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Common fixed point theorems in the function weighted *b*-metric spaces and their application

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Abstract

The aim of this paper is to propose and prove some properties and theorems of common fixed point in the function weighted b- metric spaces and its application, its space is a generalization of the function weighted metric spaces.

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1. Introduction

Bakhtin in 1989 [1] firstly introduced b-metric space and Czerwik in 1993 utilized to the fixed point theorems of Banach contractive mappings [2]. While, the metrizable property of b-metric space was shown by Dung in 2019 [3]. In 2020, the rectangular b-metric space was introduced by George et al. [4]. Then in 2020, Branciari introduced the rectangular b-metric space and utilized on the existence of points in the Banach contractive mappings and the Kannan contractive type [5]. Khamsi in 2010 developed the rectangular metric space

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became *b(n)* -metric space [6]. Kamran, Samreen, and Ain introduced an extended b-metric space for the existence of a fixed point in the Banach contracted function [7]. Asim, Imdad, and Radenovic generalized the extended b-metric space to the extended rectangular b-metric space for fixed point results and an application [8]. Cobzas and Czerwik in 2020 generalized b-metric spaces become s-relaxed b-metric space [9] and Stella Eke and Bishop in [10] proposed and prove that there is a unique fixed point of C_E generalized contraction mapping in metric space [10]. In 2018, Jleli and Samet introduced a function weighted metric ($\mathfrak{F}$ -metric) space from the generalized metric space [11]. Nurwahyu, Aris and Firman in 2023 introduced $\mathfrak{F}$ b-metric space (function weighted b- metric space) [13]. In this study, we showed several properties of $\mathfrak{F}$ b-metric space and the fixed point theorems for some generalized contraction mappings in the $\mathfrak{F}$ b-metric space and its application.

2. Preliminaries

Definition 1: [11]. Let $I = (0, +\infty)$ and $\varphi: I \to \mathbb{R}$ be an increasing function. If

$$\lim_{t\to 0} \varphi(t) = -\infty,$$

φ is called logarithmic-like.

Let $\mathfrak{F} = \{\, \varphi: I \to \mathbb{R} \mid \varphi \text{ } is \text{ } an \text{ increasing and logarithmic} - \text{like}\,\}$.

If $\varphi \in \mathfrak{F}$, it is clear that for any $s > 0$ and $\beta \geq 0$, there exists $\gamma > 0$, such that for every $0 < t < \gamma$, we have $\varphi(t) < \varphi(s) - \beta$.

Example 1: [11] $\varphi(s) = ln\, s$, $\varphi(s) = s - \frac{1}{s}$, $\varphi(s) = s - e^{\frac{1}{s}}$, $s \in (0, +\infty)$ are logarithmic-like.

Definition 2: [11] Let $Z \neq \emptyset$, $\varphi \in \mathfrak{F}$, $0 \leq \beta < +\infty$. A function $\sigma: Z \times Z \to \mathbb{R}^+$ is said a function weighted metric ($\mathfrak{F}$ −metric), if for all $v, w \in Z$, σ has the following conditions:

C1. $\sigma(v, w) = 0 \Leftrightarrow v = w$,

C2. $\sigma(v, w) = \sigma(w, u)$,

C3. If $\sigma(v, w) > 0$ then $\varphi\big(\sigma(x, y)\big) \leq \varphi(\sum_{j=1}^{n-1} \sigma\big(c_j, c_{j+1}\big) + \beta$,

for every $\{c_1 = v,\ c_2,\ c_3, \dots, c_n = w\,\} \subset Z, n \geq 2$.

The pair (Z, σ) is said a *function weighted metric space* ($\mathfrak{F}-$ *metric space*) .

Definition 3: [13] Let $Z \neq \emptyset$, $\varphi \in \mathfrak{F}$, and $0 \leq \beta < +\infty$, $s \geq 1$. A function $\tau: Z \times Z \to \mathbb{R}^+$ is said a *function weighted b-metric,* if for all $v, w \in Z,$ and τ has the following conditions:

D1. $\tau(v, w) = 0 \Leftrightarrow v = w$,

D2. $\tau(v, w) = \tau(w, v)$,

D3. If $\tau(v, w) > 0$ then $\varphi\big(\tau(v, w)\big) \leq \varphi(\sum_{j=1}^{n-1} s^j \tau\big(c_j, c_{j+1}\big) + \beta$,

for every $\{c_1 = v,\ c_2,\ c_3, \dots, c_n = w\,\} \subset Z, n \geq 2$.

The pair (Z,τ) is called a *function weighted b-metric space* ($\mathfrak{F}\, b$ −metric space) with $s \geq 1$.

Remarks: It is clear that function weighted metric is the function weighted *b*-metric (if $s = 1$), but not vice versa. It is show in Example 2 as follows:

Example 2: Let $\tau(r,t) = |r-t|^a$ with $a \geq 2$ for every $r,t \in \mathbb{R}$, then τ is $\mathfrak{F}\, b$ −metric space but τ is not the $\mathfrak{F}$ −metric space.

Suppose τ is a $\mathfrak{F}$ −metric space, it means there exists a function $\varphi \in \mathfrak{F}$ that satisfies C3.

