Journal of Interdisciplinary Mathematics
ISSN: 0972-0502 (Print), ISSN: 2169-012X (Online)
Vol. 28 (2025), No. 3-B, pp. 1097–1108
DOI : 10.47974/JIM-2086

Present study on fourth order differential subordination associated with differential operator

Suad Y. Al-Mayyahi *
Department of Mathematics
College of Education of Pure Science
University of Wasit
Wasit
Iraq

Kassim A. Jassim †
Department of Mathematics
College of Science
University of Baghdad
Baghdad
Iraq

Abstract

Univalent functions represent a crucial area in complicated analysis applicable to both single and many variables. Researchers have performed a traditional analysis of this subject since at least 1907. Geometric function theory is synthesis for geometry & analysis. Various research has examined first-order differential subordination, subsequently leading to inquiries into second-order differential subordination within the unit disc. Abdulhalim and Miller [3] presented third-order differential subordination by reinterpreting essential inequalities of realvalued function inside framework ofor complexvalued function. This research seeks to formulate fourth-order differential subordination & superordination by application for generalized derivative operator $D^{s,m,k}_{\gamma,\sigma} f(z)$ such that numerous theorems are stated and proved involving the generalized derivative operator $D^{s,m,k}_{\gamma,\sigma} f(z)$ for analytic function [1] in open-unit disc $\mathfrak{A}$, which using generalised Srivastava-Attiya operator, generalised Al-Oboudi operator, binomial series. Based on this new operator, we derive results regarding fourth differential subordination and superordination technique for analytic functions related to operator $D^{s,m,k}_{\gamma,\sigma} f(z)$ within the open unit disk $\mathfrak{A}$ and we discuss these insights by examining relevant categories of admissible functions, some theories and colleries.

Subject Classification: *Primary 30C45, Secondary 3080.*
Keywords: *Univalent functions, Differential operator, Admissible functions, Univalent function, Convolution.*

This article has been corrected with minor changes. These changes do not impact the academic content of the article.

* *E-mail:* **syounis@uowasit.edu.iq (Corresponding Author)**
† *E-mail:* **kassim.jassim@sc.uobaghdad.edu.iq**

1. Introduction

In the nineteenth century, Riemann developed the concept of regular geometric functions in complex analysis in 1850, defining functions with ranges that provide unique geometries. In 1907, Koebe advanced the field by proposing the idea of univalent functions in the course of his studies. This idea pertains to many categories of standardizing univalent fundamental functions within the open unit disc, specifically $\boldsymbol{p}$-valent, meromorphic, meromorphic $\boldsymbol{p}$ -valent, and harmonic functions. A theory for analytic, univalent, multivalent functions inside unit disc classifies functions into categories based on their unique geometric & analytical properties. The theory of differential subordination in $\mathfrak{C}$ generalizes difference inequalities in $\mathbb{R}$.. Initially presented by Mocanu and Mocanu [2]. In 2011, Abdulhalim and Miller [3] introduced fundamental principles and expanded idea for second-order differential subordination. Numerous writers have effectively used concept for first & second order differential subordination in the past few years to tackle numerous significant issues in this area, as seen in [4], [5], [6], [7], [8]. Furthermore, several others as [9], [10], [11], [12] examined a notion for differential subordination &superordination in third order. A paper on differential subordinations, an extension of differential inequalities, was issued by Mocanu and Mocanu [2]. Some researchers [13], [14] developed third order to investigate fourth-order differential subordination. A primary emphasis for this study is to generate differential subordination & superordination outcomes, relevant types for admissible function, theories and several sandwich-type theorems combine the subordination and superordination employing the generalized differential operator in open unit disc. The fundamental concepts covered in this study are represented using standard annotations:

Let's take $Ḥ = Ḥ(\mathfrak{A})$ represents class all function which is analytic within $\mathfrak{A} = \{z: z \in \mathfrak{C};\ |z| < 1\}$, that $\mathfrak{C}$ is a plane of complex number. Assume $n \in z^+$ and the number $c \in \mathfrak{C}$. We define $Ḥ[c, n]$ as subclass of functions characterized by the structure $f(z) = c + c_n z^n + c_{n+1} z^{n+1} + \cdots, (z \in \mathfrak{A})$, where z is an element of $\mathfrak{A}$, and let $Ḥ_1 = [1,1]$. Given two analytic functions, f and Ḱ in $\mathfrak{A}$. Suppose $\mathcal{W}$ be Schwarz function belongs to $\mathfrak{A}$ with $\mathcal{W}(0) = 0$, then we define that f is subordinate to Ḱ or Ḱ is superordinate to f, and $|\mathcal{W}(z)| < 1$ in which $f(z) = Ḱ(\mathcal{W}(z))$ $(z \in \mathfrak{A})$. Upon that happening, we would write

$$f \prec Ḱ \text{ or } f(z) \prec Ḱ(z)\ (z \in \mathfrak{A})$$

To be more precise, in situations when a function Ḱ univalent in $\mathfrak{A}$, so the following equivalent [2], [15] the function f is subordinate to the function Ḱ if and only if $f(0) = Ḱ(0)$ and $f(\mathfrak{A}) \subset Ḱ(\mathfrak{A})$.

