Journal of Interdisciplinary Mathematics
ISSN: 0972-0502 (Print), ISSN: 2169-012X (Online)
Vol. 28 (2025), No. 4, pp. 1417–1428
DOI : 10.47974/JIM-1919

Absolute linear method of summation with weighted mean

Aradhana Dutt Jauhari *
Divison of Mathematics
School of Basic Sciences
Galgotias University
Greater Noida 203201
Uttar Pradesh
India

Alka Munjal †
Department of Mathematics
School of Advanced Sciences
Vellore Institute of Technology
Andra Pradesh 522241
India

Santosh Kumar §
Department of Mathematics
School of Physical Sciences
North-Eastern Hill University
Shillong 793022
Meghalaya
India

Abstract

The study uses a summability method is utilized to characterize the set $(|R, p_n, \delta|_k, |R, q_n, \delta|_k)$ for $k \geq 1$ and any arbitrary positive sequences (p_n). Advanced results have been derived from previously established findings. This is significant because absolute summability is crucial for assessing the Bounded Input Bounded Output (BIBO) stability of systems. By imposing specific conditions on the main result, sufficient criteria for the summability of

* *E-mail:* **draradhana27@gmail.com (Corresponding Author)**
† *E-mail:* **alkamunjal8@gmail.com**
§ *E-mail:* **drsengar2002@gmail.com**

© Taru Publications

$|R, p_n|_k$ and $|C, 1|_k$ have been established, reinforcing the relevance and application of this study.

Subject Classification (2010): *46A45, 40D09, 40F05.*

Keywords: *Summation methods, Banach space, Riesz mean, Equivalance theorem.*

1. Introduction

One of the most important methods of exploration in the study of the many functions that arise in mathematical theory and its applications to science and engineering is expansion in series. The main advantage of this strategy is that it allows to express new and more intricate functions in terms of simpler and more familiar ones.

In 1817, Bolzano provided a clear description of convergence, while Cauchy independently and precisely defined it in 1821. Abel and Cauchy highlighted the importance of distinguishing between convergent and divergent series, expressing strong concerns and cautioning against the use of divergent series. Due to the issues they identified, divergent series lost favor within the mathematical community. Mathematicians have resisted conducting systematic research on these series for about 75 years. Abel and Cauchy were adamant in their warnings, but they weren't quite certain that the choice to completely ban the Divergent series was the right one. They probably understood that even the study of divergent series may offer possible mathematical insights and applications. Even though convergence was the main focus, dealing with convergent series alone posed difficulties. However, with time and developments in mathematical analysis, scholars were able to resolve these issues and provide more thorough explanations for diverse findings.

The study of mathematics is a dynamic field that constantly changes as fresh ideas, methods, and viewpoints are discovered. Even though some theories might have been dismissed or met with scepticism in the past, they can be reviewed and reassessed as our grasp of mathematics expands. In the study of mathematical series and sequences, convergence and divergence continue to be fundamental ideas. They also play a significant role in other disciplines of mathematics and its applications.

The total value of an infinite series,

$$u_0 + u_1 + u_2 + \dots \tag{1.1}$$

It is common to use Cauchy's method with the sum s_n of the first $n+1$ terms, specifically:

$$s_n = u_0 + u_1 + u_2 + \ldots + u_n. \tag{1.2}$$

The series (1.1) is said to have the sum s if the limit $\lim_{n\to\infty} s_n$ exists and is equal to the finite value s.

A key objective in advancing the theory of infinite series is to establish the fundamental rules for their operations. For instance, examine the product of (1.1) with another series.

$$v_0 + v_1 + v_2 + \ldots,$$

When both series are absolutely convergent with sums u and v respectively, it can be easily demonstrated that the Cauchy product series

$$w_0 + w_1 + w_2 + \ldots, \tag{1.3}$$

where,

$$w_n = u_0 v_n + u_1 v_{n-1} + \ldots + u_n v_0$$

The series converges and its sum w equals uv (a similar result holds under less stringent conditions). However, the convergence of the u-series and the v-series is not required for the convergence of the w-series.

Cesàro summability is a technique for assigning a sum to a divergent series by averaging its partial sums rather than directly considering them. This method often extends the concept of convergence, enabling some divergent series to be assigned meaningful sums.