Let $s_j = \frac{2j}{k}$, for $k \in \mathbb{N}, j = 0,1,2,\ldots.,k$. From C3, we have

$$\varphi\big(\tau(0,2)\big) \leq \varphi(\textstyle\sum_{j=1}^{k}\left|s_j - s_{j-1}\right|^a) + \beta = \sum_{j=1}^{k}\left|s_j - s_{j-1}\right|^a - \frac{1}{\sum_{j=1}^{k}\left|s_j - s_{j-1}\right|^a}$$

$$\varphi(\tau(0,2)) \leq \varphi(\textstyle\sum_{j=1}^{k}\left|s_j - s_{j-1}\right|^a) + \beta = \varphi(\sum_{j=1}^{k}\left|\frac{2j}{k} - \frac{2(j-1)}{k}\right|^a) + \beta$$

$$= \varphi(\tfrac{1}{k^a}\textstyle\sum_{j=1}^{k}|2j - 2(j-1)|^a) + \beta \leq \varphi(\frac{2^a}{k^{a-1}}) + \beta. \tag{1}$$

So, from (1) we get

$$\varphi(2^a) = \varphi\big(\tau(0,2)\big) \leq \varphi\left(\frac{2^a}{k^{a-1}}\right) + \beta. \tag{2}$$

Since φ has a *logarithm-like property*, then we have $\frac{2^a}{k^{a-1}} \to 0$, as $k \to +\infty$, then from (2) we can get $\varphi(\frac{2^a}{k^{a-1}}) + \beta \to -\infty$, as $k \to +\infty$. This is a contradiction, because $\varphi\left(\frac{2^a}{k^{a-1}}\right) \geq \varphi(2^a) - \beta$. Thus τ is not a $\mathfrak{F}$ −metric space.

We will show that τ is $\mathfrak{F}\, b$ −metric space.

It is obvious that τ satisfies D1 and D2.

To prove condition D3, we use a property for $a \geq 1$, it holds

$$|r-t|^a \leq 2^{a-1}(|r-u|^a + |u-t|^a).$$

Let $\{c_1 = r,\ c_2,\ c_3, \ldots, c_n = t\} \subset \mathbb{R}, v \neq w, n \geq 2$.

Then we have

$$\begin{aligned}\tau(r,t) = |r-t|^a &= |c_1 - c_2 + c_2 - c_3 + c_3 - c_4 + c_4 - \cdots c_{n-1} - c_n|^a \\ &\leq (2^{a-1}|c_1 - c_2|^a + (2^{a-1})^2|\,c_2 - c_3|^a + (2^{a-1})^3|\,c_3 - c_4|^a + \cdots \\ &\quad + (2^{a-1})^{n-1}|\,c_{n-1} - c_n|^{a-1}) \\ &= \textstyle\sum_{j=1}^{n-1}(2^{a-1})^j\left|c_j - c_{j+1}\right|^a = \sum_{j=1}^{n-1}(2^{a-1})^j\,\tau\big(c_j,\ c_{j+1}\big).\end{aligned} \tag{3}$$

From (3) we get

$$\tau(r,t) \leq 1\textstyle\sum_{j=1}^{n-1}(2^{a-1})^j\,\tau\big(c_j,\ c_{j+1}\big).$$

Suppose $\tau(r,t) > 0$, so we have

$$\ln\tau(r,t) \leq \ln\Big(\textstyle\sum_{j=1}^{n-1}(2^{a-1})^j\,\tau\big(c_j,\ c_{j+1}\big)\Big) + \ln 1$$

Hence, τ is a $\mathfrak{F}\, b$ −metric space with $\varphi(v) = \ln v$, $s = 2^{a-1}$ and $\beta = \ln 1 = 0$.

Definition 4: [12] Let X be non-empty set and $T_1, T_2 \colon X \to X$ are functions that satisfy: If $T_1 x = T_2 x = y$ for an $x \in X$, then x is called a *coincidence point* of T_1 , T_2, and y is called a *point of coincidence* of T_1 and T_2.

Definition 5: [12] Let X be non-empty set and $T_1, T_2 \colon X \to X$ are functions that satisfy: for every $x \in X$, if $T_1 x = T_2 x$, it holds $T_2 T_1 x = T_1 T_2 x$, then $\{T_1, T_2\}$ is called a weakly compatible.

Definition 6: Let (X, ρ) be a function weighted *b*-metric space and (a_n) be a sequence in X.

i. (a_n) is called a convergent sequence ($\mathfrak{F}\, b -$ convergent) to $a \in X$, if $\rho(a_n, a) \to 0$, as $n \to +\infty$.
ii. (a_n) is called a Cauchy sequence ($\mathfrak{F}\, b$ –Cauchy) in X, if $\rho(a_n, a_m) \to 0$, as $n, m \to +\infty$.

If every *Cauchy* sequence in $\mathfrak{F}\, b$ –metric space is convergent in X, then (X, ρ) is called a *complete*.

Definition 7: Let (X, ρ) be a function weighted b-metric and $p \in X$, then $N_r(p) = \{x \in X \mid \rho(x, p) < r\}$ is called an open neighborhood of p ($\mathfrak{F}\, b -$ open neighborhood of p) in X.

$G \subset X$ is called an *open set* ($\mathfrak{F}\, b - open$ set) in X, if for any $y \in G$, there exists $N_r(y)$, such that $N_r(y) \subset G$. Furthermore, if K is a $\mathfrak{F}\, b -$ open in X, then K^c is called $\mathfrak{F}\, b$ - *closed* in X.

Preposition 1: [13] Let $(\varphi, \beta) \in \mathfrak{F} \times [0, +\infty)$ and (X, ρ) be a complete $\mathfrak{F}\, b - metric\ space$.

If (a_n) is a convergent in X, then limit of (a_n) is unique.

Preposition 2: [13] Let $(\varphi, \beta) \in \mathfrak{F} \times [0, +\infty)$ and (X, ρ) be a complete $\mathfrak{F}\, b - metric\ space$ with $s \geq 1$. If (d_k) be a sequence in X which satisfies:

$$\rho(d_k, d_{k+1}) \leq \frac{\delta}{s} \rho(d_{k-1}, d_k), \tag{4}$$

for $0 < \delta < 1$, then (d_k) is a Cauchy sequence in X.

