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Qualitative property to type of neutral differential equation

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Abstract

Our paper dealt with the study of oscillatory phenomena in neutral linear differential equations (NDEs). We obtained sufficient conditions to prove oscillatory phenomena. Some delays were explained to obtain some conditions for oscillation. In fact, all new results and conditions innovated and improved some oscillation properties, appearing in the literature with examples.

Subject Classification: *34K11.*

Keywords: *Oscillation phenomena, Several delays, Linear NDE, Differential operators.*

1. Introduction

The huge development of different fields in science has given rise to several physical laws, which are formulated as DEs. The applied problems mathematically using DEs, thus DEs play an important effect in solving practical problems [1-3]. The theory of DEs is a great tool in modeling lots of scientific problems in population, control theory and nonlinear problems [4, 5]. There are several papers studied some behaviors of solutions like asymptotic behavior, oscillatory property and stability to different kinds of DEs [4-8]. In publications, different authors concluded some conditions to asymptotic property and oscillation behavior for NDEs [9-11].

2. Main Results

Consider the differential equation:

$$\frac{d}{dt}\left[\pounds(t)\frac{d}{dt}Z(t)\right] + \Upsilon_2(t)\,x(\gamma(t)) + \Upsilon_3(t)\,x(\sigma(t)) = 0,\ t \geq t_0 > 0 \qquad (1)$$

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Suppose that: $Z(t) = x(t) + \Upsilon_1(t)x(\Im(t))$,

$$\pounds, \Upsilon_1, \Im, \gamma, \sigma \in C^1[t_0, \infty)$$

Supposing the below conditions hold:

$H1$: Let $\pounds(t),\ \Upsilon_2(t), \Upsilon_3(t) > 0$ with $0 \le \Upsilon_1(t) \le \Upsilon_0 < \infty$.

$H2$: Let $\frac{d}{dt}\Im(t) \le \Im_0,\ \sigma(t), \gamma(t) < t$,

$$\lim_{t\to\infty}\Im(t) = \lim_{t\to\infty}\gamma(t) = \lim_{t\to\infty}\sigma(t) = \infty$$

$H3$: Let $\eta(t) = \int_{t_0}^{t} \frac{ds}{\pounds(s)} \to \infty$, as $t \to \infty$

Definition 1.1: [11] A nontrivial solution $x(t)$ to eq. (1) is called nonoscillatory if it is either eventually a positive solution or it is eventually a negative, if it doesn't, so $x(t)$ satisfies oscillatory property.

Lemma 1: *Let x is positive solution to equation (1) then $Z(t)$ satisfies the following:*

$$Z(t) > 0, \frac{d}{dt}Z(t) > 0,\ \frac{d}{dt}[\pounds(t)\frac{d}{dt}Z(t)] < 0, \text{ moreover}$$

$\mathfrak{H}(t) = \pounds(t)\frac{d}{dt}Z(t)$ is decreasing.

Proof: Suppose that $x(t)$ is positive to equation (1). Then:

$$\frac{d}{dt}\left[\pounds(t)\frac{d}{dt}Z(t)\right] = -\Upsilon_2(t)\,x\,(\gamma(t)) - \Upsilon_3(t)\,x\,(\sigma(t)) < 0$$

$\Rightarrow \pounds(t)\frac{d}{dt}Z(t)$ is decreasing.

So either $\pounds(t)\frac{d}{dt}Z(t) > 0$ or $\pounds(t)\frac{d}{dt}Z(t) < 0$

If $\pounds(t)\frac{d}{dt}Z(t) < 0$, then:$\exists\ \Lambda$ a positive constant:

$$\pounds(t)\frac{d}{dt}Z(t) \le -\Lambda < 0$$

$$\frac{d}{dt}Z(t) \le \frac{-\Lambda}{\pounds(t)} \qquad (2)$$

Integrating the last inequality(2)from $t_1 \to t$

$$\int_{t_1}^{t}\frac{d}{ds}Z(s)\,ds \le \int_{t_1}^{t}\frac{-\Lambda}{\pounds(s)}ds$$

$$Z(t) - Z(t_1) \le \Lambda\int_{t_1}^{t}\frac{-1}{\pounds(s)}ds \to \infty \text{ as } t \to \infty$$

So $Z(t) \to -\infty$ as $t \to \infty$

This a contradiction with assumption $Z(t) > 0$.

Then $\pounds(t)\frac{d}{dt}Z(t) > 0$, so we get: $\frac{d}{dt}Z(t) > 0$

Theorem 2.1: *Let the conditions $H1$-$H3$ hold and*

$\Im o\gamma = \gamma o\Im, \Im o\sigma = \sigma o\Im$ with $\gamma(t) \le \sigma(t)$

Assume that the first order neutral differential inequality:

$$\frac{d}{dt}\left[5(t) + \frac{\Upsilon_0}{\Im_0}\, 5(\Im(t))\right] + 5(\sigma(t)$$

$$\{\Gamma_1(t)\,[\eta(t) - \eta(t_0)] + \Gamma_2(t)\,[\eta(t) - \eta(t_0)]\} \le 0 \quad (3)$$

Does not have positive solution then eq. (1) oscillates.