Bohr and Riesz independently investigated the behavior of Dirichlet series under Cesàro summability. They showed that the range where Cesàro means converge to a finite value can exceed the range of convergence where the original series converges. Consequently, there are regions where the Dirichlet series diverges, yet Cesàro means still provide a convergent result. Although Cesàro summability was valuable for Dirichlet series, Riesz noted its limitations and introduced a new definition based on "typical means" [3] to address these issues while preserving some beneficial properties of Cesàro summability. Hardy and Riesz have extensively explored this summation method and its application to Dirichlet series.

Many authors like Bor [1, 2], Orhan and Sarigol [5] Sarigol [6, 7, 8, 9], Sulaiman [10, 11, 12] and Yeldiz [13, 14] worked on absolute summability factors by different means also Mishra and Rathour [15] worked on

Comparison of summability and Cesaro summability. Whereas this work shows the result on absolute summability with Reisz means in new dimensions with applications in the form of example and graph as well as corollaries also.

2. Preliminaries

Following definitions and theorems are important to establishing the main results of this paper.

Definition 2.1: Suppose $A=(a_{nk})$ be an infinite matrix and Σa_n be an infinite series with a sequence of its partial sums (s_n). Let

$$T_n = \sum_{\omega=0}^{\infty} a_{n\omega} s_\omega,\ n=0,1,... \tag{2.1}$$

exists (i.e. the series) by u_n, we denote the n^{th} (C,1) mean of the sequence $\{s_n\}$ is denoted as follows. The series Σa_n is said to be summable $|C,1;\delta|_k$, where, $k \geq 1, \delta \geq 0$ if

$$\sum_{n=1}^{\infty} n^{\delta k+k-1} |u_n - u_{n-1}|^k < \infty. \tag{2.2}$$

Definition 2.2: Consider the sequence (p_n) of positive numbers satisfying

$$P_n = \sum_{\omega=0}^{n} p_\omega \to \infty\, n \to \infty (P_{-i} = p_{-i} = 0, i \geq 1). \tag{2.3}$$

The transformation process,

$$t_n = \frac{1}{P_n} \sum_{\omega=0}^{n} p_\omega s_\omega; (P_n = 0), \tag{2.4}$$

The terms of the sequence (t_n) are given as the (R, p_n) means of the sequence (s_n) with the coefficients defined by (p_n). We define the series Σa_n is said to be summable under $|R, p_n; \delta|_k$, provided that $k \geq 1$ and $\delta \geq 0$, if

$$\sum_{n=1}^{\infty} n^{\delta k+k-1} |t_n - t_{n-1}|^k < \infty. \tag{2.5}$$

If $(p_n)=1$ for all n in $(R, p_n, \delta)_k$ summability reduces to $(C,1;\delta)_k$. Additionally, when $\delta = 0$, it further simplifies to $(R, p_n)_k$.

Remark 2.1: Suppose $k \geq 1$ and $A=(a_{nv})$ is an infinite matrix. In order that $A \in (l^k, l^k)$, required that $a_{nv} = O(1)$ for all $n, v \leq 0$. (Sarigol [7])

In 2022, Sarigol [7] proved results for $(|C,0|_k)$ and $(|R,p_n|_s)$ summability and provided the sufficient conditions.

Theorem 2.1: *For* $1<k\leq s<\infty$ *and* $\lambda=(\lambda_v)$ *is a sequence of numbers. Then,* $\lambda\in(|C,0|_k, |R,p_n|_s)$ *if and only if*

$$\left(\sum_{v=1}^{m}\frac{P_{v-1}^k}{v}|\lambda|^k\right)^{1/k}\left(\sum_{n=m}^{\infty}\left(\frac{n^{\frac{1}{s}}p_n}{P_nP_{n-1}}\right)^s\right)^{\frac{1}{s}}=O(1). \tag{2.6}$$

Theorem 2.2: *Let* $1<k\leq s<\infty$ *and* $\lambda=(\lambda_v)$ *be a sequence of numbers. Then,* $\lambda\in(|R,p_n|_k, |C,0|_s)$ *if and only if*

$$\left(\sum_{v=m-1}^{m}\frac{1}{v}\left(\frac{P_{v-1}P_v}{p_v}\right)^k\right)^{1/k}\left(\sum_{n=m}^{m+1}|\frac{n^{\frac{1}{s}}\lambda_n}{P_n}|^s\right)^{\frac{1}{s}}=O(1). \tag{2.7}$$