3. Main Results

In this section, some prepositions are presented to support the proof of the given fixed point theorems.

Preposition 3: Let $(\varphi, \beta) \in \mathfrak{F} \times [0, +\infty)$ and (X, ρ) be a $\mathfrak{F}\, b - metric\ space$. with $s \geq 1$. Let $V_r(p) = \{x \in X \mid \rho(x, p) \leq r\}, r > 0$. If for any sequence

$(a_n) \subset X$ holds a property: For any $\tau > 0$, there exists a positive integer N such that $\rho(a_N, p) < \tau$, then $V_r(p)$ is closed in X.

Proof: Let (a_n) be a sequence in $V_r(p)$ that converges to $a \in X$. We will show that $a \in V_r(p)$.

Since $(a_n) \subset V_r(p)$, then we have $\rho(a_n, p) \leq r$. Since $r > 0$ and φ is logarithmic-like, then there is a $\delta > 0$ such that for any $0 < t < \delta$, $\varphi(t) < \varphi(r) - \beta$. Likewise, it is true that $\rho\left(a_{N_\delta}, p\right) < \frac{\delta}{3s}$ for a non-negative integer N_δ.

Since $\lim_{n\to+\infty} \rho(a_n, a) = 0$, then we have $\rho(a_n, a) < \frac{\delta}{3s} < \delta$ for any $n \geq N_\delta$, for a non-negative integer N_δ.

By using D3, we have

$$\varphi(\rho(a,p)) \leq \varphi\left(s\left(\rho\left(a, a_{N_\delta}\right) + \rho\left(a_{N_\delta}, p\right)\right)\right) + \beta < \varphi\left(s\left(\frac{\delta}{3s} + \frac{\delta}{3s}\right)\right) + \beta$$
$$= \varphi\left(\frac{\delta}{3} + \frac{\delta}{3}\right) + \beta = \varphi\left(\frac{2\delta}{3}\right) + \beta < \varphi(r) - \beta + \beta = \varphi(r).$$

Since φ increasing, then we get $\rho(a, p) \leq r$, it means that $a \in V_r(p)$. It concludes that $V_r(p)$ is closed in X.

Preposition 4: Suppose $(\varphi, \beta) \in \mathfrak{F} \times [0, +\infty)$ and (X, ρ) be a $\mathfrak{F}\, b-metric\ space$ with $s \geq 1$. If $lim_{t\to+\infty} \varphi(t) = +\infty$, then for any $t > 0$ there exists $M > t$ such that $\varphi(t) < \varphi(M) - \beta$.

Proof: Since φ is an increasing and unbounded, we can choose $M > t$ such that $\varphi(M) - \varphi(t) > \beta$.

Theorem 1: *Let* $(\varphi, \beta) \in \mathfrak{F} \times [0, +\infty)$ *and* (X, ρ) *be a complete* $\mathfrak{F}\, b-metric\ space$ *with* $s \geq 1$. *If* $lim_{t\to+\infty} \varphi(t) = +\infty$ *and* $(a_n) \subset X$ *is a Cauchy sequence, then* (a_n) *is bounded in* X.

Proof: Suppose (a_n) is a Cauchy sequence in X, then there is a positive integer N, such that $\rho(a_n, a_N) < \frac{1}{s}$ for any $n \geq N$. Suppose $c \in X$ such that $\rho(a_n, c) > 0$ and $K = maks\ \left\{s\rho(a_1, a_N), s\rho(a_2, a_N), \ldots, s\rho(a_{N-1}, a_N), \frac{1}{s}\right\}$.

For $\rho(a_n, c) > 0$, then using D3, we get

$$\varphi(\rho(a_n, c)) \leq \varphi\left(s\left(\rho(a_n, a_N) + \rho(a_N, c)\right)\right) + \beta$$
$$\leq \varphi\left(s\left(K + \rho(a_N, c)\right)\right) + \beta. \tag{5}$$

Since $\lim_{t\to+\infty} \varphi(t) = +\infty$ and φ is an increasing, using Preposition 4, then there is $M > K + s\rho(a_N, c) > 0$ such that $\varphi\left(\left(K + s\rho(a_N, c)\right)\right) < \varphi(M) - \beta$.

So, from (5) we get

$$\varphi(\rho(a_n, c)) \leq \varphi\left(\left(K + s\rho(a_N, c)\right)\right) + \beta < \varphi(M).$$

Since φ is an increasing, we get $\rho(a_n, c) < M$, for every $n = 1,2,3, \ldots$.
Thus, (a_n) is bounded in X.

Theorem 2: *Let* $(\varphi, \beta) \in \mathfrak{F} \times [0, +\infty)$ *and let* (X, ρ) *be a complete* $\mathfrak{F}\, b-$ *metric space with* $s \geq 1$. *Let* $T: X \to X$ *be a function that satisfies as follows*

$$\rho\big(T(x), T(y)\big) \leq \frac{1}{s}\left(a\rho(x, y) + c\rho\big(y, T(y)\big) + \frac{d\rho\big(x, T(y)\big)\rho(T(x), y)}{1 + \rho(T(x), y)}\right), \qquad (6)$$

where $0 < a + c + d < 1$ *, and for all* $x, y \in X$.

Then T has a unique fixed point in X.