Proof: Suppose that $x(t)$ is a positive solution of eq. (1)

By replacing $\Im(t)$ in equation(1),

We Obtain:

$$\frac{d}{d\Im(t)}\left[\pounds(\Im(t))\frac{d}{d\Im(t)} z(\Im(t))\right] + \Upsilon_2(\Im(t))\, x(\gamma(\Im(t))) +$$

$$\Upsilon_3(\Im(t))\, x(\sigma(\Im(t))) = 0$$

$$\frac{1}{d\Im/_{dt}}\frac{d}{dt}\left[\pounds(\Im(t))\frac{d}{dt} z(\Im(t))\right] + \Upsilon_2(\Im(t))\, x(\gamma(\Im(t)))$$

$$+\Upsilon_3(\Im(t))\, x(\sigma(\Im(t))) = 0$$

We multiply the last equation by Υ_0 and we use H2 so we get:

$$0 = \frac{\Upsilon_0}{d\Im/_{dt}}\frac{d}{dt}\left[\pounds(\Im(t))\frac{d}{dt} z(\Im(t))\right] + \Upsilon_0\, \Upsilon_2(\Im(t))\, x(\gamma(\Im(t)))$$

$$+\Upsilon_0\Upsilon_3(\Im(t))\, x(\sigma(\Im(t)))$$

$$\ge \frac{\Upsilon_0}{\Im_0}\frac{d}{dt}\left[\pounds(\Im(t))\frac{d}{dt} z(\Im(t))\right] + \Upsilon_0\, \Upsilon_2(\Im(t))\, x(\gamma(\Im(t)))$$

$$+\Upsilon_0\Upsilon_3(\Im(t))\, x(\sigma(\Im(t))) \quad (4)$$

From (1) and (3) we can find:

$$\Upsilon_2(t)\, x(\gamma(t)) + \Upsilon_3(t)\, x(\sigma(t)) + \Upsilon_0\, \Upsilon_2(\Im(t))\, x(\gamma(\Im(t))) +$$

$$\Upsilon_0\Upsilon_3(\Im(t))\, x(\sigma(\Im(t))) \ge \Gamma_1(t)(x(\gamma(t)) + \Upsilon_0 x(\gamma(\Im(t))))$$

$$\Gamma_2(t)\left(x(\sigma(t) + y_0\, x(\sigma(\Im(t)))\right)$$

$$\ge \Gamma_1(t)\, z(\gamma(t)) + \Gamma_2(t)\, z(\sigma(t)) \quad (5)$$

Combing (1), (4) and (5), we can obtain:

$$\frac{d}{dt}\left[\pounds(t)\frac{d}{dt} z(t)\right] + \frac{\Upsilon_0}{\Im_0}\frac{d}{dt}\left[\pounds(\Im(t))\frac{d}{dt} z(\Im(t))\right] + \Gamma_1(t)\, z(\gamma(t)) +$$

$$\Gamma_2(t)\, z(\sigma(t)) \le 0 \quad (6)$$

Since $\pounds(t)\frac{d}{dt} z(t)$ is decreasing:

Let $5(t) = \pounds(t)\frac{d}{dt} z(t)$, $\frac{d}{dt} z(t) = \frac{5(t)}{\pounds(t)}$

$$\int_{t_1}^{t} \frac{d}{ds} z(s)\,ds = \int_{t_1}^{t} \frac{5(s)}{\pounds(s)}\,ds$$

$$z(t) - z(t_1) = \int_{t_1}^{t} \frac{5(s)}{\pounds(s)}\,ds$$

But $5(t) \le 5(t_1)$, $\forall\, t \ge t_1$

$$Z(t) \geq Z(t) - Z(t_1) \leq 5(t) \int_{t_1}^{t} \frac{1}{\pounds(s)}\, ds$$
$$= 5(t)[\eta(t) - \eta(t_0)]$$
$$Z(t) \geq 5(t)[\eta(t) - \eta(t_0)] \tag{7}$$

By setting (7) in (6) we get:

$$\frac{d}{dt}\left[5(t) + \frac{\Upsilon_0}{\Im_0} 5(\Im(t))\right] + \Gamma_1(t)\, 5(\gamma(t))\; [\eta(\gamma(t))$$
$$-\eta(\gamma(t_0)] + \Gamma_2(t)\, 5(\sigma(t))[\eta(\sigma(t)) - \eta(\sigma(t_0))] \leq 0$$

Since $\gamma(t) \leq \sigma(t)$ then $5(\gamma(t)) \geq 5(\sigma(t))$

$$\frac{d}{dt}\left[5(t) + \frac{\Upsilon_0}{\Im_0} 5(\Im(t))\right] + 5(\sigma(t))$$
$$\{\Gamma_1(t)[\eta(\gamma(t)) - \eta(\gamma(t_0)] + \Gamma_2(t)[\eta(\sigma(t)) - \eta(\sigma(t_0))]\} \leq 0$$

But $5(t)$ is a positive, a contradiction.