3. Main Result

A result has been established on $|R,p_n,\delta|_k$ and $|R,q_n,\delta|_k$ summability with a minimum set of sufficient conditions. A collection (X,Y) of all sequences such that- (X,Y) = $(\alpha=(\alpha_w)$: $\Sigma a_{nw}\alpha_w$ is summable by Y whenever Σa_w is summable X) is used in the present work for the $|R,p_n,\delta|_k$ summable factor, where X and Y are methods for summability.

Theorem 3.1: *The* $|R,p_n,\delta|_k$ *summability implies the* $|R,q_n,\delta|_k$, $k\geq 1$ *summability if and only if the following conditions are satisfied:*

(i) $\frac{q_vP_v}{Q_vp_v}=O(v^{\delta k+k-1})$,

(ii) $|\Delta Q_{v-1}|W_v=O\left(\frac{p_v}{P_v}\right)$,

(iii) $Q_vW_v=O(1)$,

where, $W_v=\left(\sum_{i=v+1}^{\infty}i^{\delta k+k-1}\left(\frac{q_i}{Q_iQ_i-1}\right)^k\right)^{\frac{1}{k}}$.

Proof: **Necessary Condition:**

We say that the series is summable under $|R,p_n;\delta|_k$ conditions, where $k\geq 1$ and $\delta\geq 0$, if

$$\sum_{n=1}^{\infty} n^{\delta k+k-1} |t_n - t_{n-1}|^k < \infty. \tag{3.1}$$

Let

$$\alpha_n = \frac{p_n}{P_n P_{n-1}} \sum_{\omega=1}^{n} P_{\omega-1} a_\omega, \tag{3.2}$$

$$\beta_n = \frac{q_n}{Q_n Q_{n-1}} \sum_{\omega=1}^{n} Q_{\omega-1} a_\omega, \tag{3.3}$$

for $n \geq 1$,

$$\gamma_n = \frac{q_n}{Q_n Q_{n-1}} \sum_{\omega=1}^{n-1} (P_\omega Q_{\omega-1} - Q_\omega P_{\omega-1}) + \frac{q_n P_n \beta_n}{p_n Q_n}. \tag{3.4}$$

We possess

$$n^{(\delta k+k-1)/k} \beta_n \Rightarrow n^{(\delta k+k-1)/k} \gamma_n. \tag{3.5}$$

Therefore, a series that is $|R, p_n; \delta|_k$ summable to be also $|R, q_n; \delta|_k$ summable if and only if A belongs to the interval (l^k, l'^k).

We are provided with the information that

$$\sum_{n=1}^{\infty} n^{\delta k+k-1} |\beta_n|^k \leq \infty, \tag{3.6}$$

whenever

$$\sum_{n=0}^{\infty} |\alpha_n| \leq \infty. \tag{3.7}$$

The space of sequences satisfying (3.7) qualifies as BK space if it is normal with respect to [4]

$$\|\alpha\| = \sum_{n=0}^{\infty} |\alpha_n| \leq \infty. \tag{3.8}$$

We are also examining the space of sequences that satisfy (3.6), which is identified as a BK space with respect to the norm

$$\|\beta_n\| = \{|\beta_0|^k + \sum_{n=1}^{\infty} n^{\delta k+k-1} |\beta_n|^k\}^{1/k}. \tag{3.9}$$

Therefore, (3.3) converts the space of sequences that satisfy (3.7) into the space that satisfies (3.6).