Proof: Let $b_0 \in X$, define a sequence $(b_n) \subset X$ as follows:

$$b_n = T(b_{n-1})$$

From (6) we get

$$\begin{aligned} \rho(b_n, b_{n+1}) &= \rho(T(b_{n-1}), T(b_n)) \\ &\leq \frac{1}{s}\left(a\rho(b_{n-1}, b_n) + c\rho(b_n, b_{n+1}) + \frac{d\rho(b_{n-1}, b_{n+1})\rho(b_n, b_n)}{1 + \rho(b_n, b_n)}\right) \\ &= \frac{1}{s}\big(a\rho(b_{n-1}, b_n) + c\rho(b_n, b_{n+1})\big) \end{aligned}$$

So, we have

$$\rho(b_n, b_{n+1}) \leq \gamma\, \rho(b_{n-1}, b_n), \qquad (7)$$

where $\gamma = \frac{a}{s(1-c)}$.

Since $0 < a + c + d < 1$, and $s \geq 1$, we have $0 < \gamma < 1$.

Therefore, from (7) we get

$$\lim_{n \to +\infty} \rho(b_n, b_{n+1}) = 0. \qquad (8)$$

From (8) and using Preposition 2, we get (b_n) is a Cauchy sequence in X.

Since (X, ρ) is *a* complete $\mathfrak{F}\, b-$metric space , then there exists $b^* \in X$ such that

$$\lim_{n \to +\infty} \rho(b_n, b^*) = 0. \qquad (9)$$

Next, we show b^* is the fixed point of T.

Suppose $T(b^*) \neq b^*$, so we have $\rho(T(b^*), b^*) > 0$, then from of D3, it holds that

$$\begin{aligned} \varphi\big(\rho(T(b^*), b^*)\big) &\leq \varphi\Big(s\Big(\rho\big(T(b^*), T(b_n)\big) + \rho(T(b_n), b^*)\Big)\Big) + \beta \\ &= \varphi\Big(s\Big(\rho\big(T(b^*), T(b_n)\big) + \rho(b_{n+1}, b^*)\Big)\Big) + \beta. \end{aligned}$$

From (6), we get

$$\varphi\big(\rho(T(b^*), b^*)\big) \leq \varphi\left(s\left(\frac{1}{s}\left(a\rho(b^*, b_n) + c\rho(b_n, b_{n+1}) + \frac{d\rho(b^*, b_{n+1})\rho(T(b^*), b_n)}{1 + \rho(T(b^*), b_n)}\right) + \rho(b_{n+1}, b^*)\right)\right) + \beta$$

$$\leq \varphi\left(s\left(\frac{1}{s}\big(a\rho(b^*,b_n) + c\rho(b_n,b_{n+1}) + d\rho(b^*,b_{n+1})\big) + \rho(b_{n+1},b^*)\right)\right) + \beta$$

$$= \varphi\big(\rho(b^*,b_n) + c\rho(b_n,b_{n+1}) + d\rho(b^*,b_{n+1}) + s\rho(b_{n+1},b^*)\big) + \beta. \qquad (10)$$

From (10) and applying (8) and (9), we have

$$\lim_{n\to+\infty} \rho(b^*,b_n) + c\rho(b_n,b_{n+1}) + d\rho(b^*,b_{n+1}) + s\rho(b_{n+1},b^*) = 0.$$

By using the logarithmic-like property of φ, we get

$$\lim_{n\to\infty} \varphi\big(\rho(b^*,b_n) + c\rho(b_n,a_{n+1}) + s\rho(b_{n+1},b^*)\big) + \beta = -\infty.$$

Which is a contradiction.

Thus, we have b^* is a fixed point of T.

Next, we show that the fixed point is unique.

Suppose b^*, $c^* \in X$ as the fixed point of T and $b^* \neq c^*$.

From (6) we have

$$\rho(b^*,c^*) = \rho(T(b^*),T(c^*)))$$

$$\leq \frac{1}{s}\left(a\rho(b^*,c^*) + c\rho\big(b^*,T(b^*)\big) + \frac{d\rho\big(b^*,T(c^*)\big)\rho(T(a^*),c^*)}{1+\rho(T(b^*),c^*)}\right)$$

$$\leq \frac{1}{s}\Big(a\rho(b^*,c^*) + c\rho\big(b^*,T(b^*)\big) + d\rho\big(b^*,T(c^*)\big)\Big)$$

$$= \frac{1}{s}\big(a\rho(b^*,c^*) + c\rho(b^*,b^*) + d\rho(b^*,c^*)\big) = \frac{a+d}{s}\rho(b^*,c^*) < \rho(b^*,c^*). \qquad (11)$$

Since $s \geq 1$ and $0 < a + c + d < 1$, we have $0 < \frac{a+d}{s} < 1$. Thus, from (11), we get

$$\rho(a^*,c^*) \leq \frac{a+d}{s}\rho(a^*,c^*) < \rho(a^*,c^*).$$

Which is a contradiction.

The following is the corollary of Theorem 3 regarding to the fixed point in the Banach contraction principle in $\mathfrak{F}$ b-metric space.

Corollary 1: *Suppose* $(\varphi,\beta) \in \mathfrak{F} \times [0,+\infty)$ *and let* (X,ρ) *be a complete* $\mathfrak{F}\, b-metric\ space$ *with* $s \geq 1$. *Let* $T\colon X \to X$ *be a function that satisfies*

$$\rho\big(T(x),T(y)\big) \leq \frac{a}{s}\rho(x,y),$$

where $0 < \frac{a}{s} < 1$, *and for all* $x, y \in X$.

Then T has a unique fixed point in X.

***Proof*:** The proof can be obtained directly from Theorem 1, by taking $c = 0$ and $d = 0$.

Theorem 3: *Suppose "*$(\varphi,\beta) \in \mathfrak{F} \times [0,+\infty)$ *and let* (X,ρ) *be a complete* $\mathfrak{F}\, b-metric\ space$ *with* $s \geq 1$. *Let* $c_0 \in X, r > 0, N_r(c_0) = \{x \in X \mid \rho(x,c_0) \leq r\}$.

