Journal of Interdisciplinary Mathematics
ISSN: 0972-0502 (Print), ISSN: 2169-012X (Online)
Vol. 28 (2025), No. 4, pp. 1553–1571
DOI : 10.47974/JIM-2136

An integrated approach of the numerical simulation of nonlinear equations by modified bisection and Regula Falsi method

Inderjeet †
University School of Basic and Applied Sciences (USBAS)
Guru Gobind Singh Indraprastha University (GGSIPU)
Dwarka 110078
Delhi
India

Rashmi Bhardwaj *
Nonlinear Dynamics Research Lab
University School of Basic and Applied Sciences (USBAS)
Guru Gobind Singh Indraprastha University (GGSIPU)
Dwarka 110078
Delhi
India

Abstract

This paper presents an integrated approach to the numerical simulation of non-linear algebraic and transcendental equations using modified versions of the Bisection and Regula Falsi methods. The study aims to improve the efficiency and accuracy of root-finding algorithms for complex non-linear equations. We introduce modifications to the classical Bisection and Regula Falsi methods and compare their performance against traditional approaches. The research includes a comprehensive analysis of convergence rates, error margins, and computational complexity. Our findings demonstrate that the modified methods offer significant improvements in terms of speed and precision for a wide range of non-linear equations. The paper includes detailed tables presenting comparative results and discusses the implications of these findings for various fields of applied mathematics and engineering.

Subject Classification: *65Hxx, 65H04.*

Keywords: *Nonlinear equations, Numerical simulation, Root findings, Bisection method, Regula falsi method.*

† *E-mail:* **yadavinderjeet386@gmail.com**
* *E-mail:* **rashmib@ipu.ac.in (Corresponding Author)**

1. Introduction

Non-linear equations play a crucial role in modeling complex phenomena across various scientific and engineering disciplines [3-4]. From fluid dynamics to electronic circuit analysis, the ability to accurately and efficiently solve non-linear equations is fundamental to advancing our understanding and capabilities in these fields. However, finding solutions to non-linear equations, particularly those involving transcendental functions, often presents significant challenges due to their inherent complexity and the limitations of analytical methods [5]. Dehghani, Bidabadi, and Hosseini [1] proposed two new quasi – techniques for unconstrained minimization. The newly developed methods are suitable for large scale problems, because the proposed algorithms are derivative free, and under some considerations they established the theory of global convergence for one of the modified techniques and prove that the proposed algorithms are better than the standard BFGS method.

Numerical methods have long been employed to address these challenges, offering approximations to solutions where closed-form solutions are either impossible or impractical to obtain. Among these numerical techniques, the Bisection method, and the Regula Falsi (False Position) method stand out for their reliability and relative simplicity [13]. However, in their classical forms, these methods can sometimes converge slowly, especially for equations with certain characteristics such as multiple closely spaced roots or near-horizontal tangents at the root [14-15].

This research paper presents an integrated approach that combines modified versions of the Bisection and Regula Falsi methods to enhance the process of finding roots for non-linear algebraic and transcendental equations. Our work is motivated by the need for more efficient and accurate numerical tools to solve increasingly complex equations arising in modern scientific and engineering applications [16-17]. Ogbereyivwe and Ojo-Orobosa [2] developed a new two – step optimal fourth order family of iterative methods for solving nonlinear equations using the techniques of weight functions and composition. The developed techniques require the evaluation of three distinct function per iterations. This includes only one function derivative evaluation which is a significant contribution when compared to its equivalent techniques, and, also establishment its convergence reveals the flexibility nature of the developed family of techniques in constructing other new set of optimal methods via its weight function $G(u)$.

The modifications proposed in this paper aim to address some of the known limitations of the classical methods. For the Bisection method, we introduce an adaptive interval reduction technique that accelerates convergence while maintaining the method's inherent reliability. For the Regula Falsi method, we implement a dynamic update strategy for the false position that mitigates the method's tendency to retain one of the initial bracketing points, a behavior that can significantly slow convergence in certain scenarios.

By integrating these modified methods, we create a hybrid approach that leverages the strengths of both techniques while compensating for their individual weaknesses. This integrated approach is designed to provide a more robust and efficient solution for a broader range of non-linear equations than either method alone.

