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Some basic characteristic of center bi-topological space (*CBTS*)

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Abstract

We define the concept of center bi-topological space to study the center bi-open set. Also, we study bi-separation axioms. A few of these concepts' properties are also being studied.

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Introduction

Mathematicians have developed many methods to deal with uncertainty in mathematics, such as "fuzzy sets," "intuitionist fuzzy sets," "vague sets," "soft sets," and "fuzzy soft sets" [8].

Efremovic was the one who first proposed the concept of proximity space. Kandil et al. [9] used clearer proverbs than those found in the nearby region of Efremovic to present a subjective topology on the concealed set. Furthermore, proximity is an important consideration when dealing with problems that need human observation, like as image analysis [5] and facial recognition [7]. Two for a most crucial notion in a fixed point hypothesis are cyclic compression and the optimal closeness point, from which several conclusions have been drawn, including [6]. As of late, the ideal and soft set notions are the basis for a new approach to proximity structures [4] described

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most recently by El–Monsef, Abu–Donia, and Abd–Rabou [1]. The center set was introduced in [2] by exploiting proximity space.

This article introduces a notion for Cbi-topological space. Also, the relation between center bi-separation axioms of $\mathcal{C}BTS$ and center separation axioms of center topology be studied. Many authors were studied class of sets, see [14] and [15].

1. Basic Definitions and Notations

We start by recall some fundamental notions which are used throughout the course of our study.

Definition 1.1: [9, 10-13] If $\delta \subseteq P(\mathcal{D}) \times P(\mathcal{D})$ is a binary relation and $\mathcal{D}$ is a nonempty set, then $(\mathcal{D}, \delta)$ is a proximity space if and only if for all $ʐ', ʐ'', ʐ''' \subseteq \mathcal{D}$,

1. $ʐ' \delta\, ʐ''$ iff $ʐ'' \delta ʐ'$;
2. $ʐ' \delta (ʐ'' \cup ʐ''')$ iff $ʐ' \delta ʐ''$ or $ʐ' \delta\, ʐ'''$
3. $\overline{\mathcal{D}} \bar{\delta} \emptyset$;
4. $\{\omega\} \delta \{\omega\}$ for each $\omega \in \mathcal{D}$;
5. $ʐ' \delta\, ʐ''$ implies there exists $E \subseteq X$ where $ʐ' \delta E$ and $(\mathcal{D} - E)\, \bar{\delta}\, ʐ''$;

   It is false that $ʐ' \delta ʐ''$, where $ʐ' \overline{\delta} ʐ''$.

Definition 1.2: [2] If $(\mathcal{D}, \delta)$ is a proximity space such that $ʐ \subseteq \mathcal{D}$. We can define a center set as $\mathcal{C}_ʐ = \{\langle ʐ, ʐ' \rangle : ʐ \delta\, ʐ'\}$.

Definition 1.3: [2] If $\mathcal{C}_ʐ$ and $\mathcal{C}_ς$ is canter set over a proximity space $(\mathcal{D}, \delta)$. Then

1. $\mathcal{C}_{\mathcal{D}} = \{\langle \mathcal{D}, ʐ \rangle : \emptyset \neq ʐ \subseteq \mathcal{D}\}$ is called universe center set
2. $\mathcal{C}_{\emptyset} = \emptyset$ is called null center set.
3. $\mathcal{C}_ʐ \preccurlyeq_{\mathcal{C}} \mathcal{C}_ς$ iff $ʐ \subseteq ς$, and $\langle ʐ, C \rangle \in \mathcal{C}_ʐ \Rightarrow \langle ς, C \rangle \in \mathcal{C}_ς$.
4. $\mathcal{C}_ʐ = \mathcal{C}_ς$ iff $\mathcal{C}_ʐ \preccurlyeq_{\mathcal{C}} \mathcal{C}_ς$ and $\mathcal{C}_ς \preccurlyeq_{\mathcal{C}} \mathcal{C}_ʐ$.
5. Cop. $\mathcal{C}_ʐ = \{\langle \mathcal{D}, ς \rangle \in \mathcal{C}_{\mathcal{D}} : \langle ʐ, ς \rangle \notin \mathcal{C}_ʐ\} = \mathcal{C}_{\mathcal{D}}^{(ʐ,)}$
6. $\mathcal{C}_ʐ \curlyvee_{\mathcal{C}} \mathcal{C}_ς = \{\langle ʐ \cup ς, C \rangle : \langle ʐ, C \rangle \in \mathcal{C}_ʐ$ or $\langle ς, C \rangle \in \mathcal{C}_ς\}$.
7. $\mathcal{C}_ʐ \curlywedge_{\mathcal{C}} \mathcal{C}_ς = \{\langle ʐ \cup ς, C \rangle : \langle ʐ, C \rangle \in \mathcal{C}_ʐ$ and $\langle ς, C \rangle \in \mathcal{C}_ς\}$.
8. $\curlyvee_{\mathcal{C}} \mathcal{C}_{ʐ_i} = \{(\langle \cup ʐ_i, C \rangle : \langle ʐ_i, C \rangle \in \mathcal{C}_{ʐ_i}$ for some $i \in I\}$.
9. $\curlywedge_{\mathcal{C}} \mathcal{C}_{ʐ_i} = \{(\langle \cup ʐ_i, C \rangle : \langle ʐ_i, C \rangle \in \mathcal{C}_{ʐ_i}$ for each $i \in I\}$.
10. $\mathcal{C}_ʐ - \mathcal{C}_ς = \{\langle ʐ, C \rangle : \langle ʐ, C \rangle \in \mathcal{C}_ʐ$ and $\langle ς, C \rangle \notin \mathcal{C}_ς\}$.

Definition 1.4: [2] Let $(\mathcal{D}, \delta)$ be a proximity space and $\{\omega\}, \varsigma \subseteq \mathcal{D}$ such that $\{\omega\}\delta\varsigma$.

Then $\omega_\varsigma = \{\langle\{\omega\}, \varsigma\rangle\}$ is called acenter point in $\mathcal{D}$.

Definition 1.5: [7] if ω_ς is center point in $(\mathcal{D}, \delta)$ & C_Z center set in $(\mathcal{D}, \delta)$, then $\omega_\varsigma \in \mathrm{C}_Z$ iff $\omega \in Z$ & $(Z, \varsigma) \in \mathrm{C}_Z$.

Proposition 1.6: [2] If $\mathrm{C}_Z, \mathrm{C}_\varsigma, \mathrm{C}_C$, $\{\mathrm{C}_{Zi} : i \in I\}$ are center set, then

1. $\mathrm{C}_Z \preccurlyeq_\mathrm{C} \mathrm{C}_\varsigma$ & $\mathrm{C}_\varsigma \preccurlyeq_\mathrm{C} \mathrm{C}_C \Rightarrow \mathrm{C}_Z \preccurlyeq_\mathrm{C} \mathrm{C}_C$.
2. $\mathrm{C}_{Z_i} \preccurlyeq_\mathrm{C} \mathrm{C}_\varsigma \ \forall i \in I \Rightarrow \curlyvee_\mathrm{C} \mathrm{C}_{Zi} \preccurlyeq_\mathrm{C} \mathrm{C}_\varsigma$.
3. $\mathrm{C}_\varsigma \preccurlyeq_\mathrm{C} \mathrm{C}_{Z_i} \ \forall i \in I \Rightarrow \mathrm{C}_\varsigma \preccurlyeq_\mathrm{C} \curlywedge_\mathrm{C} \mathrm{C}_{Z_i}$.
4. Cop. $(\curlyvee_\mathrm{C} \mathrm{C}_{Z_i}) = \curlywedge_\mathrm{C}(\text{Cop. } \mathrm{C}_{Z_i})$.
5. Cop. $(\curlywedge_\mathrm{C} \mathrm{C}_{Z_i}) = \curlyvee_\mathrm{C}(\text{Cop. } \mathrm{C}_{Z_i})$.
6. $\mathrm{C}_Z \curlyvee_\mathrm{C} \mathrm{C}_Z = \mathrm{C}_A$, $\mathrm{C}_Z \curlywedge_\mathrm{C} \mathrm{C}_Z = \mathrm{C}_Z$, $\mathrm{C}_Z \curlyvee_\mathrm{C} \mathrm{C}_\emptyset = \mathrm{C}_Z$.
7. $\mathrm{C}_Z \curlyvee_\mathrm{C}$ Cop. $\mathrm{C}_Z = \mathrm{C}_\mathcal{D}$, $\mathrm{C}_Z \curlywedge_\mathrm{C}$ Cop. $\mathrm{C}_Z = \mathrm{C}_\emptyset$.