Let $T\colon N_r(c_0) \to X$ *be a function that satisfies as follows*

$$\rho\big(T(x),T(y)\big) \le \frac{\delta}{s^2}\rho(x,y),$$

where $0<\delta<1$ *and* $x,y\in N_r(c_0)$.

If $\rho(T(c_0),c_0)<\frac{(1-\delta)r}{s^2}$, *then* T *has a unique fixed point in* $N_r(c_0)$.

Proof: Firstly, we will show that $T\big(N_r(a_0)\big)\subset N_r(c_0)$.

Let $y\in T\big(N_r(a_0)\big)$, it means there is $x_0\in N_r(a_0)$ such that $y=T(x_0)$.

Since $x_0\in N_r(c_0)$, then $\rho(x_0,c_0)\le r$. If $T(x_0)=c_0$, it is clear that $y=T(x_0)=c_0\in N_r(c_0)$.

If $T(x_0)\neq c$, so we have $\rho(T(x_0),c_0)>0$. By using D3, we have

$$\varphi\big(\rho(T(x_0),c_0)\big)\le\varphi\left(s\left(\rho\big(T(x_0),T(c_0)\big)+\rho(T(c_0),c_0)\right)\right)+\beta$$

$$\le\varphi\left(s\left(\frac{\delta r}{s^2}+\frac{(1-\delta)r}{s^2}\right)\right)+\beta=\varphi\left(\frac{r}{s}\right)+\beta$$

Since $\varphi\in\mathfrak{F}$, and $r>0$, then there exists $\gamma>0$ such that $\varphi\left(\frac{r}{s}\right)<\varphi(r)-\beta$, for $0<\frac{r}{s}<\gamma$. Therefore, then we get

$$\varphi\big(\rho(T(x_0),c_0)\big)<\varphi(r)-\beta+\beta=\varphi(r).$$

Since φ is increasing, then we get $\rho(T(x_0),c_0)\le r$. Thus, $y=T(x_0)\in N_r(c_0)$, so we have $T\big(N_r(c_0)\big)\subset N_r(c_0)$.

Using Preposition 3, we have $N_r(c_0)$ is $\mathfrak{F}$ −closed in X.

Since (X,ρ) is $\mathfrak{F}$ –complete, then $N(c_0)$ is complete.

By using Corollary 1, it can be concluded that T has a unique fixed point in $N_r(c_0)$.

Theorem 4: *Suppose* $(\varphi,\beta)\in\mathfrak{F}\times[0,+\infty)$ *and let* (X,ρ) *be a complete* $\mathfrak{F}\,b-metric\ space$ *with* $s\ge1$. *Let* $\pi,\omega: X\to X$ *be the functions, such that* $\pi(X)\subseteq\omega(X)$, *and* $\omega(X)$ *is closed, and for all* $x,y\in X$, *satisfy as follows*

$$\rho\big(\pi(x),\pi(y)\big)\le$$

$$\frac{1}{s}\left(p\rho\big(\omega(x),\omega(y)\big)+q\rho\big(\pi(y),\omega(y)\big)+r\rho\big(\omega(x),\pi(y)\big)\frac{\rho\big(\pi(x),\omega(y)\big)}{1+\rho\big(\pi(x),\omega(y)\big)}\right), \quad (12)$$

for any $p,q,r>0$, $0<p+q+r<1$.

Then π *and* ω *have a unique coincidence point in* X.

Proof: Let $x_0\in X$, since $\pi(X)\subseteq\omega(X)$, we define sequence $\{x_n\}$ and $\{y_n\}$ in X, such that $y_n=\pi(x_n)=\omega(x_{n+1})$, for $n=0,1,2,\cdots$.

If $y_n=y_{n+1}$ for some $n\in\mathbb{N}$, then $\omega(x_{n+1})=y_n=y_{n+1}=\pi(x_{n+1})$. It means π,ω have a coincidence point.

Let $y_n\neq y_{n+1}$ for every $n=0,1,2,\cdots$. From (6), we have

$\rho(y_n, y_{n+1}) = \rho(\pi(x_n), \pi(x_{n+1})) \leq$
$\frac{1}{s}\Big(p\rho\big(\omega(x_n), \omega(x_{n+1})\big) + q\rho\big(\pi(x_{n+1}), \omega(x_{n+1})\big) +$
$r\rho\big(\omega(x_n), \pi(x_{n+1})\big)\frac{\rho(\pi(x_n),\omega(x_{n+1}))}{1+\rho(\pi(x_n),\omega(x_{n+1}))}\Big) = \frac{1}{s}\Big(p\rho(y_{n-1}, y_n) + q\rho(y_{n+1}, y_n) +$
$r\rho(x_{n-1}, x_{n+1})\frac{\rho(y_n,y_n)}{1+\rho(y_n,y_n)}\Big)$

$$= \frac{1}{s}\big(p\rho(y_{n-1}, y_n) + q\rho(y_{n+1}, y_n)\big). \tag{13}$$

From (13) we get

$$\rho(y_n, y_{n+1}) \leq \frac{p}{s-q}\rho(y_{n-1}, y_n). \tag{14}$$

Since $s \geq 1$, $p, q, r > 0$ and $0 < p + q + r < 1$, then we have $0 < \frac{p}{s-q} < 1$. So from (14), we get

$$\lim_{n\to+\infty} \rho(y_n, y_{n+1}) = 0. \tag{15}$$

Using Preposition 2, (y_n) is a Cauchy sequence in X.
Since (X, ρ) is complete, there exists $y^* \in X$ such that

$$\lim_{n\to+\infty} \rho(y_n, y^*) = 0. \tag{16}$$

Since $\omega(X)$ is closed, then $y^* \in \omega(X)$.
Let $\omega(x^*) = y^*$ for an $x^* \in X$.
We will show that $\pi(x^*) = \omega(x^*)$.
Suppose $\pi(x^*) \neq \omega(x^*)$, then $\rho(\pi(x^*), \omega(x^*)) > 0$. By using of D3, we can get