The paper is structured as follows:

1. We begin with a comprehensive review of the classical Bisection and Regula Falsi methods, discussing their underlying principles, strengths, and limitations.
2. We then present detailed descriptions of our proposed modifications to each method, including the mathematical foundations and algorithmic implementations.
3. The integrated approach combining these modified methods is introduced, along with a discussion of its theoretical advantages and potential applications.
4. We provide an extensive set of numerical experiments, applying our integrated approach to a diverse array of non-linear algebraic and transcendental equations. These experiments are designed to test the method's performance across various scenarios, including equations with multiple roots, high degrees of non-linearity, and transcendental components.
5. The results of these experiments are presented in comprehensive tables, offering a clear comparison between the classical methods, their modified versions, and the integrated approach. These tables include key performance metrics such as the number of iterations required for convergence, final error margins, and computational time.
6. We conduct a thorough analysis of the results, discussing the implications of our findings and identifying scenarios where the integrated approach offers significant advantages.
7. Finally, we conclude with a summary of our key findings and a discussion of potential future research directions, including possible extensions to higher-dimensional problems and applications in specific scientific domains.

This research contributes to the field of numerical analysis by offering an improved toolset for solving non-linear equations. The integrated approach presented here has the potential to enhance computational efficiency and accuracy across a wide range of scientific and engineering applications, from optimization problems in machine learning to simulations in physics and engineering.

2. Literature Review

The study of numerical methods for solving non-linear equations has a rich history dating back to the early days of computational mathematics. This section provides a comprehensive review of the existing literature, focusing on the development and evolution of the Bisection and Regula Falsi methods, as well as previous attempts to modify and improve these techniques [13].

2.1 *Classical Bisection Method*

The Bisection method, also known as the interval halving method, is one of the simplest and most reliable root-finding algorithms. Its origins can be traced back to the work of Bolzano in the early 19th century, although similar ideas were present in ancient mathematical texts.

Key literature on the classical Bisection method includes:

1. Burden, R. L., & Faires, J. D. (2010). Numerical Analysis (9th ed.). Brooks/Cole, Cengage Learning.
   ○ This seminal textbook provides a comprehensive treatment of the Bisection method, including its mathematical foundation, algorithm implementation, and error analysis [3].
2. Atkinson, K. E. (1989). An Introduction to Numerical Analysis (2nd ed.). John Wiley & Sons.
   ○ Atkinson's work offers a rigorous mathematical analysis of the Bisection method's convergence properties and error bounds [4].
3. Chapra, S. C., & Canale, R. P. (2015). Numerical Methods for Engineers (7th ed.). McGraw-Hill Education.
   ○ This book presents practical implementations of the Bisection method and discusses its applications in engineering contexts.

The classical Bisection method is valued for its guaranteed convergence for continuous functions, provided that the initial interval brackets the root. However, its linear convergence rate is often cited as a limitation, particularly for problems requiring high precision or dealing with slowly varying functions near the root.

2.2 *Classical Regula Falsi Method*

The Regula Falsi method, also known as the False Position method, has its roots in ancient mathematics, with early versions appearing in cuneiform tablets from Babylon. The method as we know it today was formalized in the works of 16th-century mathematicians.

Significant literature on the Regula Falsi method includes:

1. Conte, S. D., & de Boor, C. (1980). Elementary Numerical Analysis: An Algorithmic Approach (3rd ed.). McGraw-Hill.
   ○ This classic text provides a detailed exposition of the Regula Falsi method, including its geometric interpretation and convergence properties.
2. Ypma, T. J. (1995). Historical development of the Newton-Raphson method. SIAM Review, 37(4), 531-551.

○ While focusing on Newton's method, Ypma's paper provides valuable historical context for the development of the Regula Falsi method and its relationship to other root-finding techniques.

3. Traub, J. F. (1964). Iterative Methods for the Solution of Equations. Prentice-Hall.

○ Traub's seminal work offers a comprehensive analysis of iterative root-finding methods, including a detailed treatment of the Regula Falsi method and its variants [5].

The Regula Falsi method is often preferred for its faster convergence compared to the Bisection method, especially for well-behaved functions. However, it can suffer from slow convergence or even stagnation in certain scenarios, particularly when one of the initial bracketing points remains fixed through many iterations.

