Journal of Interdisciplinary Mathematics
ISSN: 0972-0502 (Print), ISSN: 2169-012X (Online)
Vol. 28 (2025), No. 3-B, pp. 1081–1088
DOI : 10.47974/JIM-2084

Topological free group-groupoids

Taghreed Hur Majeed
Department of Mathematics
College of Education
Mustensiriyah University
Baghdad
Iraq

Abstract

In this work, we introduced some new properties on topological group-groupoid and topological free group-groupoid. We obtain some results related to topological group-groupoid and topological free group-groupoid. In the end of this study, we get results written as propositions. Between the quantity y demand and the price.

Subject Classification: *Primary 93A30, Secondary 49K15.*

Keywords: *Topological group-groupoid, Topological free group-groupoid, Topological groupoid, Group-groupoid.*

1. Introduction

A group-groupoid Γ is a groupoid with a morphism of groupoids, the group multiplication m: $\Gamma\times\Gamma\to\Gamma$, defined as $(a,b)\to a.b$ for all $a,b\in\Gamma$ yielding a group structure internal to the category of groupoids, if the identity for the group structure on O_Γ is written e, then 1_e is the identity for the group structure on the set of arrows. In a group-groupoid Γ for a, $b\in\Gamma$, the groupoid composition is denoted by $a\circ b$ where $\alpha(a)=\beta(b)$ and the group multiplication by a.b,[3][5]. A morphism group-groupoid $h:\Gamma\_1\to\Gamma\_2$ is a morphism of the underling groupoids preserving the group structure, [2][5]. In a group interchange law hold:$(a\circ b)(c\circ d)=(ac)\circ(bd)$ where (ac) and (bd) are defined, [1]. Let Γ be group-groupoid, then the target, source and the object maps are morphism of groups, the group inversion μ: $\Gamma\to\Gamma$ defined by $a\to \bar{a}$ is a morphism of groupoids. The

This research was conducted during the NREUP at Michigan State University and it was sponsored by NSF grants DMS-1156582 and DMS-1359016.

E-mail: **taghreedmajeed@uomustansiriyah.edu.iq**

groupoid inversion δ: $\Gamma\rightarrow\Gamma$ is a morphism of groupoids, [5]. Let Γ topological groupoid, in case Γ be additionally a topological group as well, the following that maps are morphisms from topological groupoids next Γ be referred regarded as a topological group-groupoid:

(i) $m:\Gamma\times\Gamma\rightarrow\Gamma,(a,b)\rightarrow a+b$ Total group.

(ii) μ: $\Gamma\rightarrow\Gamma, a\rightarrow -a$ reverse map from the group.

(iii) $e:(*)\rightarrow\Gamma$ Where $*$ be an individual groupoid, [7]. We've got to interchange law $(a*b)+(c*d)=(a+c)*(b+d)$ where the two of them were $a*b$, $c*d$ are outlined, [8]. Let $(\Gamma, \oplus)$ is topological group that we acquire topological group-groupoid. $\Gamma\times\Gamma$ Given the object set of object Γ, the set of morphisms that be defined by $\Gamma\times\Gamma=\{(a,b):a,b\in\Gamma\}$,[5][8]. The composition be given by $(a,b)\oplus(c,d)=(a,d)$, Total group is given by $(a,b)\oplus(c,d)=(a\oplus c),(b\oplus d)$ Since Γ is a topological group, all structure maps of $\Gamma\times\Gamma$ become continuous since $\Gamma\times\Gamma$ has a product. be a topological group groupoid [5], [8]. Let Γ and Γ' be two topological group-groupoid, h: $\Gamma\rightarrow\Gamma'$ a morphism on the underlying topological groupoid called covering morphism of topological group-groupoid, [1], [7]. Let Γ a topological group-groupoid, a subset N of Γ is said to be a topological subgroup-groupoid if N is a subgroup -groupoid and topological subgroup of Γ. see [1], [6] and [4-11].

2. Topological Free Group-Groupoid

Topological group-groupoid base β of topological group-groupoid be the family $\{\Gamma_n\}_{n\in N}$ of open topological period subgroup-groupoid of Γ satisfy those conditions:

(i) $\{\Gamma_n\}_{n\in N}$ is represented essential system of neighborhood for zero in Γ. [9].

(ii) $\bigcap_{n\in N}\Gamma_n=\{0\}$.

(iii) $\Gamma_1\supset\Gamma_2\supset\Gamma_3\ldots$

(iv) $\bigoplus_{i\neq n}\Gamma_n=\Gamma$ if $\mu_j:\Gamma\rightarrow\Gamma_j$ be open continuous. Projective map then $\ker(\mu_j)=\bigoplus_{i\neq n}\Gamma_n$ and $\Gamma=Ker(\mu_j)\oplus\Gamma_n$ such that $n\geq j\geq 1$.