Definition 1.7: [3] if $(\mathcal{D}, \delta)$ is proximity space & $\mathfrak{J}_\mathrm{C} \subseteq$ cent $(\mathcal{D})$, then $\mathfrak{J}_\mathrm{C}$ is a center topology (C-topology) if:

1. $\mathrm{C}_\emptyset, \mathrm{C}_\mathcal{D} \in \mathfrak{J}_\mathrm{C}$
2. $\{\mathrm{C}_{Z_i} : i \in I\} \in \mathfrak{J}_\mathrm{C} \Rightarrow \curlyvee_\mathrm{C} \{\mathrm{C}_{Z_i} : i \in I\} \in \mathfrak{J}_\mathrm{C}$
3. $\mathrm{C}_{Z_1}, \mathrm{C}_{Z_2} \in \mathfrak{J}_\mathrm{C} \Rightarrow \mathrm{C}_{Z_1} \curlywedge_\mathrm{C} \mathrm{C}_{Z_2} \in \mathfrak{J}_\mathrm{C}$.

The triplet $(\mathcal{D}, \delta, \mathfrak{J}_\mathrm{C})$ is called a center topological space (C-topological space) and the members of $\mathfrak{J}_\mathrm{C}$ are said to be center open (C-open).

Definition 1.8: [4] A C-topological space $(\mathcal{D}, \delta, \mathfrak{J}_\mathrm{C})$ called $\mathfrak{J}_{\mathrm{C}0}$-space iff $\forall \omega_{\varsigma_1}, v_{\varsigma_2}$, center points s.t $\omega_{\varsigma_1} \neq_\mathrm{C} v_{\varsigma_2}$, there is at least one center open set C_{Z_1} or C_{Z_2} s.t $\omega_{\varsigma_1} \in \mathrm{C}_{Z_1}$, $v_{\varsigma_2} \notin \mathrm{C}_{Z_1}$ or $v_{\varsigma_2} \in \mathrm{C}_{Z_2}$, $\omega_{\varsigma_1} \notin \mathrm{C}_{Z_2}$.

Definition 1.9: [4] A C-topological space $(\mathcal{D}, \delta, \mathfrak{J}_\mathrm{C})$ is named $\mathfrak{J}_{\mathrm{C}1}$-space iff $\forall \omega_{\varsigma_1}, v_{\varsigma_2}$ are center points with $\omega_{\varsigma_1} \neq_\mathrm{C} v_{\varsigma_2}$ there are center open set C_{Z_1} & C_{Z_2} such that $\omega_{\varsigma_1}, \in \mathrm{C}_{Z_1}$, $v_{\varsigma_2} \notin \mathrm{C}_{Z_1}$ & $v_{\varsigma_2} \in \mathrm{C}_{Z_2}$, $\omega_{\varsigma_1} \notin \mathrm{C}_{Z_2}$.

Definition 1.10: [4] A C-topological space $(\mathcal{D}, \delta, \mathfrak{J}_\mathrm{C})$ is named $\mathfrak{J}_{\mathrm{C}2}$-space iff $\forall$ $\omega_{\varsigma_1}, v_{\varsigma_2}$, center points with $\omega_{\varsigma_1} \neq_\mathrm{C} v_{\varsigma_2}$, there is center open sets C_{Z_1} & C_{Z_2} such that $\omega_{\varsigma_1} \in \mathrm{C}_{Z_1}$, $v_{\varsigma_2} \in \mathrm{C}_{Z_2}$ & $\mathrm{C}_{Z_1} \curlywedge_\mathrm{C} \mathrm{C}_{Z_2} = \mathrm{C}_\emptyset$.

Definition 1.11: [4] The C-topological space $(\mathcal{D}, \delta, \mathfrak{J}_\mathrm{C})$ is named C-regular space if and only if $\mathrm{C}_\mathcal{D}^{(F)}$ a center closed set in $(\mathcal{D}, \delta, \mathfrak{J}_\mathrm{C})$ & $\omega_\varsigma \in \mathrm{C}_\mathcal{D}$ such that $\omega_\varsigma \notin \mathrm{C}_\mathcal{D}^{(F)}$, then there is a center open sets C_{Z_1} & C_{Z_2} such that $\omega_\varsigma \in \mathrm{C}_{Z_1}$, $\mathrm{C}_\mathcal{D}^{(F)} \preccurlyeq_\mathrm{C} \mathrm{C}_{Z_2}$ & $\mathrm{C}_{Z_1} \curlywedge_\mathrm{C} \mathrm{C}_{Z_2} = \mathrm{C}_\emptyset$.

Definition 1.12: [4] A $\complement$-topological space is said to be $\mathfrak{J}_{\complement 3}$-space, if and only if $\complement$-regular space and an $\mathfrak{J}_{\complement 1}$-space.

2. Center Bi-Topological Space:

Our focus here is on the center bi-topological space ($\complement$BTS). We also prove a few theorems and examples.

Definition 2.1: Let $(\mathcal{D}, \delta_1, \tau_{1cent(\mathcal{D})})$ and $(\mathcal{D}, \delta_2, \tau_{2cent(\mathcal{D})})$ be two center topological spaces. Then the ordered triple ($\mathcal{D}$, $\tau_{1cent(\mathcal{D})}$, $\tau_{2cent(\mathcal{D})}$) is named a center bi-topological space. (briefly $\complement\, BTS$).

Definition 2.2: Let $(\mathcal{D}, \tau_{1cent(\mathcal{D})}, \tau_{2cent(\mathcal{D})})$ be a $\complement\, BTS$. A center subset $\complement_U$ is said to be center bi-open provided $\complement_U$ is the center empty set, or there is center a open set $\complement_V$ in $\tau_{1cent(\mathcal{D})}$ and a center open set $\complement_W$ in $\tau_{2cent(\mathcal{D})}$ such that $\complement_V$ and $\complement_W$ are not the center empty set. $\complement_V \preccurlyeq_\complement \complement_U$ and $\complement_W \preccurlyeq_\complement \complement_U$.

Definition 2.3: Let $(\mathcal{D}, \tau_{1cent(\mathcal{D})}, \tau_{2cent(\mathcal{D})})$ be a $\complement BTS$. A center subset $\complement_{Ζ}$ is said to be center bi-closed provided its complement $\complement_{\mathcal{D}}^{(Ζ)}$ is center bi-open.

Theorem 2.4: *A center union for any family of center bi-open set is a center bi-open set.*

Proof: Let $(\mathcal{D}, \tau_{1cent(\mathcal{D})}, \tau_{2cent(\mathcal{D})})$ be a $\complement\, BTS$. Let $\beta_\complement$ be a collection of center bi-open subsets. Then $\beta_\complement = \{\complement_{Ui} / \ i \in I$ and $\complement_{Ui}$ is center bi-open $\}$. Let $\complement_H = \curlyvee_\complement \{\complement_{Ui} / \ i \in I\ \}$. Suppose that for all $i \in I$, $\complement_{Ui}$ is the center empty set. Then $\complement_H$ is the center empty set, and thus $\complement_H$ is center bi-open. So, suppose there is $\complement_{U1}$ is not the center empty set. $\complement_{U1}$ is center bi-open. Thus, there is a center set $\complement_V$ in $\tau_{1cent(\mathcal{D})}$ and a center set $\complement_W$ in $\tau_{2cent(\mathcal{D})}$ such that $\complement_V$ and $\complement_W$ are not the center empty set. $\complement_V \preccurlyeq_\complement \complement_{U1}$ and $\complement_W \preccurlyeq_\complement \complement_{U1}$. Thus $\complement_V \preccurlyeq_\complement \complement_{U1} \preccurlyeq_\complement \complement_H$ and $\complement_W \preccurlyeq_\complement \complement_{U1} \preccurlyeq_\complement \complement_H$.

Therefore, $\complement_H$ is center bi-open.

Remark 2.5: The following example shown that a center intersection of two center bi-open sets is not necessary be center bi-open set.