Applying B-S theorem in the usual way, we find there is a constant $M > o$ s.t.

$$\| \beta \| \le M \| \alpha \| \tag{3.10}$$

for all sequences satisfying (3.7). Taking any $\omega \ge 0$ using (3.10), we get $a_{\omega+1} = 1,\ \ a_i = 0,\ \ i \ne \omega + 1.$ then (3.2) and (3.3) becomes

$$\alpha_n = \begin{cases} 0 & \text{if } n < \omega + 1; \\ \frac{P_\omega p_n}{P_n P_{n-1}} & \text{if } n \ge \omega + 1. \end{cases}$$

$$\beta_n = \begin{cases} 0 & \text{if } n < \omega + 1; \\ \frac{Q_\omega q_n}{Q_n Q_{n-1}} & \text{if } n \ge \omega + 1. \end{cases}$$

by (3.8) and (3.9) $\| \alpha \| = 1$ and $\| \beta \| = Q_\omega W_\omega$ Now (3.10)

$$Q_\omega W_\omega \le M. \tag{3.11}$$

Now, from (3.10)

$$a_\omega = 1, a_{\omega+1} = -1, a_i = 0, i \ne \omega, i \ne \omega + 1. \tag{3.12}$$

Also for any fixed $\omega \ge 0$ then we have frome (3.7) and (3.8),

$$\alpha_n = \begin{cases} 0 & \text{if } n < \omega; \\ \frac{p_\omega}{P_\omega} & \text{if } n = \omega; \\ \frac{-p_\omega p_n}{P_n P_{n-1}} & \text{if } n > \omega. \end{cases}$$

$$\beta_n = \begin{cases} 0 & \text{if } n < \omega; \\ \frac{q_\omega}{Q_\omega} & \text{if } n = \omega; \\ \frac{q_n \Delta Q_{\omega-1}}{Q_n Q_{n-1}} & \text{if } n > \omega. \end{cases}$$

Thus, from (3.7) and (3.8)

$$\| \alpha \| = \frac{2p_\omega}{P_\omega},$$

$$\| \beta \| = \{ \omega^{\delta k + k - 1} \left(\frac{q_\omega}{Q_\omega} \right)^k + | \Delta Q_{\omega-1} |^k W_\omega^k \}^{1/k},$$

from (3.9), we obtain

$$\omega^{\delta k + k - 1} \left(\frac{q_\omega}{Q_\omega} \right)^k + | \Delta Q_{\omega-1} |^k W_\omega^k \le (2M)^k \left(\frac{p_\omega}{P_\omega} \right)^k.$$

This hold fo any $\omega \geq 0$, we get

$$\omega^{\delta k+k-1}\left(\frac{q_\omega}{Q_\omega}\right)^k + |\Delta Q_{\omega-1}|^k W_\omega^k = O\left(\frac{p_\omega}{P_\omega}\right)^k,$$

which gives condition (i) and (ii).

Sufficient Condition

By (3.2)

$$a_\omega = \frac{P_\omega}{p_\omega}\alpha_\omega - \frac{P_{\omega-2}}{p_{\omega-1}}\alpha_{\omega-1}, P_{-2} = P_{-1} = 0, \alpha_{-1} = 0. \tag{3.13}$$

By (3.2), (3.3) and (3.13),

$$\beta_n = \frac{q_n}{Q_n Q_{n-1}} \sum_{\omega=1}^{n} Q_{\omega-1} a_\omega, \tag{3.14}$$

$$= \frac{q_n}{Q_n Q_{n-1}} \sum_{\omega=1}^{n} Q_{\omega-1}\left(\frac{P_\omega}{p_\omega}\alpha_\omega - \frac{P_{\omega-2}}{p_{\omega-1}}\alpha_{\omega-1}\right), \tag{3.15}$$

$$= \frac{q_n}{Q_n Q_{n-1}} \sum_{\omega=1}^{n-1} (Q_{\omega-1}P_\omega - Q_\omega P_{\omega-1})\frac{\alpha_\omega}{p_\omega} + \frac{q_n P_n}{Q_n p_n}\alpha_n. \tag{3.16}$$

Let

$$\kappa_n = n^{(\delta k+k-1)/k}\beta_n, n \geq 1.$$

Then, we have

$$\kappa_n = \sum_{\omega=1}^{n} a_{n\omega}\alpha_\omega$$

$$a_{n\omega} = \begin{cases} n^{(\delta k+k-1)/k}\dfrac{q_n}{Q_n Q_{n-1}}\left(\dfrac{P_\omega}{p_\omega}\Delta Q_{\omega-1} + Q_\omega\right), & \text{if } 1 \leq \omega \leq n-1; \\ n^{(\delta k+k-1)/k}\dfrac{q_n P_n}{Q_n p_n}, & \text{if } \omega = n; \\ 0, & \text{if } \omega > n. \end{cases}$$

Hence, Σa_n is said to be summable by $|R, q_n, \delta|_k$, $k \geq 1$, when Σa_n is $|R, p_n, \delta|_k$ if and only if $\Sigma|\kappa_n|^k < \infty$, whenever $\Sigma|\alpha_n| < \infty$ and $sup_\omega \Sigma|a_{n\omega}|^k < \infty$.