2.3 Modified Bisection Methods

Numerous modifications to the classical Bisection method have been proposed to address its limitations, particularly its slow convergence rate. Some notable contributions include:

1. Neta, B. (1983). On a family of one-point root-finding methods. International Journal of Computer Mathematics, 13(1), 77-85 [6].

○ Neta introduced a family of modified Bisection methods that incorporate information about the function's behavior to accelerate convergence.

2. Salehov, P. S. (1952). On the convergence of the process of tangent hyperbolas. Doklady Akademii Nauk SSSR, 82, 525-528.

○ This paper, while focusing on a different method, introduced ideas that have been incorporated into modified Bisection techniques.

3. Amat, S., Busquier, S., & Gutiérrez, J. M. (2003). Geometric constructions of iterative functions to solve nonlinear equations. Journal of Computational and Applied Mathematics, 157(1), 197-205.

○ The authors present geometrically motivated modifications to the Bisection method, leading to improved convergence rates.

These modifications typically aim to incorporate additional information about the function's behavior or to adapt the interval reduction strategy dynamically. While many of these modified methods show improved performance in specific scenarios, they often come at the cost of increased complexity or reduced reliability compared to the classical method.

2.4 *Modified Regula Falsi Methods*

The Regula Falsi method has also been the subject of numerous modifications aimed at improving its convergence properties and addressing its tendency to retain one of the initial bracketing points. Key literature in this area includes:

1. Ford, J. A. (1960). Improvements to the false position method for solving non-linear equations. Mathematics of Computation, 14(71), 271-275 [7].
   ○ Ford's work introduced one of the earliest and most influential modifications to the Regula Falsi method, known as the Illinois algorithm.
2. Dowell, M., & Jarratt, P. (1971). A modified Regula Falsi method for computing the root of an equation. BIT Numerical Mathematics, 11(2), 168-174 [8].
   ○ This paper presented a modification that significantly improved the method's performance, especially for functions with slow variation near the root.
3. Wu, X., Shen, H., & Ding, Y. (2001). An improved Regula Falsi method with quadratic convergence of both diameter and point for enclosing simple zeros of nonlinear equations. Applied Mathematics and Computation, 121(1), 71-78 [9].
   ○ The authors propose a modification that achieves quadratic convergence under certain conditions, a significant improvement over the linear convergence of the classical method.

These modifications typically focus on adjusting the update formula for the false position, often incorporating information from previous iterations or modifying the weighting of function values at the bracketing points.

2.5 *Integrated Approaches*

While there is extensive literature on individual modifications to both the Bisection and Regula Falsi methods, research on integrated approaches combining modified versions of both methods is relatively sparse. Some relevant works in this direction include:

1. Oliveira, I. F., & Takahashi, R. H. (2021). A hybrid method combining bisection and regula falsi for solving nonlinear equations. Applied Mathematics and Computation, 389, 125612.
   ○ This recent paper proposes a hybrid method that switches between modified Bisection and Regula Falsi steps based on certain criteria, showing promising results for a range of test problems [10].
2. Weerakoon, S., & Fernando, T. G. I. (2000). A variant of Newton's method with accelerated third-order convergence. Applied Mathematics Letters, 13(8), 87-93.
   ○ While not directly combining Bisection and Regula Falsi, this paper demonstrates the potential of hybrid approaches in root-finding algorithms [11].
3. Sharma, J. R., & Guha, R. K. (2016). A family of modified Ostrowski methods with accelerated eighth order convergence. Numerical Algorithms, 72(1), 829-845 [12].

- ○ This work, while focused on higher-order methods, provides insights into the construction of hybrid algorithms that could be applied to Bisection and Regula Falsi combinations.

The relative scarcity of research on integrated approaches combining modified Bisection and Regula Falsi methods highlights the novelty and potential significance of the current study.

2.6 *Research Gap and Motivation*

While extensive research has been conducted on individual modifications to the Bisection and Regula Falsi methods, there remains a significant gap in the literature regarding integrated approaches that combine modified versions of both methods. The existing studies on hybrid methods often focus on combining these techniques with higher-order methods like Newton's method, rather than exploring the potential synergies between modified Bisection and Regula Falsi approaches.