Group-oid topological structure Γ is called topological free group-groupoid if it has topological group-groupoid base. Zero topological group-groupoid be topological free group-groupoid and the base of topological group-groupoid be empty set. Every group-groupoid base is topological group-groupoid base but the opposite is not true. Every free group-groupoid base is group-groupoid base and base of topology. Every free group -groupoid is free topology but the opposite is not true.

Proposition 2-1: Let $\Gamma\cong M\oplus N$ be topological direct summand of M and N respectively, Γ be topological free group-groupoid, then N and M be Hausdorff - spaces.

Proof: By every topological direct summand of topological free group-groupoid be topological free group-groupoid thus M and N are topological free group-groupoid and topological trivial subgroup-groupoid is represent direction of Γ in zero then M and N be Hausdorff-space.

Proposition 2-2: Let $\{\Gamma_\alpha\}_{\alpha\in L}$ is family of topological group-groupoid, $\Gamma = \bigoplus_{\alpha\in L}\Gamma_\alpha$ be topological free group-groupoid if and only if Γ_α be Hausdorff-space for all $\alpha \in L$.

Proof: $\rightarrow$ Let for all Γ_α be Hausdorff -space thus for all Γ_α have topological group-groupoid base denoted by $\tilde{\beta}_\alpha$ that $\tilde{\beta} = \bigoplus_{\alpha\in L}\beta_\alpha$ be represent topological group-groupoid base then Γ be topological free group-groupoid by topological group-groupoid be topological free group-groupoid if it has topological group-groupoid base.

$\leftarrow$ The second part suppose Γ be topological free to show Γ_α is Hausdorff-space for all $\alpha \in L$, topological group-groupoid base $\tilde{\beta}$ of topological group-groupoid be the family $\{\Gamma_n\}_{n\in N}$ of open topological period subgroup-groupoid of Γ satisfy those conditions:

(i) $\{\Gamma_n\}_{n\in N}$ is represented essential system of neighborhood for zero in Γ.

(ii) $\bigcap_{n\in N}\Gamma_n = \{0\}$

(iii) $\Gamma_1 \supset \Gamma_2 \supset \Gamma_3 \ldots$

(iv) $\bigoplus_{i\neq n}\Gamma_n = \Gamma$ if $\mu_j: \Gamma \rightarrow \Gamma_j$ be open continuous. Projective map then $\mathrm{Ker}(\mu_j) = \bigoplus_{i\neq n}\Gamma_n$ and $\Gamma = Ker(\mu_j) \oplus \Gamma_n$ such that $n \geq j \geq 1$then Γ has topological group-groupoid base denoted by $\tilde{\beta}$, where $\tilde{\beta}$ is topological direct summand of open topological period sub group-groupoid from Γ that mean $\Gamma \cong \bigoplus_{n\in N}\Gamma_n$. Now, we obtain $\Gamma \cong \bigoplus_{\alpha\in\Gamma}\Gamma_\alpha$, then $\Gamma \cong \bigoplus_{\alpha\in L}\Gamma_\alpha \cong \bigoplus_{n\in N}\Gamma_n$ and by definition of topological direct summand, thus Γ_α be Hausdorff since $\bigoplus_{\alpha\in L}\Gamma_\alpha = \{e\}$ where Γ_α subset of Γ {e} closed set.

Proposition 2-3: Let $\{\Gamma_\alpha\}_{\alpha\in L}$ is family of topological group-groupoid, $\Gamma = \bigoplus_{\alpha\in L}\Gamma_\alpha$,for all $\alpha \in L$ be Hausdorff space if and only if for all Γ_α is topological free group groupoid.

Proof: $\leftarrow$ Let for all Γ be Hausdorff -space thus for all Γ have topological group-groupoid base denoted by β be represent topological group-groupoid base then Γ be topological free group-groupoid by topological group-groupoid be topological free group-groupoid if it has topological group-groupoid base.

$\rightarrow$The second part suppose Γ_α for all $\alpha \in L$ be topological free to show Γ is Hausdorff-space, topological group-groupoid base $\tilde{\beta}_\alpha$ that $\tilde{\beta} = \bigoplus_{\alpha\in L}\beta_\alpha$ of topological group-groupoid be the family $\{\Gamma_n\}_{n\in N}$ of open topological period subgroup-groupoid of Γ satisfy those conditions:

(i) $\{\Gamma_n\}_{n\in N}$ is represented essential system of neighborhood for zero in Γ.