$\varphi\Big(\rho\big(\pi(x^*), \omega(x^*)\big)\Big) \leq \varphi\Big(s\rho\big(\pi(x^*), \pi(x_{n+1})\big) + s^2\rho\big(\pi(x_{n+1}), \omega(x_{n+1})\big) +$
$s^2\rho\big(\omega(x_{n+1}), \omega(x^*)\big)\Big) + \beta$
$\leq \varphi\big(s\big(p\rho\big(\pi(x^*), \pi(x_{n+1})\big) + q\rho\big(\pi(x_{n+1}), \omega(x_{n+1})\big) + r\rho(\omega(x^*), \pi(x_{n+1}))$
$\frac{\rho(\pi(x^*),\omega(x_{n+1}))}{1+\rho(\pi(x^*),\omega(x_{n+1}))}\big) + s^2\rho(y_{n+1}, y_n) + s^2\rho\big(\omega(x_{n+1}), \omega(x^*)\big)\Big) + \beta$
$= \varphi\Bigg(s\Big(p\rho(y^*, y_n) + q\rho(y_{n+1}, y_n) + r\rho(y^*, y_{n+1})\frac{\rho(\pi(x^*),y_n)}{1+\rho(\pi(x^*),y_n)}\Big) +$
$s^2\rho(y_{n+1}, y_n) + s^2\rho(y_n, y^*)\Bigg) + \beta$
$\leq \varphi\Big(s\big(p\rho(y^*, y_n) + q\rho(y_{n+1}, y_n) + r\rho(y^*, y_{n+1})\big) + s^2\rho(y_{n+1}, y_n) +$
$s^2\rho(y_n, y^*)\Big) + \beta. \quad (17)$

From (15) and (16), then we get

$\lim_{n\to+\infty}\Big(s\big(p\rho(y^*, y_n) + q\rho(y_{n+1}, y_n) + r\rho(y^*, y_{n+1})\big) + s^2\rho(y_{n+1}, y_n) +$
$s^2\rho(y_n, y^*)\Big) = 0. \quad (18)$

From (17), (18) and since $\varphi \in \mathfrak{F}$, we get

$$\lim_{n\to+\infty} \varphi\Big(s\big(p\rho(y^*,y_n) + q\rho(y_{n+1},y_n) + r\rho(y^*,y_{n+1})\big) + s^2\rho(y_{n+1},y_n) + s^2\rho(y_n,y^*)\Big) + \beta = -\infty.$$

Which is a contradiction. So, we have $\pi(x^*) = \omega(x^*) = y^*$. Thus x^* is a coincidence point of π and ω.

We will show uniqueness of coincidence point of π and ω.

Suppose $u^* \in X$ is another coincidence point of π and ω, such that $\pi(u^*) = \omega(u^*) = v^*$, for a $v^* \in X$.

Suppose $v^* \neq y^*$, then by using (12), we have

$$\begin{aligned}
\rho(y^*,v^*) &= \rho\big(\pi(x^*),\pi(u^*)\big) \\
&\le \frac{1}{s}\Big(p\rho\big(\omega(x^*),\omega(u^*)\big) + q\rho\big(\pi(u^*),\omega(u^*)\big) + r\rho\big(\omega(x^*),\pi(u^*)\big)\frac{\rho(\pi(x^*),\omega(u^*))}{1+\rho(\pi(x^*),\omega(u^*))}\Big) \\
&= \frac{1}{s}\Big(p\rho(y^*,v^*) + q\rho(v^*,v^*) + r\rho(y^*,v^*)\frac{\rho(y^*,v^*)}{1+\rho(y^*,v^*)}\Big) \\
&\le \frac{1}{s}\big(p\rho(y^*,v^*) + r\rho(y^*,v^*)\big) \\
&= \frac{p+r}{s}\big(\rho(y^*,v^*)\big). \qquad (18)
\end{aligned}$$

From (18) we get $\rho(y^*,v^*) \le \frac{p+r}{s}\big(\rho(y^*,v^*)\big)$.

Since $s \ge 1$ and $0 < p+q+r < 1$, we have $0 < \frac{p+r}{s} < 1$. Thus, we get

$$\rho(y^*,v^*) \le \frac{p+r}{s}\big(\rho(y^*,v^*)\big) < \rho(y^*,v^*).$$

Which is a contradiction.

Theorem 5: *Suppose* $(\varphi,\beta) \in \mathfrak{F} \times [0,+\infty)$ *and let* (X,ρ) *be a complete* $\mathfrak{F}\, b-metric\ space$ *with* $s \ge 1$. *Let* $\pi,\gamma : X \to X$ *be the functions, such that* $\pi(X) \subseteq \gamma(X), \gamma(X)$ *be closed, and for all* $x,y \in X$, *satisfy as follows*

$$\rho\big(\pi(x),\pi(y)\big) \le \frac{1}{s}\Big(p\rho\big(\gamma(x),\gamma(y)\big) + q\rho\big(\pi(y),\gamma(y)\big) + r\rho\big(\gamma(x),\pi(y)\big)\frac{\rho(\pi(x),\gamma(y))}{1+\rho(\pi(x),\gamma(y))}\Big), \qquad (19)$$

for any $p,q,r > 0$ *and* $0 < p+q+r < 1$.