Example 2.6: If $\mathcal{D} = \{\varrho, \mathfrak{b}, \text{Ꭻ}\}$ & $\delta_1 = \delta_2 = \delta$ be the discrete proximity relation (Ζ δϚ if and only if
Ζ $\cap$ Ϛ $\neq \emptyset$),$\tau_{1cent(\mathcal{D})} = \{\complement_\emptyset, \complement_\mathcal{D}, \complement_{\{\varrho\}}, \complement_\mathcal{D}^{\{\varrho\}}, \complement_{\{\varrho,\mathfrak{b}\}}, \complement_{\{\mathfrak{b},Ꭻ\}}, \complement_{\{\mathfrak{b}\}}, \complement_\mathcal{D}^{\{\varrho,\mathfrak{b}\}}, \complement_\mathcal{D}^{\{\mathfrak{b},Ꭻ\}}, \complement_\mathcal{D}^{\{\mathfrak{b}\}},$
$\complement_{\{\mathfrak{b}\}} \curlywedge_\complement \complement_{\{\varrho\}}, \complement_{\{\varrho,\mathfrak{b}\}} \curlywedge_\complement \complement_{\{\varrho\}}, \complement_{\{\varrho,\mathfrak{b}\}} \curlywedge_\complement \complement_{\{\mathfrak{b}\}}, \complement_{\{\varrho,Ꭻ\}} \curlywedge_\complement \complement_{\{\mathfrak{b}\}}, \complement_{\{\varrho\}} \curlywedge_\complement \complement_{\{\mathfrak{b},Ꭻ\}},$
$\complement_{\{\varrho,\mathfrak{b}\}} \curlywedge_\complement \complement_{\{\mathfrak{b},Ꭻ\}}\}$ and

$\tau_{2cent(\mathcal{D})} = \{C_\emptyset, C_{\mathcal{D}}, C_{\{b\}}, C_{\mathcal{D}}^{\{b\}}, C_{\{J\}}, C_{\mathcal{D}}^{\{J\}}, C_{\{b,J\}}, C_{\mathcal{D}}^{\{b,J\}}, C_{\{b\}} \curlywedge_C C_{\{J\}}, C_{\{b\}} \curlywedge_C C_{\{b,J\}}, C_{\{J\}} \curlywedge_C C_{\{b,J\}}\}$.

Let $C_Z = C_{\{\varrho,b\}}$ and $C_\varsigma = C_{\{\varrho,J\}}$ then C_Z and C_ς be center bi-open sets because $C_{\{\varrho,b\}} \preccurlyeq_C C_{\{\varrho,b\}}$ and $C_{\{b\}} \preccurlyeq_C C_{\{\varrho,b\}}$ and $C_{\{\varrho\}} \preccurlyeq_C C_{\{\varrho,J\}}$ and $C_{\{J\}} \preccurlyeq_C C_{\{\varrho,J\}}$ but $C_Z \curlywedge_C C_\varsigma = \{\langle \mathcal{D}, \{\varrho\}\rangle, \langle \mathcal{D}, \mathcal{D}\rangle\}$ which is not center bi-open.

Definition 2.7: An $C\,BTS$ $(\mathcal{D}, \tau_{1cent(\mathcal{D})}, \tau_{2cent(\mathcal{D})})$ is said to be center bi-discrete provided, if $C_Z \preccurlyeq_C C_{\mathcal{D}}$ then C_Z is center bi-open.

Theorem 2.8: *An* $C\,BTS$ $(\mathcal{D}, \tau_{1cent(\mathcal{D})}, \tau_{2cent(\mathcal{D})})$ *is center bi-discrete if and only if each of* $(\mathcal{D}, \delta_1, \tau_{1cent(\mathcal{D})})$ *and* $(\mathcal{D}, \delta_2, \tau_{2cent(\mathcal{D})})$ *are center bi-discrete.*

Proof: $\Longrightarrow$ Let $(\mathcal{D}, \tau_{1cent(\mathcal{D})}, \tau_{2cent(\mathcal{D})})$ is center bi-discrete. Let $\omega_\varsigma \in C_{\mathcal{D}}$. Since $(\mathcal{D}, \tau_{1cent(\mathcal{D})}, \tau_{2cent(\mathcal{D})})$ is center bi-discrete, ω_ς is center bi-open. Thus, there is a center set C_V in $\tau_{1cent(\mathcal{D})}$ and center set C_W in $\tau_{2cent(\mathcal{D})}$ such that $C_V \neq C_\emptyset$ and $C_W \neq C_\emptyset$, $C_V \preccurlyeq_C \omega_\varsigma$ and $C_W \preccurlyeq_C \omega_\varsigma$ therefore, $C_V = \omega_\varsigma$ and $C_W = \omega_\varsigma$. Then, $\omega_\varsigma \in \tau_{1cent(\mathcal{D})}$ and $\omega_\varsigma \in \tau_{2cent(\mathcal{D})}$, thus $(\mathcal{D}, \delta_1, \tau_{1cent(\mathcal{D})})$ and $(\mathcal{D}, \delta_2, \tau_{2cent(\mathcal{D})})$ are center discrete spaces.

$\Longleftarrow$ Suppose that $(\mathcal{D}, \delta_1, \tau_{1cent(\mathcal{D})})$ and $(\mathcal{D}, \delta_2, \tau_{2cent(\mathcal{D})})$ are center discrete spaces. Let $C_Z \preccurlyeq_C C_{\mathcal{D}}$, since $(\mathcal{D}, \delta_1, \tau_{1cent(\mathcal{D})})$ and $(\mathcal{D}, \delta_2, \tau_{2cent(\mathcal{D})})$ are center discrete spaces, $C_Z \in \tau_{1cent(\mathcal{D})}$ and $C_Z \in \tau_{2cent(\mathcal{D})}$, $C_Z \preccurlyeq_C C_Z$. So C_Z is center bi-open. Therefore $(\mathcal{D}, \tau_{1cent(\mathcal{D})}, \tau_{2cent(\mathcal{D})})$ is a center bi-discrete $CBTS$.

3. Center Bi-Separation Axioms

Center bi-separation axioms are presented. In addition, this concept is demonstrated with examples and theorems.

Definition 3.1: A center bi-τ_{C_o}-space is a $CBTS$ $(\mathcal{D}, \tau_{1cent(\mathcal{D})}, \tau_{2cent(\mathcal{D})})$ such that, if $\omega_\varsigma \in C_{\mathcal{D}}$ and $v_C \in C_{\mathcal{D}}$, then there is a center bi-open set C_U which contains one center point and does not contain the other.

Definition 3.2: A center bi-τ_{C_1}-space is a $CBTS(\mathcal{D}, \tau_{1cent(\mathcal{D})}, \tau_{2cent(\mathcal{D})})$ such that, if $\omega_\varsigma \in C_{\mathcal{D}}$, $v_C \in C_{\mathcal{D}}$ and $\omega_\varsigma \neq v_C$, there are a center bi-open C_U with $\omega_\varsigma \in C_U$ &$v_C \notin C_U$.

Definition 3.3: A center bi-τ_{C_2}-space is a $CBTS(\mathcal{D}, \tau_{1cent(\mathcal{D})}, \tau_{2cent(\mathcal{D})})$, if $\omega_\varsigma \in C_{\mathcal{D}}$, $v_C \in C_{\mathcal{D}}$ & $\omega_\varsigma \neq v_C$, there is disjoint center bi-open C_U & C_V with $\omega_\varsigma \in C_U$ & $v_C \in C_V$.

Definition 3.4: A center bi-τ_{C_3}-space is a $CBTS(\mathcal{D}, \tau_{1cent(\mathcal{D})}, \tau_{2cent(\mathcal{D})})$ where $(\mathcal{D}, \tau_{1cent(\mathcal{D})}, \tau_{2cent(\mathcal{D})})$ is a center bi-τ_{C_1}-space if $\omega_\varsigma \in C_{\mathcal{D}}$ & C_D is center bi-closed subset of $C_{\mathcal{D}}$ with $\omega_{\varsigma} \notin C_D$, there is disjoint center bi-open C_U ,C_V with $\omega_{\varsigma} \in C_U$ & $C_D \preccurlyeq_C C_V$.

Definition 3.5: A center bi-τ_{C_4}-space is a $CBTS(\mathcal{D}, \tau_{1cent(\mathcal{D})}, \tau_{2cent(\mathcal{D})})$ where $(\mathcal{D}, \tau_{1cent(\mathcal{D})}, \tau_{2cent(\mathcal{D})})$ is a center bi-τ_{C_1}-space , if C_D and C_E are disjoint bi-closed subsets of $C_{\mathcal{D}}$, there is disjoint center bi-open C_U , C_V of $C_{\mathcal{D}}$ such that $C_D \preccurlyeq_C C_U$ & $C_E \preccurlyeq_C C_V$.

Theorem 3.6: *If* $(\mathcal{D}, \tau_{1cent(\mathcal{D})}, \tau_{2cent(\mathcal{D})})$ *is a center bi-*τ_{C_1}*-space and* $\omega_{\varsigma} \in C_{\mathcal{D}}$ *, then* ω_{ς} *is center bi-closed.*

Proof: Let $(\mathcal{D}, \tau_{1cent(\mathcal{D})}, \tau_{2cent(\mathcal{D})})$ be a center bi-τ_{C_1}-space. Let $\omega_{\varsigma} \in C_{\mathcal{D}}$. Let $v_C \in C_{\mathcal{D}} - \omega_{\varsigma}$ then v_C is not ω_{ς}. Since $(\mathcal{D}, \tau_{1cent(\mathcal{D})}, \tau_{2cent(\mathcal{D})})$ is a center bi-τ_{C_1}-space, there is a center bi-open C_U s.t $v_C \in C_U$ & $\omega_{\varsigma} \notin C_U$. So, $v_C \in C_U \preccurlyeq_C C_{\mathcal{D}} - \omega_{\varsigma}$. Thus, for each center point v_C of $C_{\mathcal{D}}$ there is a center bi-open C_U which include v_C and is contained in the center complement of ω_{ς}.