(ii) $\bigcap_{n\in N}\Gamma_n = \{0\}$.

(iii) $\Gamma_1 \supset \Gamma_2 \supset \Gamma_3 \ldots$

(iv) $\bigoplus_{i \neq n} \Gamma_n = \Gamma$ if $\mu_j : \Gamma \to \Gamma_j$ be open continuous. Projective map then $\mathrm{Ker}(\mu_j) = \bigoplus_{i \neq n} \Gamma_n$ and $\Gamma = Ker(\mu_j) \oplus \Gamma_n$ such that $n \geq j \geq 1$then Γ has topological group-groupoid base denoted by $\tilde{\beta}$, where $\tilde{\beta}$ is topological direct summand of open topological period sub group-groupoid from Γ that mean $\Gamma \cong \bigoplus_{n \in N} \Gamma_n$. Now, we obtain $\Gamma \cong \bigoplus_{\alpha \in \Gamma} \Gamma_\alpha$, then $\Gamma \cong \bigoplus_{\alpha \in L} \Gamma_\alpha \cong \bigoplus_{n \in N} \Gamma_n$ and by definition of topological direct summand, thus Γ be Hausdorff since $\bigoplus_{\alpha \in L} \Gamma_\alpha = \{e\}$ where Γ_α subset of Γ {e} closed set.

Proposition 2-4: Let $\Gamma \cong M \oplus N$ is topological direct summand of M and N respectively, M is topological free group-groupoid then Γ/N be topological free group-groupoid too.

Proof: By the following diagram:

$$\begin{tikzcd}
 & M \arrow[d, "i"] \arrow[ddl, dashed, "h^*"'] \\
 & \Gamma/N \arrow[d, "h_1"] \arrow[dl, dashed, "h_3"'] \arrow[u, bend right, "i^{-1}"'] \\
D \arrow[r, "h_2"'] & B
\end{tikzcd}$$

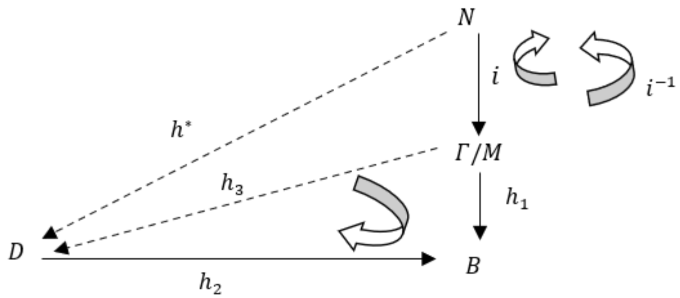
and $i: M \to \Gamma/N$ be homeomorphism and $i^{-1}: \Gamma/N \to M$ such that B and D be topological group-groupoid, $i: M \to \Gamma/N$ be topological group-groupoid homeomorphism and $h_2: D \to B$ be surjective topological group-groupoid homeomorphism, since N is topological group-groupoid, there exist $h^*: M \to D$ is topological group-groupoid homeomorphism such that:

$h_2 \circ h^* = h_1 \circ i$. Now we define
$h_3: \Gamma/N \to D$ by the form
$h_3 = h^* \circ i^{-1}$
We prove that $h_2 \circ h_3 = h_1$ by the formally

$$\begin{aligned}
h_2 \circ h_3 &= h_2 \circ (h^* \circ i^{-1}) \\
&= (h_2 \circ h^*) \circ i^{-1} \\
&= (h_1 \circ i) \circ i^{-1} \\
&= h_1 \circ (i \circ i^{-1}) \\
&= h_1
\end{aligned}$$
thus Γ/N be topological free group-groupoid.

Proposition 2-5: Let $\Gamma \cong N \oplus M$ is topological direct summand of N and M respectively, N is topological free group-groupoid topological free group-groupoid, then Γ/M be topological free group-groupoid topological free group-groupoid too.

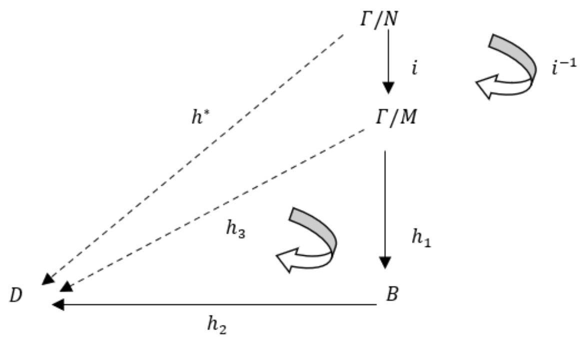
Proof: By proposition (2-4) and the diagram:



$i: N \rightarrow \Gamma/M$ and $i^{-1}: \Gamma/M \rightarrow N$

Proposition 2-6: Let Γ/N and Γ/M are topological group groupoid, Γ/N be a topological free group-groupoid, then Γ/M be topological free group-groupoid too.