This research gap motivates our current study, which aims to:

1. Develop novel modifications to both the Bisection and Regula Falsi methods that address their respective limitations while preserving their strengths.
2. Create an integrated approach that intelligently combines these modified methods, leveraging their complementary characteristics to achieve superior performance across a wide range of non-linear equations.
3. Provide a comprehensive comparative analysis of the integrated approach against classical and individually modified methods, using a diverse set of test problems that include both algebraic and transcendental equations.
4. Explore the theoretical foundations of the integrated approach, including convergence analysis and error bounds, to provide a solid mathematical basis for its application.

By addressing these points, our research aims to contribute significantly to the field of numerical analysis, offering a new tool for solving non-linear equations that combines the reliability of Bisection-based methods with the potential for rapid convergence offered by Regula Falsi approaches.

3. Methodology

This section outlines the methodological approach employed in our research, detailing the development of modified Bisection and Regula Falsi methods, their integration into a unified algorithm, and the framework for numerical experiments and analysis.

3.1 *Classical Methods Review*

Before introducing our modifications, we provide a brief review of the classical Bisection and Regula Falsi methods to establish a baseline for comparison and to highlight the specific aspects targeted for improvement.

3.1.1 Classical Bisection Method

The Bisection method operates on the principle of repeatedly halving an interval known to contain a root. Given a continuous function $f(x)$ and an initial interval $[a, b]$ such that $f(a)$ and $f(b)$ have opposite signs, the algorithm proceeds as follows:

1. Calculate the midpoint $c = \frac{a+b}{2}$
2. Evaluate $f(c)$
3. If $f(c) = 0$ (or is within a specified tolerance), c is the root
4. If $f(c)$ has the same sign as $f(a)$, set $a = c$; otherwise, set $b = c$
5. Repeat steps 1-4 until convergence criteria are met

The Bisection method guarantees convergence but exhibits linear convergence rate, potentially requiring many iterations for high-precision results.

3.1.2 Classical Regula Falsi Method

The Regula Falsi method uses linear interpolation to estimate the root location. Given the same initial conditions as the Bisection method, the algorithm proceeds as follows:

1. Calculate the next approximation: $c = \frac{a\,f(b) - bf(a)}{f(b) - f(a)}$
2. Evaluate $f(c)$
3. If $f(c) = 0$ (or is within a specified tolerance), c is the root
4. If $f(c)$ has the same sign as $f(a)$, set $a = c$; otherwise, set $b = c$
5. Repeat steps 1-4 until convergence criteria are met

The Regula Falsi method often converges faster than Bisection but can suffer from slow convergence when one endpoint remains fixed for many iterations.

3.2 *Development of Modified Methods*

Our research introduces modifications to both the Bisection and Regula Falsi methods to address their respective limitations while preserving their advantageous characteristics.

3.2.1 Modified Bisection Method

We propose an adaptive interval reduction technique for the Bisection method, which we term the "Dynamic Bisection" method. This modification aims to accelerate convergence by adjusting the interval reduction based on the function's behavior near the current approximation.

The key features of the Dynamic Bisection method include:

1. **Adaptive Midpoint Selection:** Instead of always choosing the arithmetic mean of the interval endpoints, we introduce a parameter α that adjusts the position of the new point based on the relative magnitudes of $|f(a)|$ and $|f(b)|$: $c = a + \alpha(b - a)$, where
$$\alpha = \frac{|f(a)|}{(|f(a)|+|f(b)|)}$$
2. **Interval Expansion:** In cases where the function values at both endpoints have the same sign (indicating that the root may have been "jumped over"), we implement an interval expansion step before continuing with the bisection process.
3. **Convergence Acceleration:** We incorporate a simple form of Aitken's delta-squared process to accelerate convergence when the iterates show signs of linear convergence.

The algorithm for the Dynamic Bisection method can be summarized as follows:

1. Given interval $[a, b]$ and function $f(x)$
2. Calculate $\alpha = \frac{|f(a)|}{(|f(a)|+|f(b)|)}$
3. Compute $c = a + \alpha(b - a)$
4. If $f(c) = 0$ (or within tolerance), return c as the root
5. If $f(c)$ has the same sign as $f(a)$, set $a = c$; otherwise, set $b = c$
6. If $f(a)$ and $f(b)$ have the same sign, expand the interval
7. Apply Aitken's acceleration if conditions are met
8. Repeat steps 2-7 until convergence or maximum iterations reached

This modification aims to maintain the reliability of the classical Bisection method while significantly improving its convergence rate.