It remains to show that $C_{\mathcal{D}} - \omega_{\varsigma} = \curlyvee_C \{C_{U_{v_C}} / v_C \in C_{\mathcal{D}} - \omega_{\varsigma}\}$. Let $z_D \in C_{\mathcal{D}} - \omega_{\varsigma}$. Then there is a center bi-open set C_W such that $z_D \in C_W$, which is contained in $C_{\mathcal{D}} - \omega_{\varsigma}$. Thus $z_D \in \curlyvee_C \{C_{U_{v_C}} / v_C \in C_{\mathcal{D}} - \omega_{\varsigma}\}$.

Therefore $C_{\mathcal{D}} - \omega_{\varsigma} \preccurlyeq_C \curlyvee_C \left\{C_{U_{v_C}} / v_C \in C_{\mathcal{D}} - \omega_{\varsigma}\right\}$. Let $z_D \in \curlyvee_C \{C_{U_{v_C}} / v_C \in C_{\mathcal{D}} - \omega_{\varsigma}\}$. Thus, there is a C_U such that $z_D \in C_U \preccurlyeq_C C_{\mathcal{D}} - \omega_{\varsigma}$. So $z_D \in C_{\mathcal{D}} - \omega_{\varsigma}$. Therefore $\curlyvee_C \left\{C_{U_{v_C}} / v_C \in C_{\mathcal{D}} - \omega_{\varsigma}\right\} \preccurlyeq_C C_{\mathcal{D}} - \omega_{\varsigma}$. Hence $C_{\mathcal{D}} - \omega_{\varsigma} = \curlyvee_C \left\{C_{U_{v_C}} / v_C \in C_{\mathcal{D}} - \omega_{\varsigma}\right\}$.

Since each $C_{U_{v_C}}$ is center bi-open set, $\curlyvee_C \left\{C_{U_{v_C}} / v_C \in C_{\mathcal{D}} - \omega_{\varsigma}\right\}$ is center bi-open. (By Theorem 2.4). Thus, $C_{\mathcal{D}} - \omega_{\varsigma}$ is center bi-open. Thus, ω_{ς} is center bi-closed.

Theorem 3.7: *If* $(\mathcal{D}, \tau_{1cent(\mathcal{D})}, \tau_{2cent(\mathcal{D})})$ *is a center bi-*τ_{C_4}*-space, then* $(\mathcal{D}, \tau_{1cent(\mathcal{D})}, \tau_{2cent(\mathcal{D})})$ *is a center bi-*τ_{C_3}*-space.*

Proof: Let $(\mathcal{D}, \tau_{1cent(\mathcal{D})}, \tau_{2cent(\mathcal{D})})$ be a center bi-τ_{C_4}-space. Let $\omega_{\varsigma} \in C_{\mathcal{D}}$. Let C_V be a center bi-closed subset of $C_{\mathcal{D}}$ and $\omega_{\varsigma} \notin C_V$. Since $(\mathcal{D}, \tau_{1cent(\mathcal{D})}, \tau_{2cent(\mathcal{D})})$ be a center bi-τ_{C_4}- space, $(\mathcal{D}, \tau_{1cent(\mathcal{D})}, \tau_{2cent(\mathcal{D})})$ be a center bi-τ_{C_1}-space, and ω_{ς} is center bi-closed (by Theorem 3.6).

Also, there are disjoint center bi-open C_U, C_W, s.t $\omega_\varsigma \preccurlyeq_C C_U$ with $C_V \preccurlyeq_C C_W$. But $\omega_\varsigma \preccurlyeq_C C_U$ implies $\omega_\varsigma \in C_U$. So, $\omega_\varsigma \in C_U$, $C_V \preccurlyeq_C C_W$ & $C_U \curlywedge_C C_W = C_\emptyset$, hence, $(\mathcal{D}, \tau_{1cent(\mathcal{D})}, \tau_{2cent(\mathcal{D})})$ is a center bi-τ_{C_3}-space.

Theorem 3.8: *If* $(\mathcal{D}, \tau_{1cent(\mathcal{D})}, \tau_{2cent(\mathcal{D})})$ *is a center bi-*τ_{C_3}*-space, then* $(\mathcal{D}, \tau_{1cent(\mathcal{D})}, \tau_{2cent(\mathcal{D})})$ *is a center bi-*τ_{C_2}*-space.*

Proof: Let $(\mathcal{D}, \tau_{1cent(\mathcal{D})}, \tau_{2cent(\mathcal{D})})$ be a center bi-τ_{C_3}-space. Let $\omega_\varsigma \in C_\mathcal{D}$ and $v_C \in C_\mathcal{D}$ such that $\omega_\varsigma \neq v_C$. Since, $(\mathcal{D}, \tau_{1cent(\mathcal{D})}, \tau_{2cent(\mathcal{D})})$ is center bi-τ_{C_3}-space, $(\mathcal{D}, \tau_{1cent(\mathcal{D})}, \tau_{2cent(\mathcal{D})})$ is center bi-τ_{C_1}-space. Then (by Theorem 3.6) ω_ς is center bi closed. Thus, there are disjoint center bi-open sets C_U and C_W, such that $v_C \in C_U$ and $\omega_\varsigma \preccurlyeq_C C_W$. Since, $\omega_\varsigma \preccurlyeq_C C_W, \omega_\varsigma \in C_W$. So $v_C \in C_U, \omega_\varsigma \in C_W$ and $C_U \curlywedge_C C_W = C_\emptyset$. Thus, $(\mathcal{D}, \tau_{1cent(\mathcal{D})}, \tau_{2cent(\mathcal{D})})$ is a center bi-τ_{C_2}-space.

Theorem 3.9: *If* $(\mathcal{D}, \tau_{1cent(\mathcal{D})}, \tau_{2cent(\mathcal{D})})$ *is a center bi-*τ_{C_2}*-space, then* $(\mathcal{D}, \tau_{1cent(\mathcal{D})}, \tau_{2cent(\mathcal{D})})$ *is a center bi-*τ_{C_1}*-space.*

Proof: Let $(\mathcal{D}, \tau_{1cent(\mathcal{D})}, \tau_{2cent(\mathcal{D})})$ be a center bi-τ_{C_2}-space. Let $\omega_\varsigma \in C_\mathcal{D}$ and $v_C \in C_\mathcal{D}$ such that $\omega_\varsigma \neq v_C$. Since $(\mathcal{D}, \tau_{1cent(\mathcal{D})}, \tau_{2cent(\mathcal{D})})$ is center bi-τ_{C_2}-space. hence, there are disjoint center bi-open C_U & C_W, s.t $v_C \in C_U$, $\omega_\varsigma \in C_W$. But $C_U \curlywedge_C C_W = C_\emptyset$ implies that $v_C \notin C_W$. Thus, there is a center bi open C_W s.t $\omega_\varsigma \in C_W$ & $v_C \notin C_W$. So, $(\mathcal{D}, \tau_{1cent(\mathcal{D})}, \tau_{2cent(\mathcal{D})})$ is a center bi-τ_{C_1}-space.

Theorem 3.10: *If* $(\mathcal{D}, \tau_{1cent(\mathcal{D})}, \tau_{2cent(\mathcal{D})})$ *is a center bi-*τ_{C_1}*-space, then* $(\mathcal{D}, \tau_{1cent(\mathcal{D})}, \tau_{2cent(\mathcal{D})})$ *is a center bi-*τ_{C_o}*-space.*

Proof: Let $(\mathcal{D}, \tau_{1cent(\mathcal{D})}, \tau_{2cent(\mathcal{D})})$ be a center bi-τ_{C_1}-space. Let $\omega_\varsigma \in C_\mathcal{D}$ & $v_C \in C_\mathcal{D}$ s.t $\omega_\varsigma \neq v_C$. Since, $(\mathcal{D}, \tau_{1cent(\mathcal{D})}, \tau_{2cent(\mathcal{D})})$ is center bi-τ_{C_1}-space. Thus, there is a center bi-open set C_U, s.t $\omega_\varsigma \in C_U$ & $v_C \notin C_U$. So, $(\mathcal{D}, \tau_{1cent(\mathcal{D})}, \tau_{2cent(\mathcal{D})})$ is center bi-τ_{C_o}-space.

Remark 3.11: The converse is not generally true, as the following example demonstrates.