Proof: By the following diagram



such that B and D be topological group-groupoid where $\Gamma/N = x + N$, $\Gamma/M = x + M$ be conjugate class for all $x \in \Gamma$, In algebra $\Gamma/N \cong \Gamma/M$, that's mean there exist map $i: \Gamma/N \rightarrow \Gamma/M$ for all x+ $\Gamma/M \neq \Gamma/M$ and for all $x, y \in \frac{of\Gamma}{M}, x = x_1 + N, y = y_1 + N\; x_1, y_1 \in \Gamma\;, x + \frac{\Gamma}{M} \neq y + \frac{\Gamma}{M}$ for all class of $\frac{\Gamma}{M}$, there exist class of $\frac{\Gamma}{M}$ thus $\frac{\Gamma}{N} \cong \frac{\Gamma}{M} i: \Gamma/M \rightarrow \Gamma/N$ be homeomorphism and $i^{-1}\Gamma/N \rightarrow \Gamma/M be$topological group-groupoid homeomorphism and $h_2: D \rightarrow B$ be surjective topological group-groupoid homeomorphism, since N is topological group-groupoid, there exist $h^*: \Gamma/M \rightarrow D$ is topological group-groupoid homeomorphism such that:

$h_2 \circ h^* = h_1 \circ i$. Now we define Γ/N

$h_3: \Gamma/N \rightarrow D$ by the form

$h_3 = h^* \circ i^{-1}$

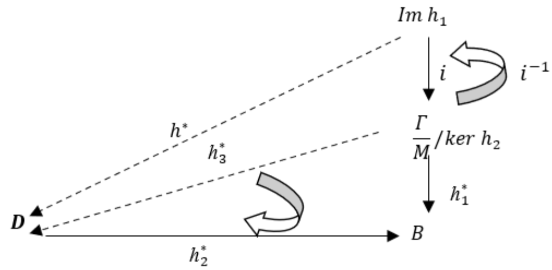
We prove that $h_2 \circ h_3 = h_1$ by the formally

$h_2 \circ h_3 = h_2 \circ (h^* \circ i^{-1})$

$= (h_2 \circ h^*) \circ i^{-1}$
$= (h_1 \circ i) \circ i^{-1}$
$= h_1 \circ (i \circ i^{-1})$
$= h_1$ thus Γ/M be topological free group-groupoid

Proposition 2-7: Let $\Gamma/M = Im\ h_1 \oplus ker\ h_2$ is direct summand of $Im\ h_1$ and $ker\ h_2$ respectively, $Im\ h_1$ is topological free group-groupoid then $\frac{\Gamma}{M}/ker\ h_2$ be topological free group-groupoid too where $\Gamma = N \oplus M$. Let $h_1: N \to \Gamma/M$, $h_2 = \frac{\Gamma}{M} \to N$.

Proof: By the following the diagram:



that B and D be topological group-groupoid where $Im\ h_1 = \{h_1(\mathrm{x}) \in \frac{\Gamma}{M}, h_1(x) = x + M, x \in N\}$ and $ker\ h_2 = \{z + M: h_2(z + M) = e$ in N for all $z \in N$, $\Gamma/M = x + M$ be conjugate class for all $x \in \Gamma$, In algebra $Im\ h_1 \frac{\cong \frac{\Gamma}{M}}{ker} h_2$, that's mean there exist map $i: Im\ h_1 \to \Gamma/M\ /ker\ h_2$ for all $x + M \neq (y + M) + (z + M)$ and for all $x, y \in M, x = x_1 + N, y = y_1 + N\ x_1, y_1 \in \Gamma, x + \frac{\Gamma}{M} \neq y + \frac{\Gamma}{M}$ for all class of $\frac{\Gamma}{M}$, there exist class of Γ/M thus $Im\ h_1 \cong \Gamma/M\ /ker\ h_2$ that be homeomorphism and $i^{-1}: \frac{\frac{\Gamma}{M}}{ker} h_2 \to Im\ h_1$ be topologicalgroup-groupoid homeomorphism and $h_2: D \to B$ be surjective topological group-groupoid homeomorphism, since N is topological group-groupoid, there exist $h^*: Im\ h_1 \to D$ is topological group-groupoid homeomorphism such that