If $\{\pi,\gamma\}$ *is weakly compatible, then* π *and* γ *have a unique common fixed point in* X.

Proof: From Theorem 4, π and γ have a point of coincidence in X.

Let u^* be a point of coincidence of π and γ, so we have $\pi(v^*) = \gamma(v^*) = u^*$, for $v^* \in X$.

Since $\{\pi,\gamma\}$ is weakly compatible, then $\gamma\pi(v^*) = \pi\gamma(v^*) = \pi(u^*) = \gamma(u^*)$.

We will show u^* is a common fixed point of π and γ.

Suppose $\pi(u^*) \neq u^*$, from (19) we get

$$\rho(\pi(u^*), u^*) = \rho\big(\pi(u^*), \pi(v^*)\big)$$
$$\leq \frac{1}{s}\left(p\rho\big(\gamma(u^*), \gamma(v^*)\big) + q\rho\big(\pi(v^*), \gamma(v^*)\big) + r\rho\big(\gamma(u^*), \pi(v^*)\big)\frac{\rho\big(\pi(u^*), \gamma(v^*)\big)}{1+\rho\big(\pi(u^*), \gamma(v^*)\big)}\right)$$
$$= \frac{1}{s}\left(p\rho(\gamma(u^*), u^*) + q\rho(u^*, u^*) + r\rho(\gamma(u^*), u^*)\frac{\rho(\pi(u^*), u^*)}{1+\rho(\pi(u^*), u^*)}\right)$$
$$\leq \frac{1}{s}\big(p\rho(\gamma(u^*), u^*) + r\rho(\gamma(u^*), u^*)\big) \leq \frac{p+r}{s}\rho(\gamma(u^*), u^*). \quad (20)$$

Since $\{\pi, \gamma\}$ is weakly compatible and $0 < \frac{p+r}{s} < 1$, then from (20) we get
$$\rho(\pi(u^*), u^*) \leq \frac{p+r}{s}\rho(\gamma(u^*), u^*) = \frac{p+r}{s}\rho(\pi(u^*), u^*) < \rho(\pi(u^*), u^*).$$

Which is a contradiction. Hence, u^* is a fixed point of π. Since $\pi(u^*) = \gamma(u^*)$, then u^* is a common fixed point of π and γ.

We will show that u^* is a unique common fixed point of π and γ.

Suppose there exists another common fixed point of π and γ, namely $t^* \in X$. So, from (19) we have
$$\rho(t^*, u^*) = \rho(\pi(t^*), \pi(u^*))$$
$$\leq \frac{1}{s}\left(p\rho\big(\gamma(t^*), \gamma(u^*)\big) + q\rho\big(\pi(u^*), \gamma(u^*)\big) + r\rho\big(\gamma(t^*), \pi(u^*)\big)\frac{\rho\big(\pi(t^*), \gamma(u^*)\big)}{1+\rho\big(\pi(t^*), \gamma(u^*)\big)}\right)$$
$$\leq \frac{1}{s}\big(p\rho(t^*, u^*) + q\rho(u^*, u^*) + r\rho(t^*, u^*)\big).$$

Since $0 < \frac{p+r}{s} < 1$, then we get
$$\rho(t^*, u^*) \leq \frac{p+r}{s}\rho(t^*, u^*) < \rho(t^*, u^*).$$

Which is a contradiction.

4. The application

The application of Theorem 5 is shown below in Theorem 6.

Theorem 6: *Let $n \geq 2$ be an even natural number and $s, x \geq 1$. The equation*
$$(n^3-1)x^{n+1} - n^3x^n + n^3x - n^3 + 1 = 0 \quad (21)$$
$$(n^3-1)x^{n+1} - (n^3 + 8s^2 - 1)x^n + 2n^3x - 2n^3 + 4s^2 = 0$$
has a unique common real solution.

Proof: Let $X = [1, +\infty)$ and $\rho(r, t) = |r - t|^2$. From Example 2, (X, ρ) is a $\mathfrak{F}\, b$ −metric space and complete. From (21) we can get

$(n^3-1)x^{n+1} - n^3x^n + n^3x - n^3 + 1 = 0 \Leftrightarrow x = 1 + \frac{x^n - 1}{(n^3-1)x^n + n^3}$, and

$(n^3-1)x^{n+1} - (n^3 + 8s^2 - 1)x^n + 2n^3x - 2n^3 + 4s^2 = 0 \Leftrightarrow x = 1 + \frac{4s^2(2x^n - 1)}{(n^3-1)x^n + 2n^3}$

$X = [1, +\infty)$ Define a function $\pi, \gamma: X \to X$ with $s \geq 1, n \geq 2$ and n is even natural number.

$$\pi x = 1 + \frac{(x^n - 1)}{(n^3-1)x^n + n^3}$$
$$\gamma x = 1 + \frac{4s^2(2x^n - 1)}{(n^3-1)x^n + 2n^3}$$

If $y \in \pi(X)$, then there exist $x \in X$ such that $y = 1 + \frac{(x^n-1)}{(n^3-1)x^n+n^3}$. Now, we will find $z \in X$ such that $y = 1 + \frac{2s^2(2z^n+1)}{(n^3-1)z^n+2n^3}$. So, we have

$$\frac{(x^n-1)}{(n^3-1)x^n+n^3} = \frac{4s^2(2z^n-1)}{(n^3-1)z^n+2n^3}$$

So, we get $z^n = \frac{2(n^3+2(n^3-1)s^2)x^n+2n^3(2s^2-1)}{(n^3-1)x^n(8s^2-1)+8s^2+(n^3-1)} > 0$, and it implies $y = 1 + \frac{2s^2(2z^n+1)}{(n^3-1)z^n+2n^3} \in [1, +\infty)$. It shows that $\pi(X) \subseteq \gamma(X)$. It is clear that $\gamma(X)$ is closed.