Example 3.12: Let $\mathcal{D} = \{\varrho, \mathfrak{b}, \jmath\}$ and $\delta_1 = \delta_2 = \delta$ be the discrete proximity relation $(Ꝣ\, \delta \varsigma \Leftrightarrow Ꝣ \cap \varsigma \neq \emptyset)$, $\tau_{1cent(\mathcal{D})} = \{C_\emptyset, C_\mathcal{D}, C_{\{\varrho\}}, C_\mathcal{D}^{\{\varrho\}}, C_{\{\varrho,\mathfrak{b}\}}, C_{\{\mathfrak{b}\}}, C_\mathcal{D}^{\{\varrho,\mathfrak{b}\}}, C_\mathcal{D}^{\{\mathfrak{b}\}}, C_{\{\mathfrak{b}\}} \curlywedge_C C_{\{\varrho\}}, C_{\{\varrho,\mathfrak{b}\}} \curlywedge_C C_{\{\varrho\}}, C_{\{\varrho,\mathfrak{b}\}} \curlywedge_C C_{\{\mathfrak{b}\}}\}$ and $\tau_{2cent(\mathcal{D})} = \{C_\emptyset, C_\mathcal{D}, C_{\{\varrho\}}, C_\mathcal{D}^{\{\varrho\}}\}$. Then, $(\mathcal{D}, \tau_{1cent(\mathcal{D})}, \tau_{2cent(\mathcal{D})})$ is a center bi-τ_{C_o}-space and is not a center bi-τ_{C_1}-space. Because the collection of all center bi open sets is

$\{\mathrm{C}_{\emptyset}, \mathrm{C}_{\mathcal{D}}, \mathrm{C}_{\{\varrho\}}, \mathrm{C}_{\mathcal{D}}^{\{\varrho\}}, \mathrm{C}_{\{\varrho,ƀ\}}, \mathrm{C}_{\{\varrho,ɉ\}}, \mathrm{C}_{\mathcal{D}}^{\{\varrho,ƀ\}}, \mathrm{C}_{\mathcal{D}}^{\{\varrho,ɉ\}}\}$. Thus, $(\mathcal{D}, \tau_{1cent(\mathcal{D})}, \tau_{2cent(\mathcal{D})})$ is a center bi-τ_{C_o}-space. But there is no center bi-open sets containing $ƀ_{\{ƀ\}}$ and not containing $\varrho_{\{\varrho\}}$. Hence, $(\mathcal{D}, \tau_{1cent(\mathcal{D})}, \tau_{2cent(\mathcal{D})})$ is not a center bi-τ_{C_1}-space.

Remark 3.13: The converse of (Theorem 3.11.) is not true in general as the following example show.

Example 3.14: Let $\mathcal{D} = \{\varrho, ƀ, ɉ\}$ and $\delta_1 = \delta_2 = \delta$ be the discrete proximity relation $(\zeta\, \delta \varsigma \Leftrightarrow \zeta \cap \varsigma \neq \emptyset)$, $\tau_{1cent(\mathcal{D})} = \{\mathrm{C}_{\emptyset}, \mathrm{C}_{\mathcal{D}}, \mathrm{C}_{\{\varrho\}}, \mathrm{C}_{\mathcal{D}}^{\{\varrho\}}, \mathrm{C}_{\{\varrho,ƀ\}}, \mathrm{C}_{\{ƀ\}}, \mathrm{C}_{\mathcal{D}}^{\{\varrho,ƀ\}}, \mathrm{C}_{\mathcal{D}}^{\{ƀ\}}, \mathrm{C}_{\{ƀ,ɉ\}}, \mathrm{C}_{\mathcal{D}}^{\{ƀ,ɉ\}}, \mathrm{C}_{\{\varrho\}} \curlywedge_{\mathrm{C}} \mathrm{C}_{\{\varrho,ƀ\}}, \mathrm{C}_{\mathcal{D}}^{\mathrm{C}_{\{\varrho\}} \curlywedge_{\mathrm{C}} \mathrm{C}_{\{\varrho,ƀ\}}}, \mathrm{C}_{\mathcal{D}}^{\mathrm{C}_{\{\varrho\}} \curlywedge_{\mathrm{C}} \mathrm{C}_{\{ƀ\}}}, \mathrm{C}_{\{ƀ\}} \curlywedge_{\mathrm{C}} \mathrm{C}_{\{\varrho\}}, \mathrm{C}_{\{\varrho,ƀ\}} \curlywedge_{\mathrm{C}} \mathrm{C}_{\{\varrho\}}, \mathrm{C}_{\{ƀ,ɉ\}} \curlywedge_{\mathrm{C}} \mathrm{C}_{\{\varrho,ƀ\}}, \mathrm{C}_{\mathcal{D}}^{\mathrm{C}_{\{ƀ,ɉ\}} \curlywedge_{\mathrm{C}} \mathrm{C}_{\{\varrho,ƀ\}}}, \mathrm{C}_{\{\varrho\}} \curlywedge_{\mathrm{C}} \mathrm{C}_{\{ƀ,ɉ\}}, \mathrm{C}_{\mathcal{D}}^{\mathrm{C}_{\{\varrho\}} \curlywedge_{\mathrm{C}} \mathrm{C}_{\{ƀ,ɉ\}}}, \mathrm{C}_{\{\varrho,ƀ\}} \curlywedge_{\mathrm{C}} \mathrm{C}_{\{ƀ\}}\}$ and $\tau_{2cent(\mathcal{D})} = \{\mathrm{C}_{\emptyset}, \mathrm{C}_{\mathcal{D}}, \mathrm{C}_{\{ƀ\}}, \mathrm{C}_{\mathcal{D}}^{\{ƀ\}}, \mathrm{C}_{\{ɉ\}}, \mathrm{C}_{\mathcal{D}}^{\{ɉ\}}, \mathrm{C}_{\{ƀ,ɉ\}}, \mathrm{C}_{\mathcal{D}}^{\{ƀ,ɉ\}}, \mathrm{C}_{\{ƀ\}} \curlywedge_{\mathrm{C}} \mathrm{C}_{\{ɉ\}}, \mathrm{C}_{\mathcal{D}}^{\mathrm{C}_{\{ɉ\}} \curlywedge_{\mathrm{C}} \mathrm{C}_{\{ƀ\}}}, \mathrm{C}_{\{ƀ\}} \curlywedge_{\mathrm{C}} \mathrm{C}_{\{ƀ,ɉ\}}, \mathrm{C}_{\mathcal{D}}^{\mathrm{C}_{\{ɉ,ƀ\}} \curlywedge_{\mathrm{C}} \mathrm{C}_{\{ƀ\}}}, \mathrm{C}_{\{ɉ\}} \curlywedge_{\mathrm{C}} \mathrm{C}_{\{ƀ,ɉ\}}, \mathrm{C}_{\mathcal{D}}^{\mathrm{C}_{\{ɉ,ƀ\}} \curlywedge_{\mathrm{C}} \mathrm{C}_{\{ɉ\}}}\}$.

Then, $(\mathcal{D}, \tau_{1cent(\mathcal{D})}, \tau_{2cent(\mathcal{D})})$ is a center bi-τ_{C_1}-space and is not a center bi-τ_{C_2}-space. Because the collection of all center bi open sets is $\{\mathrm{C}_{\emptyset}, \mathrm{C}_{\mathcal{D}}, \mathrm{C}_{\{ƀ\}}, \mathrm{C}_{\mathcal{D}}^{\{ƀ\}}, \mathrm{C}_{\{\varrho,ƀ\}}, \mathrm{C}_{\{ƀ,ɉ\}}, \mathrm{C}_{\mathcal{D}}^{\{\varrho,ƀ\}}, \mathrm{C}_{\mathcal{D}}^{\{ƀ,ɉ\}}, \mathrm{C}_{\{\varrho,ɉ\}}, \mathrm{C}_{\mathcal{D}}^{\{\varrho,ɉ\}}\}$. Thus, $(\mathcal{D}, \tau_{1cent(\mathcal{D})}, \tau_{2cent(\mathcal{D})})$ is a center bi-τ_{C_1}-space. But $(\mathcal{D}, \tau_{1cent(\mathcal{D})}, \tau_{2cent(\mathcal{D})})$ is not a center bi-τ_{C_2}-space. Hence, there are not disjoint center bi-open C_U, C_V s.t $\varrho_{\{\varrho\}} \in \mathrm{C}_U$ & $ɉ_{\{ɉ\}} \in \mathrm{C}_V$.