$h_2 \circ h^* = h_1 \circ i$. Now we define$\Gamma/M\ /ker\ h_2$
$h_3: \Gamma/M\ /ker\ h_2 \to D$ by the form
$h_3 = h^* \circ i^{-1}$
We prove that $h_2 \circ h_3 = h_1$ by the formally
$h_2 \circ h_3 = h_2 \circ (h^* \circ i^{-1}) = (h_2 \circ h^*) \circ i^{-1}$
$= (h_1 \circ i) \circ i^{-1} = h_1 \circ (i \circ i^{-1})$
$= h_1$ thus $\Gamma/M\ /ker\ h_2$ be topological free group-groupoid

Proposition 2-8: Let $\{\Gamma_\alpha\}_{\alpha \in L}$ be family of topological group-groupoid then $\Gamma = \oplus_{\alpha \in L} \Gamma_\alpha$ be topological free group-groupoid if and only if for all Γ_α is not totally disconnected for all $\alpha \in L$.

Proof: Let for all Γ_α is not totally disconnected $\forall\alpha \in L$ the space is not totally disconnected if the direct of space is contain one point only for all element in the space. Then every direct in zero, Γ_α contains every topological group-groupoid in zero thus the topological group-groupoid is open, this base denoted by $\tilde{B}_\alpha$ then $\tilde{B} = \bigoplus_{\alpha\in L} \tilde{B}_\alpha$ is represent topological group-groupoid base of Γ, that means Γ be topological group-groupoid.

Let Γ be topological free group-groupoid thus Γ has topological group-groupoid base denoted by $\tilde{B}$ then $\tilde{B}$ be topological direct sum of open topological period subgroup- groupoid from Γ, that's mean $\Gamma \cong \bigoplus_{\alpha\in L} \Gamma_n$, we obtain $\Gamma \cong \bigoplus_{\alpha\in L} \Gamma_\alpha$ then $\Gamma \cong \bigoplus_{\alpha\in L} \Gamma_\alpha = \Gamma \cong \bigoplus_{\alpha\in L} \Gamma_n$, by definition of topological direct sum. We have topological free group-groupoid.

3. Conclusion

In this paper, we study topological group-groupoid and topological free group-groupoid. We give a new result have been obtained about topological group-groupoid and topological free group-groupoid were written in the form of new propositions.

References

[1] A. F. Özcan, "Actions and coverings of topological group-groupoids," *Math. Sci. Appl. E-Notes* (MSAEN), vol. 1, no. 2, pp. 72–83 (2013).

[2] A. F. Özcan and I. Icen, "Topological group-groupoids and equivalent categories," *J. Inst. Nat. Appl. Sci.*, vol. 27, no. 3, pp. 667–673 (Dec. 2022).

[3] A. M. Kumar, P. G. Chandra, and B. A. Priya, "Quotient on some generalization of topological group," *Baghdad Sci. J.*, vol. 20, no. 51 (Special Issue: ICAAM), pp. 400–408 (2023).

[4] A. R. Sadek and R. B. Esmaeel, "Some games via (D, DL) compact topological groups," *Iraqi J. Sci.*, vol. 65, no. 6, pp. 3395–3402 (2024).

[5] I. Icen, A. Fatih, and M. Habil, "Topological group-groupoids and their coverings," *Indian J. Pure Appl. Math.* (Sep. 2005).

[6] S. D. Mohsen, "Some generalizations of fuzzy soft (k∗–A^)(k^* – \hat{A})(k∗–A^)-quasinormal operators in fuzzy soft Hilbert spaces," J. Interdiscip. Math., vol. 25, no. 2, pp. 521–525 (2022).

[7] T. H. Majeed, "Algebraic topology on some results of category for source proper groupoid space," *J. Interdiscip. Math.*, vol. 26, no. 6, pp. 1133–1143 (2023).

[8] T. H. Majeed, "Some type of topological groupoids and their action," *AIP Conf. Proc.,* vol. 2834, Art. no. 080121 (Dec. 2023).

[9] M. G. Tkachenko, "On topologies of free groups," *Czechoslovak Math. J.,* vol. 34, no. 4, pp. 541–551 (1984).

[10] N. Taheri and A. K. K. Kaleel (Trans.), "Symbolic dynamics and topological complexity in discrete dynamical systems," *Babylonian J. Math.*, pp. 66–71 (2023). [Online]. Available: https://doi.org/10.58496/BJM/2023/013

[11] M. M. Abbood, H. H. Ebrahim, A. Al-Fayadh, and A. J. Obaid, "Measure defined on Γ–algebra and some of their generalizations," *J. Interdiscip. Math.,* vol. 26, no. 5, pp. 821–827 (2023), doi: 10.47974/JIM-1502.

Received October, 2024