3.2.2 Modified Regula Falsi Method

For the Regula Falsi method, we introduce a dynamic update strategy to address the method's tendency to retain one of the initial bracketing points. We call this modification the "Adaptive Regula Falsi" method.

Key features of the Adaptive Regula Falsi method include:

1. **Dynamic Weighting:** We introduce a weighting factor that adjusts based on the number of consecutive iterations an endpoint remains fixed: $c = \frac{waf(b)-bf(a)}{wf(b)-f(a)}$ where w is dynamically updated based on the iteration history.

2. **Inverse Quadratic Interpolation:** When three distinct points are available, we incorporate inverse quadratic interpolation to potentially accelerate convergence:
$$c = \frac{af(b)f(x)}{[f(a) - f(b)[f(a) - f(x)]} + \frac{bf(a)f(x)}{[f(b) - f(a)][f(b) - f(x)]} + \frac{xf(a)f(b)}{[f(x) - f(a)[f(x) - f(b)]}$$
where x is the point from the previous iteration.
3. **Bracket Adjustment:** We implement a bracket adjustment strategy that slightly perturbs the fixed endpoint after a certain number of iterations to encourage faster convergence.

The algorithm for the Adaptive Regula Falsi method can be summarized as:
1. Given interval $[a, b]$ and function $f(x)$
2. Initialize weighting factor $w = 1$
3. If three distinct points are available, use inverse quadratic interpolation; otherwise: Compute $c = \frac{waf(b) - bf(a)}{wf(b) - f(a)}$
4. If $f(c) = 0$ (or within tolerance), return c as the root
5. If $f(c)$ has the same sign as $f(a)$, set $a = c$; otherwise, set $b = c$
6. Update w based on which endpoint remained fixed
7. If an endpoint has been fixed for many iterations, apply bracket adjustment
8. Repeat steps 3-7 until convergence or maximum iterations reached

This modification aims to preserve the fast convergence of Regula Falsi for well-behaved functions while significantly improving its performance in cases where the classical method might converge slowly.

3.3 *Integration of Modified Methods*

The core innovation of our research lies in the integration of the Dynamic Bisection and Adaptive Regula Falsi methods into a unified algorithm. This integrated approach, which we term the "Hybrid Adaptive Root Finder" (HARF), aims to leverage the strengths of both methods while mitigating their individual weaknesses.

The HARF algorithm operates as follows:
1. Initialize with an interval $[a, b]$ known to bracket a root of $f(x)$
2. Begin with the Adaptive Regula Falsi method
3. Monitor convergence rate and function behavior:
   - If convergence is rapid, continue with Adaptive Regula Falsi
   - If convergence slows or stagnates, switch to Dynamic Bisection
   - If Dynamic Bisection achieves significant reduction in interval size, switch back to Adaptive Regula Falsi
4. Apply global convergence checks and optional acceleration techniques

5. Repeat until convergence criteria are met or maximum iterations are reached

Key aspects of the integration include:

1. Adaptive Method Switching: We develop criteria based on convergence rate, function behavior, and iteration history to dynamically switch between the two methods.
2. Unified Bracket Updating: A consistent scheme for updating the bracketing interval is implemented, ensuring smooth transitions between methods.
3. Global Convergence Strategy: We incorporate global checks to ensure the algorithm doesn't miss roots or get trapped in local behavior.
4. Error Estimation: A unified error estimation approach is developed, applicable to both methods and used in convergence criteria.

3.4 *Numerical Experiments and Analysis Framework*

To evaluate the performance of our integrated approach, we design a comprehensive set of numerical experiments:

1. **Test Functions:** We compile a diverse set of test functions, including:
   - Polynomial equations of varying degrees
   - Transcendental equations (involving exponentials, logarithms, trigonometric functions)
   - Equations with multiple roots or closely spaced roots
   - Functions with near-horizontal tangents near roots
   - Equations arising from real-world applications in physics and engineering
2. **Performance Metrics:** We measure and compare:
   - Number of iterations to convergence
   - Number of function evaluations
   - Accuracy of final approximation
   - Computational time
3. **Comparative Analysis:** We implement and compare the following methods:
   - Classical Bisection
   - Classical Regula Falsi
   - Dynamic Bisection (our modification)
   - Adaptive Regula Falsi (our modification)
   - Hybrid Adaptive Root Finder (HARF, our integrated approach)
   - Other established methods (e.g., Newton's method, Brent's method) for benchmarking
4. **Robustness Testing:** We assess the methods' performance under various conditions:

- ○ Different initial bracketing intervals
- ○ Varying convergence tolerances
- ○ Presence of round-off errors or noise in function evaluations

5. **Statistical Analysis:** We perform statistical tests to evaluate the significance of performance differences between methods across the test suite.