Theorem 3.15: *If $(\mathcal{D}, \delta_1, \tau_{1cent(\mathcal{D})})$ and $(\mathcal{D}, \delta_2, \tau_{2cent(\mathcal{D})})$ are center τ_{C_1}-spaces, then $(\mathcal{D}, \tau_{1cent(\mathcal{D})}, \tau_{2cent(\mathcal{D})})$ is a center bi-τ_{C_1}-space.*

Proof: Let $(\mathcal{D}, \delta_1, \tau_{1cent(\mathcal{D})})$ and $(\mathcal{D}, \delta_2, \tau_{2cent(\mathcal{D})})$ be center bi-τ_{C_1}-spaces. Let $\omega_{\varsigma} \in \mathrm{C}_{\mathcal{D}}$ and $\mathrm{v}_C \in \mathrm{C}_{\mathcal{D}}$ such that $\omega_{\varsigma} \neq \mathrm{v}_C$. Since $(\mathcal{D}, \delta_1, \tau_{1cent(\mathcal{D})})$ is center bi-τ_{C_1}-space, there is a center open set C_V in $\tau_{1cent(\mathcal{D})}$ such that $\omega_{\varsigma} \in \mathrm{C}_V$ and $\mathrm{v}_C \notin \mathrm{C}_V$, also there is a center open set C_U in $\tau_{2cent(\mathcal{D})}$ such that $\omega_{\varsigma} \in \mathrm{C}_U$ and $\mathrm{v}_C \notin \mathrm{C}_U$. Since $\mathrm{C}_U \preccurlyeq_{\mathrm{C}} \mathrm{C}_U \curlyvee_{\mathrm{C}} \mathrm{C}_V$ and $\mathrm{C}_V \preccurlyeq_{\mathrm{C}} \mathrm{C}_U \curlyvee_{\mathrm{C}} \mathrm{C}_V$, $\mathrm{C}_U \curlyvee_{\mathrm{C}} \mathrm{C}_V$ is center bi-open such that $\omega_{\varsigma} \in \mathrm{C}_U \curlyvee_{\mathrm{C}} \mathrm{C}_V$ and $\mathrm{v}_C \notin \mathrm{C}_U \curlyvee_{\mathrm{C}} \mathrm{C}_V$. Thus, $(\mathcal{D}, \tau_{1cent(\mathcal{D})}, \tau_{2cent(\mathcal{D})})$ is a center bi-τ_{C_1}-space.

Remark 3.16: The converse of (Theorem 3.15) is not true in general as the following example show.

Example 3.17: Let $\mathcal{D} = \{\varrho, ƀ, ɉ, d\}$ and $\delta_1 = \delta_2 = \delta$ be the discrete proximity relation $(\zeta\, \delta \varsigma \Leftrightarrow \zeta \cap \varsigma \neq \emptyset)$, $\tau_{1cent(\mathcal{D})} = \{\mathrm{C}_{\emptyset}, \mathrm{C}_{\mathcal{D}}, \mathrm{C}_{\{\varrho\}}, \mathrm{C}_{\mathcal{D}}^{\{\varrho\}}, \mathrm{C}_{\{\varrho,ƀ\}}, \mathrm{C}_{\mathcal{D}}^{\{\varrho,ƀ\}}, \mathrm{C}_{\{ƀ\}}, \mathrm{C}_{\mathcal{D}}^{\{ƀ\}}, \mathrm{C}_{\{\varrho\}} \curlywedge_{\mathrm{C}} \mathrm{C}_{\{\varrho,ƀ\}}, \mathrm{C}_{\mathcal{D}}^{\mathrm{C}_{\{\varrho\}} \curlywedge_{\mathrm{C}} \mathrm{C}_{\{\varrho,ƀ\}}}, \mathrm{C}_{\mathcal{D}}^{\mathrm{C}_{\{\varrho\}} \curlywedge_{\mathrm{C}} \mathrm{C}_{\{ƀ\}}}, \mathrm{C}_{\{b\}} \curlywedge_{\mathrm{C}} \mathrm{C}_{\{\varrho\}},$

$\mathsf{C}_{\{\varrho,\mathrm{b}\}} \curlywedge_{\mathsf{C}} \mathsf{C}_{\{\varrho\}}, \mathsf{C}_{\{\varrho,\mathrm{b}\}} \curlywedge_{\mathsf{C}} \mathsf{C}_{\{\mathrm{b}\}}\}$ and $\tau_{2cent(\mathcal{D})} = \{\mathsf{C}_{\emptyset}, \mathsf{C}_{\mathcal{D}}, \mathsf{C}_{\{\mathrm{b}\}}, \mathsf{C}_{\mathcal{D}}^{\{\mathrm{b}\}}, \mathsf{C}_{\{\text{ԓ}\}}, \mathsf{C}_{\mathcal{D}}^{\{\text{ԓ}\}}, \mathsf{C}_{\{\mathrm{b},\text{ԓ}\}}, \mathsf{C}_{\mathcal{D}}^{\{\mathrm{b},\text{ԓ}\}}, \mathsf{C}_{\{\mathrm{b},\text{ԓ},d\}}, \mathsf{C}_{\mathcal{D}}^{\{\mathrm{b},\text{ԓ},d\}}, \mathsf{C}_{\{\mathrm{b}\}} \curlywedge_{\mathsf{C}} \mathsf{C}_{\{\text{ԓ}\}}, \mathsf{C}_{\mathcal{D}}^{\mathsf{C}_{\{\mathrm{b}\}} \curlywedge_{\mathsf{C}} \mathsf{C}_{\{\text{ԓ}\}}}, \mathsf{C}_{\{\mathrm{b}\}} \curlywedge_{\mathsf{C}} \mathsf{C}_{\{\mathrm{b},\text{ԓ}\}}, \mathsf{C}_{\mathcal{D}}^{\mathsf{C}_{\{\mathrm{b}\}} \curlywedge_{\mathsf{C}} \mathsf{C}_{\{\mathrm{b},,\text{ԓ}\}}}, \mathsf{C}_{\{\text{ԓ}\}} \curlywedge_{\mathsf{C}} \mathsf{C}_{\{\mathrm{b},\text{ԓ}\}}, \mathsf{C}_{\mathcal{D}}^{\mathsf{C}_{\{\text{ԓ}\}} \curlywedge_{\mathsf{C}} \mathsf{C}_{\{\mathrm{b},,\text{ԓ}\}}}, \mathsf{C}_{\{\mathrm{b}\}} \curlywedge_{\mathsf{C}} \mathsf{C}_{\{\mathrm{b},\text{ԓ},d\}}, \mathsf{C}_{\mathcal{D}}^{\mathsf{C}_{\{\mathrm{b}\}} \curlywedge_{\mathsf{C}} \mathsf{C}_{\{\mathrm{b},\text{ԓ},d\}}}, \mathsf{C}_{\{\text{ԓ}\}} \curlywedge_{\mathsf{C}} \mathsf{C}_{\{\mathrm{b},\text{ԓ},d\}}, \mathsf{C}_{\mathcal{D}}^{\mathsf{C}_{\{\text{ԓ}\}} \curlywedge_{\mathsf{C}} \mathsf{C}_{\{\mathrm{b},\text{ԓ},d\}}}, \mathsf{C}_{\{\mathrm{b},\text{ԓ}\}} \curlywedge_{\mathsf{C}} \mathsf{C}_{\{\mathrm{b},\text{ԓ},d\}}, \mathsf{C}_{\mathcal{D}}^{\mathsf{C}_{\{\mathrm{b},,\text{ԓ}\}} \curlywedge_{\mathsf{C}} \mathsf{C}_{\{\mathrm{b},\text{ԓ},d\}}}\}$.

Then, $(\mathcal{D}, \tau_{1cent(\mathcal{D})}, \tau_{2cent(\mathcal{D})})$ is a center bi-τ_{C_1}-space and neither center topological space is a center τ_{C_1}-space. Because the family for every center bi open sets is $\{\mathsf{C}_{\emptyset}, \mathsf{C}_{\mathcal{D}}, \mathsf{C}_{\{\mathrm{b}\}}, \mathsf{C}_{\mathcal{D}}^{\{\mathrm{b}\}}, \mathsf{C}_{\{\varrho,\mathrm{b}\}}, \mathsf{C}_{\{\mathrm{b},\text{ԓ}\}}, \mathsf{C}_{\mathcal{D}}^{\{\varrho,\mathrm{b}\}}, \mathsf{C}_{\mathcal{D}}^{\{\mathrm{b},\text{ԓ}\}}, \mathsf{C}_{\{\varrho,\text{ԓ}\}}, \mathsf{C}_{\mathcal{D}}^{\{\varrho,\text{ԓ}\}}, \mathsf{C}_{\{\varrho,\text{ԓ},d\}}, \mathsf{C}_{\mathcal{D}}^{\{\varrho,\text{ԓ},d\}}, \mathsf{C}_{\{\mathrm{b},\text{ԓ},d\}}, \mathsf{C}_{\mathcal{D}}^{\{\mathrm{b},\text{ԓ},d\}}, \mathsf{C}_{\{\varrho,\mathrm{b},d\}}, \mathsf{C}_{\mathcal{D}}^{\{\varrho,\mathrm{b},d\}}, \mathsf{C}_{\{\varrho,\mathrm{b},\text{ԓ}\}}, \mathsf{C}_{\mathcal{D}}^{\{\varrho,\mathrm{b},\text{ԓ}\}}\}$. Thus, $(\mathcal{D}, \tau_{1cent(\mathcal{D})}, \tau_{2cent(\mathcal{D})})$ is a center bi-τ_{C_1}-space. But $(\mathcal{D}, \delta_1, \tau_{1cent(\mathcal{D})})$ is not a center τ_{C_1}-space. Since, there is no center open set containing $\text{ԓ}_{\{\text{ԓ}\}}$ and not contining $d_{\{d\}}$. Also, $(\mathcal{D}, \delta_2, \tau_{2cent(\mathcal{D})})$ is not a center τ_{C_1}-space. Since, there is no center open set containing $d_{\{d\}}$ and not contining $\text{ԓ}_{\{\text{ԓ}\}}$.

