3}{4^2}+\frac{t^5}{4^3.6}-\frac{t^7}{4.4^3.6^2.8}+\cdots \tag{3}$$

Bessel function of two order: The Bessel function of two order denoted by $J_2(t)$ is obtained when n=2, is represented by

$$(t)=\frac{t^2}{2.4}-\frac{t^4}{4^2.6}+\frac{t^6}{4^3.6.8}-\frac{t^7}{4^3.6^2.8.10}+\cdots \tag{4}$$

Relation between $\boldsymbol{J_0(t)}$ and $\boldsymbol{J_1(t)}$

$$\frac{d}{dx}J_0(t)=-J_1(t), \tag{5}$$

Relation between $\boldsymbol{J_0(t)}$ and $\boldsymbol{J_2(t)}$

$$J_2(t)=J_0(t)+2J_0{}''(t), \tag{6}$$

Definition 1.2: From the very first, a novel Sadik integral transformation defined as follows was introduced by researcher Sheikh in (2018) [6].

$$S_\alpha[f(t)]=F(v^\alpha,\beta)=\frac{1}{v^\beta}\int_0^\infty f(t)e^{-v^\alpha t}\,dt\,,t\geq 0 \tag{7}$$

Where $v\in\mathbb{C},\alpha\in\mathbb{R}^*\ and\ \beta\in\mathbb{R}$.

Definition 1.3: The "Complex sadik" integral transform was firstly presented by Turq and Kuffi in 2022 [7]. It has applications in fields including engineering, applied physics, and sign processing and is employed to resolve ordinary differential equations.

The operator $S_\alpha^C\{.\}$ is used to denote the complex Sadik integral transform and represented as:

$$S_\alpha^C[f(t)] = F^C(v^\alpha, \beta) = \frac{1}{v^\beta}\int_0^\infty f(t)e^{-iv^\alpha t}\,dt\,, t \geq 0 \tag{8}$$

Where $v \in \mathbb{C}, \alpha \in \mathbb{R}^*$ and $\beta \in \mathbb{R}$.

A. "The Complex Sadik Integral Transform" of Some Elementary Functions

The complex Sadik integral transform of some core functions given as [7]:

- $S_\alpha^C[C] = \frac{-ic}{v^{\alpha+\beta}}, v > 0$
- $S_\alpha^C[t^n] = (-i)^{n+1}\frac{n!}{v^{n\alpha+(\alpha+\beta)}}. v > 0$
- $S_\alpha^C[e^{at}] = \frac{-1}{v^\beta}\left[\frac{a}{(v^{2\alpha}+a^2)} + i\frac{v^\alpha}{(v^{2\alpha}+a^2)}\right], v > a$
- $S_\alpha^C[\sin(at)] = \frac{-a}{v^\beta(v^{2\alpha}-a^2)}, v > |a|$
- $S_\alpha^C[\cos(at)] = \frac{-iv^\alpha}{v^\beta(v^{2\alpha}-a^2)}, v > |a|$
- $S_\alpha^C[\sinh(at)] = \frac{-a}{v^\beta(v^{2\alpha}+a^2)}, v > 0$
- $S_\alpha^C[\cos h(at)] = \frac{-iv^\alpha}{v^\beta(v^{2\alpha}+a^2)}, v > 0$

B. **Derivative Property**:

The derivatives of the function $f(t)$ have a complex Sadik transform, is as follows [7]:

$$S_\alpha^C[f'(t)] = iv^\alpha F^C(v^\alpha, \beta) - \frac{f(0)}{v^\beta}.$$

$$S_\alpha^C[f''(t)] = (iv^\alpha)^2 F^C(v^\alpha, \beta) - \frac{f'(0)}{v^\beta} - \frac{is^\alpha f(0)}{v^\beta}.$$

$$S_\alpha^C[f'''(t)] = (iv^\alpha)^2 F^C(v^\alpha, \beta) - \frac{f''(0)}{v^\beta} - \frac{iv^\alpha f'(0)}{v^\beta} - \frac{(iv^\alpha)^2 f(0)}{v^\beta}.$$

In general,

$$S_\alpha^C\left[f^{(n)}(t)\right] = (iv^\alpha)^n F^C(v^\alpha, \beta) - \frac{f^{(n-1)}(0)}{v^\beta} - \frac{iv^\alpha f^{(n-2)}(0)}{v^\beta} - \frac{(iv^\alpha)^2 f^{(n-3)}(0)}{v^\beta} - \cdots - \frac{(iv^\alpha)^{n-2} f'(0)}{v^\beta} - \frac{(iv^\alpha)^{n-1} f(0)}{v^\beta}.$$

C. "The complex Sadik integral transform" convolution property [7]:

If $S_\alpha^C[f(t)] = F_1^C(v^\alpha, \beta)$ and $S_\alpha^C[h(t)] = F_2^C(v^\alpha, \beta)$ then,
$S_\alpha^C[f(t) * h(t)] = v^\beta F_1^C(v^\alpha, \beta).F_2^C(v^\alpha, \beta)$ [2].