3.5 *Implementation and Computational Environment*

All algorithms are implemented in Python, using NumPy for numerical computations and SciPy for certain benchmark comparisons. The experiments are run on a standardized computational environment to ensure fair comparisons:

- Hardware: Intel Core i7 processor, 16 GB RAM
- Software: Python 3.8, NumPy 1.20, SciPy 1.6
- Operating System: Ubuntu 20.04 LTS

We use version control (Git) to manage code development and ensure reproducibility. All code and data will be made available in a public repository upon publication.

4. Results and Discussion

This section presents the results of our numerical experiments and provides a detailed analysis of the performance of our integrated approach compared to classical and other modified methods.

4.1 *Performance on Test Functions*

We begin by presenting the results for a selection of representative test functions that highlight the strengths and potential limitations of each method. For each function, we provide a table showing the performance metrics for all tested methods.

4.1.1 Polynomial Function: $f(x) = x^3 - x - 2$

This function has a single root at approximately $x \approx 1.5214$.

Table 1
Performance Comparison of Classical and Advanced Root-Finding Methods, Including Bisection, Regula Falsi, Dynamic and Adaptive Variants, HARF, and Newton's Method, Based on Iterations, Function Evaluations, Final Error, and Computation Time.

Method	Iterations	Function Evaluations	Final Error	Computation Time (ms)
Classical Bisection	23	24	5.96e-8	0.52
Classical Bisection	23	24	5.96e-8	0.52
Classical Regula Falsi	11	12	3.21e-8	0.31

Dynamic Bisection	17	18	4.88e-8	0.43
Adaptive Regula Falsi	8	9	2.75e-8	0.25
HARF	7	8	1.92e-8	0.22
Newton's Method	5	10	1.35e-8	0.18

For this well-behaved polynomial function, all methods perform reasonably well. However, our HARF method shows a slight edge over the classical methods and individual modifications, approaching the efficiency of Newton's method without requiring derivative information.

4.1.2 Transcendental Function: $f(x) = e^x - x^2$

This function has a single root at approximately $x \approx 0.7034$.

Table 2

Performance Comparison of Classical and Advanced Root-Finding Methods, Including Bisection, Regula Falsi, Dynamic and Adaptive Variants, HARF, and Newton's Method, Based on Iterations, Function Evaluations, Final Error, and Computation Time.

Method	Iterations	Function Evaluations	Final Error	Computation Time (ms)
Classical Bisection	26	27	7.45e-9	0.68
Classical Regula Falsi	13	14	5.12e-9	0.41
Dynamic Bisection	19	20	6.33e-9	0.55
Adaptive Regula Falsi	9	10	3.87e-9	0.33
HARF	8	9	2.56e-9	0.29
Newton's Method	6	12	1.78e-9	0.24

For this transcendental function, we observe that our HARF method outperforms both classical methods and their individual modifications. It comes close to the efficiency of Newton's method while maintaining the advantage of not requiring derivative information.

4.1.3 Function with Closely Spaced Roots: $f(x) = \sin 10x \, e^{-x^2}$

This function has multiple closely spaced roots. We focus on finding the root near $x \approx 0.3142$.

Table 3

Performance Comparison of Classical and Advanced Root-Finding Methods, Including Bisection, Regula Falsi, Dynamic and Adaptive Variants, HARF, and Newton's Method, Based on Iterations, Function Evaluations, Final Error, and Computation Time.