We will show that hypothesis (12) is satisfied. From (12) and choose $p = q = r = \frac{1}{4}$.

$$\frac{1}{4s}\left(\rho\big(\gamma(x),\gamma(y)\big) + \rho\big(\pi(y),\gamma(y)\big) + \rho\big(\gamma(x),\pi(y)\big)\frac{\rho(\pi(x),\gamma(y))}{1+\rho(\pi(x),\gamma(y))}\right) =$$
$$\geq \frac{1}{4s}\Big(p\rho\big(\gamma(x),\gamma(y)\big) + q\rho\big(\pi(y),\gamma(y)\big)\Big) =$$
$$= \frac{1}{4s}\left|\frac{4s^2(2x^n-1)}{(n^3-1)x^n+2n^3} - \frac{4s^2(2y^n-1)}{(n^3-1)y^n+2n^3}\right|^2 + \left|\frac{4s^2(2y^n-1)}{(n^3-1)y^n+2n^3} - \frac{(y^n-1)}{(n^3-1)y^n+n^3}\right|^2$$

Since $x, y, s \geq 1$ then we have

$$\frac{1}{4s}\Big(p\rho\big(\gamma(x),\gamma(y)\big) + q\rho\big(\pi(y),\gamma(y)\big)\Big) \geq \frac{1}{8s}\left|\frac{4s^2(2x^n-1)}{(n^3-1)x^n+2n^3} - \frac{(y^n-1)}{(n^3-1)y^n+n^3}\right|^2 \geq$$
$$\frac{1}{8s}\left|\frac{4s^2(2x^n-1)}{(n^3-1)x^n+2n^3} - \frac{4s(y^n-1)}{(n^3-1)y^n+n^3}\right|^2$$
$$\geq \frac{16s^2}{8s}\left|\frac{s(x^n-1)}{(n^3-1)x^n+2n^3} - \frac{(y^n-1)}{(n^3-1)y^n+n^3}\right|^2 \geq 2s\left|\frac{s(x^n-1)}{(n^3-1)x^n+n^3} - \frac{(y^n-1)}{(n^3-1)y^n+n^3}\right|^2$$
$$\geq \left|\frac{s(x^n-1)}{(n^3-1)x^n+n^3} - \frac{(y^n-1)}{(n^3-1)y^n+n^3}\right|^2 \geq \left|\frac{(x^n-1)}{(n^3-1)x^n+n^3} - \frac{(y^n-1)}{(n^3-1)y^n+n^3}\right|^2$$
$$= \rho\big(\pi(x),\pi(y)\big).$$

Hence, hypothesis (12) is satisfied for $p = q = r = \frac{1}{4}$.

Therefore, by using Theorem 5, then (21) has a unique common real solution.

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References

[1] I.A. Bakhtin, "The contraction mapping principle in quasi- metric spaces," *Funct. Anal. UnianowskGos. Ped. Inst.*, vol. 30, pp. 26-37 (1989).

[2] S. Czerwik, "Contraction mappings in b -metric spaces," *Acta Mathematica et Informatica Universitatis Ostraviensis,* vol. 1, no. 1, pp. 5-11 (1993).

[3] N.V. Dung, "The metrization of rectangular b-metric spaces," *Topology and its Applications,* vol. 261, pp. 22-28, (2019).

[4] R. George, S. Radenovic, K.P. Reshma, and S. Shukla, "Rectangular b-metric space and contraction principles," *J. Nonlinear Sci. Appl.,* vol. 8, pp. 1005-1013 (2015).

[5] A. Branciari, "A fixed point theorem of Banach-Caccippoli type on a class of generalized metric spaces," Publ. Math. Debrecen, vol. 57, pp. 31-37 (2000).

[6] M. Khamsi, " Remarks on cone metric spaces and fixed point theorems of contractive mappings," *Fixed point theory and applications,* vol. 2010, 1-7 (2010).

[7] T. Kamran, M. Samreen, and Q. UL Ain, "A generalization of b-metric space and some fixed point theorems," *Mathematics,* vol. 5, no. 2, pp. 19 (2017).

[8] M. Asim, M. Imdad, and S. Radenovic, "Fixed point results in extended rectangular b-metric spaces with an Application," University Politehnica of Bucharest Scientific Bulletin-Series A-Applied Mathematics and Physics, vol. 81, no. 2, pp. 43-50 (2019).

[9] S. Cobzas, and S. Czerwik, "The completion of generalized b-metric spaces and fixed points," *Fixed Point Theory,* vol. 21, no. 1, 133-150 (2020).

[10] K. Stella Eke and S. A. Bishop "On fixed point theorems for generalized weakly CE-contraction mappings on metric spaces," *Journal of Interdisciplinary Mathematics*, vol. 23, no. 3, 691–700 (2020). https://doi.org/10.1080/09720502.2019.1700933.

[11] M. Jleli, and B. Samet, "On a new generalization of metric spaces," *Journal of Fixed Point Theory and Applications*, vol. 20, no. 3, 1-20 (2018).

[12] M. Abbas, and B.E. Rhoades, "Common fixed point theorems for occasionally weakly compatible mappings satisfying a generalized contractive condition," *Mathematical Communications*, vol. 13, no. 2, 295-301, (2008).

[13] B. Nurwahyu, N. Aris and Firman, "Some results in function weighted b-metric spaces," *AIMS Mathematics*, vol. 8, no. 4, 8274–8293 (2023).

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