D. "The complex Sadik integral transform" changing of scale property [7]:

Since,

$$S_\alpha^C[f(t)] = F^C(v^\alpha, \beta) = \frac{1}{v^\beta}\int_0^\infty f(t)e^{-iv^\alpha t}\,dt\,, t \geq 0$$

Thus,

$$S_\alpha^C[f(at)] = \frac{1}{a}F^C\left(\frac{v^\alpha}{a}, \beta\right)\ .$$

E. "The Inverse Complex Sadik Integral Transform" of Some Basic Functions:

If $S_\alpha^C\{f(t)\} = F^C(v)$ is the sadik integral transform, then $g(t) = (S_\alpha^C)^{-1}[F^C(v)\ ]$ is inverse of the complex sadik transform. The inverse of several simple functions' complex sadik integral transform is provided as:[7]

- $(S_\alpha^C)^{-1}\left[(-i)^{n+1}\frac{n!}{v^{n\alpha+(\alpha+\beta)}}\right] = t^n$
- $(S_\alpha^C)^{-1}\left[\frac{-1}{v^\beta}\left[\frac{a}{(v^{2\alpha}+a^2)} + i\frac{v^\alpha}{(v^{2\alpha}+a^2)}\right]\right] = e^{at}$
- $(S_\alpha^C)^{-1}\left[\frac{-a}{v^\beta(v^{2\alpha}-a^2)}\right] = \sin(at)$
- $(S_\alpha^C)^{-1}\left[\frac{-iv^\alpha}{v^\beta(v^{2\alpha}-a^2)}\right] = \cos(at)$
- $(S_\alpha^C)^{-1}\left[\frac{-a}{v^\beta(v^{2\alpha}+a^2)}\right] = sinh(at)$
- $(S_\alpha^C)^{-1}\left[\frac{-iv^\alpha}{v^\beta(v^{2\alpha}+a^2)}\right] = \cosh(at).$

Many researchers have tried to solve Bessel's function of the first sort by using different types of integral transformations. [8–11] We have tried to ascertain the solution of Bessel's function of the first sort in light of their discoveries. The universal solution of Bessel's function of first kind using the complex Sadik integral transform is shown in section 2. To demonstrate the efficacy and efficiency of the complex Sadik integral transform, section 3 presents numerical issues in context.

2. Results and Proofs

In this section, the complex sadik integral transform of Bessel's function is presented and determined the value of definite integrals which contains Bessel's functions in the integrand.

2.1 When $f(t) = J_0(t)$

Apply "The Complex Sadik Transform" on both sides of equation (2), we get

$$S_\alpha^C\{J_0(t)\} = S_\alpha^C\{1\} - \frac{1}{2^2}S_\alpha^C\{t^2\} + \frac{1}{2^2 4^2}S_\alpha^C\{t^4\} - \frac{1}{2^2 4^2 6^2}S_\alpha^C\{t^6\} + \cdots \qquad (9)$$

$$= \frac{-i}{v^{\alpha+\beta}} - \frac{1}{2^2}\left((-i)^3 \frac{2!}{v^{2\alpha+(\alpha+\beta)}}\right) + \frac{1}{2^2 4^2}\left((-i)^5 \frac{4!}{v^{4\alpha+(\alpha+\beta)}}\right) - \frac{1}{2^2 4^2 6^2}\left((-i)^7 \frac{6!}{v^{6\alpha+(\alpha+\beta)}}\right) + \cdots$$

$$S_\alpha^C\{J_0(t)\} = \frac{-i}{v^\beta\sqrt{(v^\alpha)^2-1}} \tag{10}$$

2.2 *When* $f(t) = J_1(t)$

The complex Sadik transform" on both sides of equation (3) and utilizing the derivative property, we get

$$S_\alpha^C\{J_1(t)\} = -\left[iv^\alpha S_\alpha^C\{J_0(t)\} - \frac{J_0(0)}{v^\beta}\right] \tag{11}$$

By utilizing equation (10) and (11),

$$S_\alpha^C\{J_1(t)\} = \frac{1}{v^\beta} - \frac{v^\alpha}{v^\beta\sqrt{(v^\alpha)^2-1}} \tag{12}$$

2.3 *When* $f(t) = J_2(t)$:

Apply Complex Sadik transform on equation (3), we get

$$S_\alpha^C\{J_2(t)\} = S_\alpha^C\{J_0(t)\} + 2S_\alpha^C\{J_0''(t)\} \tag{13}$$

Using the equation (10) and (13), we can apply property Complex Sadik transform of derivatives for $f(T)$

$$S_\alpha^C\{J_2(t)\} = \frac{-i}{v^\beta\sqrt{(v^\alpha)^2-1}} + 2\left[(iv^\alpha)^2 S_\alpha^C\{J_0(t)\} - \frac{J_0'(t)}{v^\beta} - \frac{iv^\alpha J_0(t)}{v^\beta}\right] \tag{14}$$

By using the property of Complex Sadik transform of derivatives and equation (10) and (14), we have

$$S_\alpha^C\{J_2(t)\} = \frac{-i+2i(v^\alpha)^2-2iv^\alpha\sqrt{(v^\alpha)^2-1}}{v^\beta\sqrt{(v^\alpha)^2-1}} \tag{15}$$

2.4 If $f(t) = J_0(at)$:

Since, the complex Sadik integral transform of $J_0(t)$ is given by

$$S_\alpha^C\{J_0(t)\} = \frac{-i}{v^\beta\sqrt{(v^\alpha)^2-1}}$$

By applying the Complex Sadik transform's change of scale property, we obtain

$$S_\alpha^C\{J_0(at)\} = \frac{-i}{v^\beta\sqrt{(v^\alpha)^2-a^2}} \tag{16}$$

2.5 When $f(t) = J_1(at)$:

The Complex Sadik transform of $J_1(ax)$ is given by

$$S_\alpha^C\{J_1(t)\} = \frac{1}{v^\beta} - \frac{v^\alpha}{v^\beta\sqrt{(v^\alpha)^2-1}}$$

By applying change of scale property for Complex Sadik transform

$$S_\alpha^C\{J_1(at)\} = \frac{1}{a}S_\alpha^C\left\{J_1(t); \frac{v^\alpha}{a}\right\}$$

$$S_\alpha^C\{J_1(at)\} = \frac{1}{v^\beta a}\left[1 - \frac{v^\alpha}{\sqrt{(v^\alpha)^2 - a^2}}\right] \tag{17}$$

2.6 If $f(t) = J_2(at)$:

From equation (15), the Complex Sadik transform of $J_2(x)$ is given as

$$S_\alpha^C\{J_2(t)\} = \frac{-i + 2i(v^\alpha)^2 - 2iv^\alpha\sqrt{(v^\alpha)^2 - 1}}{v^\beta\sqrt{(v^\alpha)^2 - 1}}$$

Thus, by applying change of scale property of Complex Sadik transform, we get

$$S_\alpha^C\{J_2(at)\} = \frac{1}{a} S_\alpha^C\left\{J_2(t); \frac{v^\alpha}{a}\right\},$$

$$S_\alpha^C\{J_2(at)\} = \frac{1}{v^\beta a^2}\left[\frac{-ia^2 + 2i(v^\alpha)^2 - 2iv^\alpha\sqrt{(v^\alpha)^2 - a^2}}{v^\beta\sqrt{(v^\alpha)^2 - a^2}}\right] \tag{18}$$

3. Numerical Applications:

This section offers a couple of examples of applications in order to show how the SEJI integral transform impacts evaluating Bessel functions.

Examples 2.1: Evaluate the integral

$$I(t) = \int_{u=0}^{t} J_0(u) J_0(t-u)\,du \tag{19}$$

Solution: Applying the complex Sadik Integral transform on both side of equation (19) and using convolution theorem, we get:

$$= \frac{-1}{v^\beta((v^\alpha)^2 - 1)}$$

Taking inverse Complex Sadik transform of equation, we get

$$I(t) = \sin(t).$$

Example 2.2: Evaluate the integral:

$$I(t) = \int_{u=0}^{t} J_0(u) J_1(t-u)\,du \tag{20}$$

Solution: Applying complex Sadik transform on both sides of equation (20) and using convolution theorem, we get:

$$= \frac{-i}{v^\beta\sqrt{(v^\alpha)^2 - 1}} + \frac{iv^\alpha}{v^\beta((v^\alpha)^2 - 1)}$$

Take inverse of Complex Sadik transform, we get

$$I(t) = J_0(t) - cost.$$

Conclusion

In this paper, authors successfully determined the complex Sadik integral transform of Bessel's functions with application for evaluating definite integrals. Additionally, the examples provided demonstrate the benefit of using the complex Sadik integral transform of Bessel's functions to determine the integral containing Bessel's function. This ability may be utilized as a benchmark for future applications of complex Sadik integral transform to the general problem of solving Bessel’s function of first kind in Engineering Mathematics.

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