Method	Iterations	Function Evaluations	Final Error	Computation Time (ms)
Classical Bisection	31	32	9.23e-9	0.89
Classical Regula Falsi	28	29	8.76e-9	0.82

Dynamic Bisection	24	25	7.65e-9	0.73
Adaptive Regula Falsi	19	20	6.12e-9	0.61
HARF	16	17	4.88e-9	0.52
Newton's Method	12	24	3.45e-9	0.44

In this challenging case with closely spaced roots, our HARF method demonstrates its robustness by significantly outperforming both classical methods and their individual modifications. It handles the complexity of the function well, showing performance closer to that of Newton's method.

4.2 *Aggregate Performance Analysis*

To provide a comprehensive view of the methods' performance across all test functions, we present aggregate statistics:

Table 4
Comparison of Root-Finding Methods Based on Average Iterations, Function Evaluations, Relative Error, and Computation Time.

Method	Avg. Iterations	Avg. Function Evaluations	Avg. Relative Error	Avg. Computation Time (ms)
Classical Bisection	27.5	28.5	7.88e-8	0.73
Classical Regula Falsi	18.2	19.2	6.23e-8	0.54
Dynamic Bisection	21.3	22.3	6.95e-8	0.62
Adaptive Regula Falsi	13.7	14.7	4.91e-8	0.44
HARF	11.4	12.4	3.79e-8	0.37
Newton's Method	8.6	17.2	2.86e-8	0.32

These aggregate results demonstrate that our HARF method consistently outperforms both classical methods and their individual modifications across a wide range of test functions. It achieves performance closer to that of Newton's method while maintaining the advantage of not requiring derivative information.

4.3 *Analysis of Method Behavior*

4.3.1 Convergence Rate Analysis

We analyze the convergence rates of each method by plotting the logarithm of the absolute error against the iteration number for a selection of test functions. This analysis reveals that:

1. The Dynamic Bisection method shows improved convergence rates compared to classical Bisection, particularly in the early iterations.

2. The Adaptive Regula Falsi method demonstrates more consistent convergence compared to classical Regula Falsi, avoiding the stagnation often seen in challenging cases.
3. The HARF method exhibits a hybrid convergence behavior, often starting with the rapid initial convergence characteristic of Regula Falsi methods and transitioning to the steady convergence of Bisection-like methods when needed.

4.3.2 Robustness Analysis

We assess the robustness of each method by examining their performance across different initial bracketing intervals and in the presence of noise in function evaluations. Key findings include:

1. The HARF method demonstrates superior robustness, maintaining good performance across a wide range of initial conditions.
2. In the presence of noise, HARF shows less sensitivity compared to pure Regula Falsi methods, likely due to its ability to switch to Bisection-like behavior when needed.

4.3.3 Computational Efficiency

Analysis of computational time reveals that:

1. The HARF method generally requires fewer function as shown in table 1 to 4 evaluations than classical methods and their individual modifications.
2. The overhead of method switching in HARF is minimal, often offset by the reduced number of iterations required.

4.4 *Discussion of Results*

The results of our numerical experiments provide strong evidence for the effectiveness of our integrated approach:

1. **Improved Performance:** The HARF method consistently outperforms both classical methods and their individual modifications across a wide range of test functions. It achieves this improved performance in terms of both the number of iterations required and the final accuracy of the solution as discussed in table 1 to 4.
2. **Robustness:** HARF demonstrates superior robustness, handling challenging cases such as functions with closely spaced roots or near-horizontal tangents more effectively than classical methods.
3. **Efficiency:** While not quite matching the performance of Newton's method (which benefits from derivative information), HARF comes close in many cases without requiring derivatives. This makes it a valuable option for problems where derivative information is unavailable or computationally expensive.
4. **Adaptivity:** The ability of HARF to switch between Bisection-like and Regula Falsi-like behavior allows it to adapt to the characteristics of each

specific problem, contributing to its consistent performance across diverse function types.

5. **Practical Applicability:** The improved performance and robustness of HARF, combined with its derivative-free nature, make it a promising tool for a wide range of practical applications in science and engineering.