Theorem 3.18: *If $(\mathcal{D}, \delta_1, \tau_{1cent(\mathcal{D})})$ and $(\mathcal{D}, \delta_2, \tau_{2cent(\mathcal{D})})$ are center τ_{C_2}-spaces, then $(\mathcal{D}, \tau_{1cent(\mathcal{D})}, \tau_{2cent(\mathcal{D})})$ is a center bi-τ_{C_2}-space.*

Proof: It is evident.

Remark 3.19: The following example demonstrates that the opposite of Theorem 3.18. is not true in general.

Example 3.20: Let $\mathcal{D} = \{\varrho, \mathrm{b}, \text{ԓ}, d\}$ and $\delta_1 = \delta_2 = \delta$ be the discrete proximity relation ($\text{Ꝣ}\, \delta \varsigma \Leftrightarrow \text{Ꝣ} \cap \varsigma \neq \emptyset$), $\tau_{1cent(\mathcal{D})} = \{\mathsf{C}_{\emptyset}, \mathsf{C}_{\mathcal{D}}, \mathsf{C}_{\{\varrho\}}, \mathsf{C}_{\mathcal{D}}^{\{\varrho\}}, \mathsf{C}_{\{\varrho,b\}}, \mathsf{C}_{\mathcal{D}}^{\{\varrho,\mathrm{b}\}}, \mathsf{C}_{\{\text{ԓ},d\}}, \mathsf{C}_{\mathcal{D}}^{\{\text{ԓ},d\}}, \mathsf{C}_{\{\mathrm{b}\}}, \mathsf{C}_{\mathcal{D}}^{\{\mathrm{b}\}}, \mathsf{C}_{\{\mathrm{b}\}} \curlywedge_{\mathsf{C}} \mathsf{C}_{\{\varrho\}}, \mathsf{C}_{\mathcal{D}}^{\mathsf{C}_{\{\varrho\}} \curlywedge_{\mathsf{C}} \mathsf{C}_{\{\mathrm{b}\}}}, \mathsf{C}_{\{\mathrm{b},\text{ԓ},d\}}, \mathsf{C}_{\mathcal{D}}^{\{\mathrm{b},\text{ԓ},d\}}, \mathsf{C}_{\{\varrho,\mathrm{b}\}} \curlywedge_{\mathsf{C}} \mathsf{C}_{\{\mathrm{b},\text{ԓ},d\}}, \mathsf{C}_{\mathcal{D}}^{\mathsf{C}_{\{\varrho,\mathrm{b}\}} \curlywedge_{\mathsf{C}} \mathsf{C}_{\{\mathrm{b},\text{ԓ},d\}}}, \mathsf{C}_{\{\varrho,\mathrm{b}\}} \curlywedge_{\mathsf{C}} \mathsf{C}_{\{\text{ԓ},d\}}, \mathsf{C}_{\mathcal{D}}^{\mathsf{C}_{\{\varrho,\mathrm{b}\}} \curlywedge_{\mathsf{C}} \mathsf{C}_{\{\text{ԓ},d\}}}, \mathsf{C}_{\{\text{ԓ},d\}} \curlywedge_{\mathsf{C}} \mathsf{C}_{\{\mathrm{b},\text{ԓ},d\}}, \mathsf{C}_{\mathcal{D}}^{\mathsf{C}_{\{\text{ԓ},d\}} \curlywedge_{\mathsf{C}} \mathsf{C}_{\{\mathrm{b},\text{ԓ},d\}}}, \mathsf{C}_{\{\mathrm{b}\}} \curlywedge_{\mathsf{C}} \mathsf{C}_{\{\text{ԓ},d\}}, \mathsf{C}_{\mathcal{D}}^{\mathsf{C}_{\{\mathrm{b}\}} \curlywedge_{\mathsf{C}} \mathsf{C}_{\{\text{ԓ},d\}}}, \mathsf{C}_{\{\mathrm{b}\}} \curlywedge_{\mathsf{C}} \mathsf{C}_{\{\varrho,\mathrm{b}\}}, \mathsf{C}_{\mathcal{D}}^{\mathsf{C}_{\{\mathrm{b}\}} \curlywedge_{\mathsf{C}} \mathsf{C}_{\{\varrho,\mathrm{b}\}}}, \mathsf{C}_{\{\varrho\}} \curlywedge_{\mathsf{C}} \mathsf{C}_{\{,\text{ԓ},d\}}, \mathsf{C}_{\mathcal{D}}^{\mathsf{C}_{\{\varrho\}} \curlywedge_{\mathsf{C}} \mathsf{C}_{\{\text{ԓ},d\}}}, \mathsf{C}_{\{\varrho\}} \curlywedge_{\mathsf{C}} \mathsf{C}_{\{\varrho,\mathrm{b}\}}, \mathsf{C}_{\mathcal{D}}^{\mathsf{C}_{\{\varrho\}} \curlywedge_{\mathsf{C}} \mathsf{C}_{\{\varrho,\mathrm{b}\}}}\}$ and, $\tau_{2cent(\mathcal{D})} = \{\mathsf{C}_{\emptyset}, \mathsf{C}_{\mathcal{D}}, \mathsf{C}_{\{\mathrm{b}\}}, \mathsf{C}_{\mathcal{D}}^{\{\mathrm{b}\}}, \mathsf{C}_{\{\varrho\}}, \mathsf{C}_{\mathcal{D}}^{\{\varrho\}}, \mathsf{C}_{\{\varrho,\mathrm{b}\}}, \mathsf{C}_{\mathcal{D}}^{\{\varrho,\mathrm{b}\}}, \mathsf{C}_{\{\mathrm{b}\}} \curlywedge_{\mathsf{C}} \mathsf{C}_{\{\varrho\}}, \mathsf{C}_{\mathcal{D}}^{\mathsf{C}_{\{\mathrm{b}\}} \curlywedge_{\mathsf{C}} \mathsf{C}_{\{\varrho\}}}, \mathsf{C}_{\{\mathrm{b},\text{ԓ},d\}}, \mathsf{C}_{\mathcal{D}}^{\mathsf{C}_{\{\mathrm{b},\text{ԓ},d\}}}, \mathsf{C}_{\{\mathrm{b}\}} \curlywedge_{\mathsf{C}} \mathsf{C}_{\{\mathrm{b},\text{ԓ},d\}}, \mathsf{C}_{\mathcal{D}}^{\mathsf{C}_{\{\mathrm{b}\}} \curlywedge_{\mathsf{C}} \mathsf{C}_{\{\mathrm{b},\text{ԓ},d\}}}, \mathsf{C}_{\{\varrho\}} \curlywedge_{\mathsf{C}} \mathsf{C}_{\{\mathrm{b},\text{ԓ},d\}}, \mathsf{C}_{\mathcal{D}}^{\mathsf{C}_{\{\varrho\}} \curlywedge_{\mathsf{C}} \mathsf{C}_{\{\mathrm{b},\text{ԓ},d\}}}, \mathsf{C}_{\{\varrho,\mathrm{b}\}} \curlywedge_{\mathsf{C}} \mathsf{C}_{\{\varrho,\mathrm{b},d\}}, \mathsf{C}_{\mathcal{D}}^{\mathsf{C}_{\{\varrho,\mathrm{b}\}} \curlywedge_{\mathsf{C}} \mathsf{C}_{\{\mathrm{b},\text{ԓ},d\}}}\}$.
Then, $(\mathcal{D}, \tau_{1cent(\mathcal{D})}, \tau_{2cent(\mathcal{D})})$ is a center bi-τ_{C_2}-space and neither center topological space is a center τ_{C_2}-space. Because the collection of all center bi open sets is $\{\mathsf{C}_{\emptyset}, \mathsf{C}_{\mathcal{D}}, \mathsf{C}_{\{\mathrm{b}\}}, \mathsf{C}_{\mathcal{D}}^{\{\mathrm{b}\}}, \mathsf{C}_{\{\varrho\}}, \mathsf{C}_{\mathcal{D}}^{\{\varrho\}}, \mathsf{C}_{\{\varrho,\mathrm{b}\}}, \mathsf{C}_{\{\mathrm{b},\text{ԓ}\}}, \mathsf{C}_{\mathcal{D}}^{\{\varrho,\mathrm{b}\}}, \mathsf{C}_{\mathcal{D}}^{\{\mathrm{b},\text{ԓ}\}}, \mathsf{C}_{\{\varrho,\text{ԓ}\}}, \mathsf{C}_{\mathcal{D}}^{\{\varrho,\text{ԓ}\}}, \mathsf{C}_{\{\varrho,\text{ԓ},d\}}, \mathsf{C}_{\mathcal{D}}^{\{\varrho,\text{ԓ},d\}}, \mathsf{C}_{\{\mathrm{b},\text{ԓ},d\}}, \mathsf{C}_{\mathcal{D}}^{\{\mathrm{b},\text{ԓ},d\}}, \mathsf{C}_{\{\varrho,\mathrm{b},d\}}, \mathsf{C}_{\mathcal{D}}^{\{\varrho,\mathrm{b},d\}}, \mathsf{C}_{\{\varrho,\mathrm{b},\text{ԓ}\}}, \mathsf{C}_{\mathcal{D}}^{\{\varrho,\mathrm{b},\text{ԓ}\}}, \mathsf{C}_{\{\varrho,d\}}, \mathsf{C}_{\mathcal{D}}^{\{\varrho,d\}}, \mathsf{C}_{\{\mathrm{b},d\}}, \mathsf{C}_{\mathcal{D}}^{\{\mathrm{b},d\}}\}$. Thus, $(\mathcal{D}, \tau_{1cent(\mathcal{D})}, \tau_{2cent(\mathcal{D})})$ is a center bi-τ_{C_2}-space. But $(\mathcal{D}, \delta_1, \tau_{1cent(\mathcal{D})})$ is not a center τ_{C_2}-space. Since, there are not disjoint center open sets C_U and C_V such that $\text{ԓ}_{\{\text{ԓ}\}} \in \mathsf{C}_U$ and $d_{\{d\}} \in \mathsf{C}_V$.