By the definition of $A = (a_{n\omega})$, we have

$$\sum_{n=\omega}^{\infty} |a_{n\omega}|^k = \omega^{(\delta k+k-1)}\left(\frac{q_\omega P_\omega}{Q_\omega p_\omega}\right)^k + \left|\frac{P_\omega}{p_\omega}\Delta Q_{\omega-1} + Q_\omega\right|^k W_\omega^k.$$

From conditions (i), (ii) and (iii)

$$\sum_{n=\omega}^{\infty} |a_{n\omega}|^k = O(1), \omega \to \infty.$$

Hence the proof completed.

Example 3.1: Let (p_n) and (q_n) defined by $p_\omega = \frac{x^{\omega+1}}{x+1}$ and $q_\omega = (\omega+1)^\alpha$ where $x > 1$ and $\alpha > -1$.

$$P_\omega = \frac{x(1-x^{\omega+1})}{(x+1)(1-x)},$$

$$\frac{P_\omega}{p_\omega} \sim \frac{x}{x-1}, Q_\omega \sim \frac{\omega q_\omega}{\alpha} \text{ and } W_\omega = O(1/\omega+1),$$

one can now see that condition (i)-(iii) hold.

4. Graphical Representation of Riesz Summable Factor

The Riesz summable factor has been applied to an infinite series, in order to obtain the summable points. The details of the investigation are as follows:

Riesz summable factor has been used for finding sum of an infinite series which is given by

$$\Sigma a_n = 1 - \frac{1}{3} + \frac{1}{5} - \ldots$$

and a summable point has been determined. The behavior of the sequences $\{a_n\}$, $\{s_n\}$ and $\{t_n\}$ has been observed and the variation is demonstrated in Figure 1.

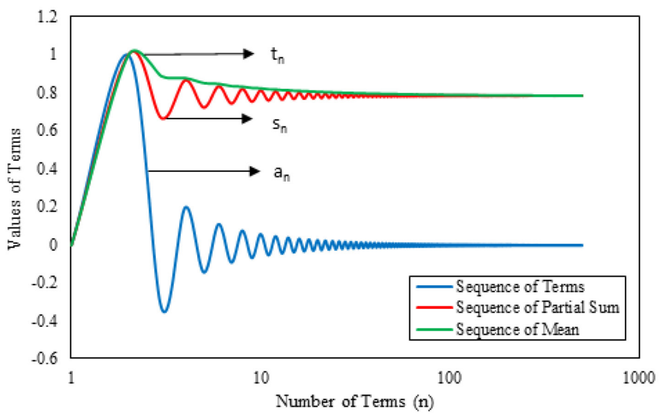

Figure 1

It has been observed that the values of terms of sequence $\{a_n\}$ oscillates and decreases continuously approaching to zero and do not approach towards the summable point (0.785398163). An impression of the summable point can be obtained by calculating the values of terms of the sequence of partial sum $\{s_n\}$, which still oscillates but approaches toward the summable point.

The values of the sequence of Riesz means $\{t_n\}$ have been calculated using the Riesz summability method, which helps in determining a sequence approaching towards the summable point.