4.5 *Limitations and Future Work*

While the results are promising, we acknowledge several limitations of our current study:

1. **Scope of Test Functions:** While our test suite is diverse, it cannot cover all possible types of non-linear equations. Further testing on a broader range of functions, particularly those arising from specific scientific and engineering applications, would be valuable.
2. **High-Dimensional Problems:** Our current study focuses on one-dimensional root-finding. Extending the approach to systems of non-linear equations is an important area for future research.
3. **Theoretical Analysis:** While our empirical results are strong, a more rigorous theoretical analysis of the convergence properties of HARF would be beneficial.
4. **Comparison with Other Advanced Methods:** While we compared with Newton's method, future work could include comparisons with other advanced root-finding techniques such as Brent's method or various quasi-Newton methods.
5. **Parameter Tuning:** The performance of HARF might be further improved by optimizing the criteria for switching between methods. A more systematic approach to tuning these parameters could be explored.

These limitations provide promising directions for future research to further refine and extend our integrated approach to non-linear equation solving.

5. Conclusion

This research presents a novel integrated approach to solving non-linear algebraic and transcendental equations, combining modified versions of the Bisection and Regula Falsi methods. Our Hybrid Adaptive Root Finder (HARF) algorithm demonstrates significant improvements in performance, robustness, and efficiency compared to classical methods and their individual modifications.

Key contributions of this work include:

1. Development of the Dynamic Bisection method, which enhances the classical Bisection method through adaptive interval reduction and convergence acceleration techniques.
2. Introduction of the Adaptive Regula Falsi method, addressing the stagnation issues of the classical Regula Falsi method through dynamic weighting and bracket adjustment strategies.

3. Creation of the HARF algorithm, which intelligently integrates the modified Bisection and Regula Falsi methods to leverage their complementary strengths.
4. Comprehensive numerical experiments as discussed in table 1 to 4 demonstrating the superior performance of HARF across a diverse range of non-linear equations, including challenging cases with closely spaced roots or near-horizontal tangents.
5. Analysis of the convergence behavior, robustness, and computational efficiency of the proposed methods, providing insights into their practical applicability.

The results of our study have several important implications:

1. **Improved Toolset for Numerical Analysis:** HARF provides a powerful new tool for solving non-linear equations, offering performance approaching that of Newton's method without requiring derivative information.
2. **Enhanced Reliability:** The robustness of HARF across diverse function types suggests it could be particularly valuable in automated systems where the nature of the equations being solved may vary or be unknown in advance.
3. **Computational Efficiency:** The reduced number of iterations and function evaluations required by HARF could lead to significant time savings in applications involving repeated solution of non-linear equations as shown in table 1 to 4.
4. **Potential for Extension:** While our current work focuses on one-dimensional root-finding, the principles underlying HARF show promise for extension to multi-dimensional problems and systems of non-linear equations.

Future research directions stemming from this work include:

1. **Theoretical Analysis:** Developing a rigorous mathematical framework for analyzing the convergence properties and error bounds of the HARF algorithm.
2. **Multi-Dimensional Extension:** Adapting the HARF approach to solve systems of non-linear equations, potentially opening up new applications in fields such as non-linear optimization and computational physics.
3. **Application-Specific Tuning:** Investigating how the HARF algorithm can be further optimized for specific classes of problems arising in various scientific and engineering disciplines.
4. **Integration with Machine Learning:** Exploring the potential for machine learning techniques to predict optimal method switching criteria or to guide the selection of initial bracketing intervals.
5. **Parallel and Distributed Implementations:** Developing versions of HARF suitable for high-performance computing environments to tackle large-scale or real-time non-linear equation solving tasks.

In conclusion, the Hybrid Adaptive Root Finder represents a significant advancement in numerical methods for solving non-linear equations. By combining the reliability of Bisection-based methods with the potential for rapid convergence offered by Regula Falsi approaches, HARF provides a robust, efficient, and widely applicable tool for tackling a broad range of non-linear equation solving problems. As computational methods continue to play an increasingly crucial role across scientific and engineering disciplines, tools like HARF have the potential to accelerate research and innovation in fields ranging from physics and engineering to economics and data science.

Acknowledgement

Authors are thankful to Guru Gobind Singh Indraprastha University and University Grants Commission for research facilities and financial support and, also both authors would like to thanks the anonymous referees for his/her comments that helped us improve this article.

Conflict of interest

The authors declare no conflict of interest.

Data Availability

This article has no associated data.

Ethical Approval

Not applicable

Authors Contributions

Rashmi Bhardwaj: Conceptualization, Supervision, Investigation, Writing – Review & Editing.

Inderjeet: Methodology, Software, Writing – Original Draft.

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Received July, 2024