Theorem 2.2: *Let the conditions Ӈ1-Ӈ3 hold and* $\Im o \gamma = \gamma o \Im, \Im o \sigma = \sigma o \Im$ *with* $\gamma(t) \geq \sigma(t)$

Assume that the first order neutral differential inequality

$$\frac{d}{dt}(5(t) + \frac{\Upsilon_0}{\Im_0} 5(\Im(t))) + 5(\gamma(t))$$
$$\{\Gamma_1(t)\,[\eta(t) - \eta(t_0)] + \Gamma_2(t)\,[\eta(t) - \eta(t_0)]\} \leq 0 \tag{8}$$

Has no positive solution then equation (1) oscillates.

Proof: The proof is similar to theorem (2.1) [end of proof]

Theorem 2.3: *We assume that the conditions Ӈ1-Ӈ3 hold and* $\Im o \gamma = \gamma o \Im, \Im o \sigma = \sigma o \Im$ *with* $\tau(t) \geq t$. *Let the first order differential inequality*

$$\frac{d}{dt}(\mathcal{W}(t) + \frac{\Upsilon_0}{\Im_0} \mathcal{W}(\Im(t))) + \mathcal{W}(\sigma(t))$$
$$\{\Gamma_1(t)\,[\eta(t) - \eta(t_0)] + \Gamma_2(t)\,[\eta(t) - \eta(t_0)]\} \leq 0 \tag{9}$$

Has no positive solution, then equation (1) oscillates.

Proof: *Let* $\mathcal{W}(t) = 5(t) + \frac{\Upsilon_0}{\Im_0} 5(\Im(t))$

Since $5(t)$ is decreasing and $\Im(t) \geq t$

Then $5(\Im(t)) \leq 5(t)$, we get: $\mathcal{W}(t) \leq 5(t) + \frac{\Upsilon_0}{\Im_0} 5(t)$

Concluding that: $\mathcal{W}(t) \leq 5(t)\left\{1 + \frac{\Upsilon_0}{\Im_0}\right\}$

$$\frac{\mathcal{W}(t)}{\left\{1 + \frac{\Upsilon_0}{\Im_0}\right\}} \leq 5(t) \tag{10}$$

Put (10) in the condition (3), we get:

$$\frac{d}{dt}\left[\frac{\mathcal{W}(t)}{\left\{1+\frac{\Upsilon_0}{\Im_0}\right\}} + \frac{\Upsilon_0}{\Im_0}\frac{\mathcal{W}(\Im(t))}{\left\{1+\frac{\Upsilon_0}{\Im_0}\right\}}\right] + \frac{\mathcal{W}(\sigma(t))}{\left\{1+\frac{\Upsilon_0}{\Im_0}\right\}}$$

$$\{\Gamma_1(t)\,[\eta(t)-\eta(t_0)] + \Gamma_2(t)\,[\eta(t)-\eta(t_0)]\} \le 0$$

We multiply the last inequality by the term $1+\frac{\Upsilon_0}{\Im_0}$

$$\frac{d}{dt}\left[\mathcal{W}(t) + \frac{\Upsilon_0}{\Im_0}\mathcal{W}(\Im(t))\right] + \mathcal{W}(\sigma(t))$$

$$\{\Gamma_1(t)\,[\eta(t)-\eta(t_0)] + \Gamma_2(t)\,[\eta(t)-\eta(t_0)]\} \le 0$$

then $\mathcal{W}(t)$ will be positive solution of (9) which contradiction.

Example 1: Study the linear neutral differential equation:

$$\frac{d}{dt}\left[t^2\frac{d}{dt}(x(t)+x(t-2\pi))\right] + 2t^2\,x(t-2\pi) + 4t\,x\left(t-\frac{\pi}{2}\right) = 0 \qquad (11)$$

Then all conditions of theorem (2.1) hold with $x(t)=\sin(t)$ is an oscillatory solution for equation (11)

When $\mathfrak{S}(t)$=$2t^2\cos t$, satisfies the inequality (3) for $t \in \{n\pi\}, n$ is positive odd number.

3. Conclusion

The oscillatory behavior of second-order NDE is considered in the linear case. We established specific conditions for known functions in the delay differential equation (1). A new relationship between the deviating arguments and known functions has been built to obtain the oscillatory property. We submitted an example to explain the importance of our results compared with relevant results in the literature.

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