The sequence p_n of the Riesz means is as follows: $p_n = 2,1,2,1,\ldots$. The typical values of the terms of the series $\{a_n\}$, $\{s_n\}$, $\{p_n\}$, $\{P_n\}$ and $\{t_n\}$ have been reported in

Typical values of the terms of the series $\{a_n\}$, $\{s_n\}$, $\{p_n\}$, $\{P_n\}$ and $\{t_n\}$

Number of terms	a_n	s_n	p_n	P_n	t_n
1	1	1	2	2	1
2	-0.333333333	0.666666667	1	3	0.888889
3	0.200000000	0.866666667	2	5	0.88
4	-0.142857143	0.723809524	1	6	0.853968
5	0.111111111	0.834920635	2	8	0.849206
6	-0.090909091	0.744011544	1	9	0.837518
7	0.076923077	0.820934621	2	11	0.834503
8	-0.066666667	0.754267954	1	12	0.827817
9	0.058823529	0.813091484	2	14	0.825713
10	-0.052631579	0.760459905	1	15	0.821363

Corollary 4.1: *If we set* $\delta = 0$ *and* $q_n = p_n$ *for all n, then* $|R, q_n|$ *summable, it implies the* $|R, p_n|_k$, $k \geq 1$ *summability which will not be Riesz summable for* $k > 1$, *Consequently,* Σa_n *which is* $|R, p_n|$ *summable but not* $|R, p_n|_k$, $k > 1$.

Corollary 4.2: *Let us take* $\delta = 0$, *then* $|R, p_n, \delta|_k$ *summability reduces to* $|R, p_n|_k$ *summability and this implies the* $|C, 1|_k$, $k \geq 1$ *summability if and only if*

$$\frac{P_\omega}{p_\omega} = O(\omega^{1/k}).$$

5. Declarations

Funding: The authors declare that no funding was received for this article.

Competing interests/Conflict of interest: The authors state that they have no conflicts of interest.

Author's contributions: Each author made equal and substantial contributions to the writing of this article, and all authors have reviewed and approved the final manuscript.

References

[1] H. Bor, "A note on two summability methods," *Proceedings Indian Academy Science* (Math. Sci.), vol. 102, no. 2, pp. 125–128 (1992).

[2] H. Bor, "On two summability methods," *Math. Proc. Cambridge Philos. Soc.,* vol. 97, pp. 147–149 (1985), doi: 0764503.

[3] R. D. Carmichael, "General Aspects of the Theory of Summable Series," *Bull. Amer. Math. Soc.,* vol. 25, pp. 97–131 (1918).

[4] I. J. Maddox, Elements of Functional Analysis, Cambridge University Press, London and New York (1970), doi: 0390692.

[5] C. Orhan and M. A. Sarigol, "On absolute weighted mean summability," *Rocky Mountain J. Math.*, vol. 23, no. 3, pp. 1091–1098 (1993).

[6] M. A. Sarigol, "Matrix transformations on fields of absolute weighted mean summability," *Studia Scientiarum Mathematicarum Hungarica,* vol. 8, no. 3, pp. 331–341 (2011).

[7] M. A. Sarigol, "Absolute Summability Factors for Cesaro and Riesz Means," *J. Applied Math and Informatics,* vol. 40, no. 1–2, pp. 283–288 (2022).

[8] M. A. Sarigol, "Extension of Mazhar's theorem on summability factors," *Kuwait J. Sci.,* vol. 42, no. 3, pp. 28–35 (2015).

[9] M. A. Sarigol, "On two absolute Riesz summability factors of infinite series," *Proceedings of the American Mathematical Society*, vol. 118, no. 2, pp. 485–488 (1993).

[10] W. T. Sulaiman, "A general theorem characterizing some absolute summability methods," *Proceedings of the Indian Academy of Sciences* (Mathematical Sciences), vol. 112, pp. 543–550 (2002).

[11] W. T. Sulaiman, "Extension on absolute summability factors of infinite series," *Journal of Mathematical Analysis and Applications*, vol. 322, no. 2, pp. 1224–1230 (2018).

[12] W. T. Sulaiman, "A recent note on the absolute Riesz summability factor of infinite series," *International Journal of Mathematical Archive,* vol. 2, no. 3, pp. 339–344 (2011).

[13] S. Yeldiz, "A new generalization on absolute Riesz summability," *AIP Conference Proceedings,* vol. 2183, p. 050017 (2019), doi: 10.1063/1.5136155.

[14] S. Yeldiz, "On absolute matrix summability factors of infinite series and Fourier series," *Bulletin des Sciences Mathematiques,* vol. 170, p. 103000 (2021).

[15] S. N. Mishra and L. Rathour, "Comparison of summability and Cesaro (C, α)-p summability," *Jnanabha,* vol. 52, no. 1, pp. 155–161 (2022).

Received January, 2024