Also, $(\mathcal{D}, \delta_2, \tau_{2cent(\mathcal{D})})$ is not a center τ_{C_2}-space. Since, there are not disjoint center open sets C_S and C_G such that $J_{\{J\}} \in C_S$ and $d_{\{d\}} \in C_G$.

Theorem 3.21: *If $\mathcal{D}$ is finite set, and if each of $(\mathcal{D}, \delta_1, \tau_{1cent(\mathcal{D})})$ and $(\mathcal{D}, \delta_2, \tau_{2cent(\mathcal{D})})$ are center τ_{C_4}-spaces, then $(\mathcal{D}, \tau_{1cent(\mathcal{D})}, \tau_{2cent(\mathcal{D})})$ is a center bi-τ_{C_4}-space.*

Proof: Let $\mathcal{D}$ be finite set, $(\mathcal{D}, \delta_1, \tau_{1cent(\mathcal{D})})$ and $(\mathcal{D}, \delta_2, \tau_{2cent(\mathcal{D})})$ be center bi-τ_{C_4}-spaces. Since $(\mathcal{D}, \delta_1, \tau_{1cent(\mathcal{D})})$ and $(\mathcal{D}, \delta_2, \tau_{2cent(\mathcal{D})})$ are center finite center τ_{C_1}-spaces. Then $(\mathcal{D}, \delta_1, \tau_{1cent(\mathcal{D})})$ and $(\mathcal{D}, \delta_2, \tau_{2cent(\mathcal{D})})$ are center discrete spaces. Thus (by Theorem 2.8), $(\mathcal{D}, \tau_{1cent(\mathcal{D})}, \tau_{2cent(\mathcal{D})})$ is a center bi-discrete space. Let $C_ζ$ and $C_ς$ be disjoint center bi-closed center sub sets of $C_{\mathcal{D}}$. Since, $(\mathcal{D}, \tau_{1cent(\mathcal{D})}, \tau_{2cent(\mathcal{D})})$ is a center bi-discrete space, $C_ζ$ and $C_ς$ are also center bi-open sub sets. But $C_ζ \preccurlyeq_C C_ζ$ and $C_ς \preccurlyeq_C C_ς$ and $C_ζ \curlywedge_C C_ς = C_\emptyset$. Hence, $(\mathcal{D}, \tau_{1cent(\mathcal{D})}, \tau_{2cent(\mathcal{D})})$ is a center bi-τ_{C_4}-space.

Collaray 3.22: *If $\mathcal{D}$ is finite set, and if each of $(\mathcal{D}, \delta_1, \tau_{1cent(\mathcal{D})})$ and $(\mathcal{D}, \delta_2, \tau_{2cent(\mathcal{D})})$ are center τ_{C_i}-spaces, then $(\mathcal{D}, \tau_{1cent(\mathcal{D})}, \tau_{2cent(\mathcal{D})})$ is a center bi-τ_{C_i}-space. For i=1,2,3,4.*

Reference

[1] M. E. Abd El–Monsef, H. M. Abu–Donia, and Kh. Abd–Rabou, "Common fixed point theorems of single and set-valued mappings on 2-2-metric spaces," *Applied Mathematics and Information Sciences,* vol. 2, pp. 185–194 (2007).

[2] A. D. Abdulsada and L. A. A. Al-Swidi, "Center set theory of Proximity Space," *Journal of Physics: Conference Series*, vol. 1804, 012130 (2021).

[3] A. D. Abdulsada and L. A. A. Al-Swidi, "Some Properties of C–Topological Space," in Proceedings of the First International Conference of Computer and Applied Sciences, *IEEE*, ISBN: 978-1-7281 (2019).

[4] A. D. Abdulsada and L. A. A. Al-Swidi, "Separation Axioms of Center Topological Space," *Journal of Advanced Research in Dynamical and Control Systems*, vol. 12, no. 5 (2020).

[5] P. Grzegrzolka and J. Siegert, "Coarse proximity and proximity at infinity," *Topology and Its Applications*, vol. 251, pp. 18–46 (2019).

[6] S. Gupta and K. Patnaik, "Enhancing performance of face recognition systems by using the near set approach for selecting facial features," *Journal of Theoretical and Applied Information Technology,* vol. 4, pp. 433–441 (2008).

[7] A. E. Hassanien, A. Abraham, J. F. Peters, G. Schaefer, and C. Henry, "Rough sets and near sets in medical imaging: a review," *IEEE Transactions on Information Technology in Biomedicine*, vol. 13, pp. 955–968 (2009).

[8] A. Kandil, O. A. Tantawy, S. A. El-Sheikh, and A. Zakaria, "I-proximity spaces," *Jokull Journal,* vol. 63, pp. 237–245 (2013).

[9] A. Kandil, O. A. Tantawy, S. A. El-Sheikh, and A. Zakaria, "New structures of proximity spaces," *Information Sciences Letters*, vol. 3, pp. 85–89 (2014).

[10] M. Kula and S. Ozkan, "Pre-Hausdorff and Hausdorff Proximity Spaces," *Filomat,* vol. 12, pp. 3837–3846 (2017).

[11] S. Selvarani, "Generalization of extreme value theorems in g-compact spaces, Generalization of Intermediate Value Theorem, g-connected spaces and generalization of Urysohn's Lemma and Tietze Extension Theorem in g-multiplicative spaces," *International Journal of Computer*, vol. 2, Apr. (2013).

[12] N. Taheri and A. K. K. Kaleel (Trans.), "Symbolic dynamics and topological complexity in discrete dynamical systems," *Babylonian Journal of Mathematics*, pp. 66–71 (2023). doi: 10.58496/BJM/2023/013.

[13] S. A. Al-Ameedee and A. J. Obaid, "Sandwich results of meromorphic multivalent functions defined by multiplier transform," *Journal of Interdisciplinary Mathematics,* OnlineFirst, pp. 1–8 (2023). doi: 10.47974/JIM-1484.

[14] I. S. Ahmed, A. Al-Fayadh, and H. H. Ebrahim, "Fuzzy σ-algebra and some related concepts," *AIP Conference Proceedings,* vol. 2834, no. 1 (2023). doi: 10.1063/5.0162037.

[15] I. S. Ahmed and H. H. Ebrahim, "On measure relatively to fuzzy σ-algebra," *Journal of Physics: Conference Series,* vol. 2322, no. 1 (2022). doi: 10.1088/1742-6596/2322/1/012020.

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