Furthermore, let $\tilde{\hat{A}}$ denote class for functions which are analytic in $\mathfrak{A}$, exhibit normalized structure:

$$f(z) = z + \sum_{n=2}^{\infty} c_n z^n \tag{1}$$

A multitude of mathematicians have examined operators and the recent extension in numerous works. Studies on operators have been ongoing since 1915. Since that time, mathematicians remain highly engaged in the discourse surrounding operators exhibiting numerous characteristics of analytic univalent functions. Consequently, this study presents a generalized differential operator [1] that examined for the class $\tilde{\hat{A}}$ of univalent functions through the infinite series $D_{\gamma,\sigma}^{s,m,k} f(z): \tilde{\hat{A}} \to \tilde{\hat{A}}$, which is described as follows:

$$D_{\gamma,\sigma}^{s,m,k} f(z) = z + \sum_{n=2}^{\infty} \left(\frac{1+\ell}{n+\ell}\right)^{s} (1 + \gamma(n-1)(1-\sigma)^{m})^{k} c_n z^n$$

$$(f \in \tilde{\hat{A}}, 0 < \gamma < 1, 0 < \sigma < 1, m \in \mathbb{N} = \{1,2,\dots\}, \ell \in \mathfrak{C} \setminus \bar{z}_0, s \in \mathfrak{C}, k \in \mathbb{N}_0)$$

From (2) it's clear that:

$$D_{\gamma,\sigma}^{s,m,k+1} f(z) = (1 - \gamma(1-\sigma)^{m}) D_{\gamma,\sigma}^{s,m,k} f(z) + \gamma(1-\sigma)^{m} z \left[D_{\gamma,\sigma}^{s,m,k} f(z)\right]' \quad (2)$$

Authors in [13], [14] developed third order to investigate fourthorder differential subordination & delineated attributes for function $\mathcal{G}$ that simply with the subsequent fourth-order differential subordination:

$$\varpi\left(\mathcal{G}(z), z\,\mathcal{G}'(z), z^2\mathcal{G}''(z), z^3\mathcal{G}^{(3)}(z), z^4\mathcal{G}^{(4)}(z); z\right) \prec h(z)$$

such that $h(z)$ be analytic univalent function within $\mathfrak{A}$, $\mathcal{G}$ be a function which analytic and $\varpi: \mathfrak{C}^5 \times \mathfrak{A} \to \mathfrak{C}$ be a function also.

Now, we give basic ideas from the theory of the fourth-order are required to validate our primary findings.

Definition 1.1: [3] Suppose Q is a collection of all functions q which are one to one and analytic specified on the collection $\mathfrak{A}$, such that

$\mathrm{E}(q) = \left\{\xi \in \partial\mathfrak{A}: \lim_{z\to\xi} q(z) = \infty\right\}$, ($\xi \in \overline{\mathfrak{A}} - E(q)$ and $q'(\xi) \neq 0$.

Definition 1.2: [13] Letting $\varpi: \mathfrak{C}^5 \times \mathfrak{A} \to \mathfrak{C}$ be function. Assume that $h(z)$ is a univalent function inside $\mathfrak{A}$. Assuming $\mathcal{G}$ is a function that is analytic within $\mathfrak{A}$ and fulfills specified fourth-order differential subordination condition:

$$\varpi\left(\mathcal{G}(z), z\,\mathcal{G}'(z), z^2\mathcal{G}''(z), z^3\mathcal{G}^{(3)}(z), z^4\mathcal{G}^{(4)}(z); z\right) \prec h(z) \quad (3)$$

Consequently, $\mathcal{G}$ represents solution to a differential subordination. The univalent functions q is dominating for solutions of the differential subordination when $\mathcal{G}$ is subordinate to q for each $\mathcal{G}$ satisfy relation (3). A dominant, denoted as $\tilde{q}(z)$, having the condition $\tilde{q}(z) \prec q(z)$ for all dominants q for (4), is the best dominant of (3).

Definition 1.3: [13] Letting $\varpi: \mathfrak{C}^5 \times \mathfrak{A} \to \mathfrak{C}$ be function. Assume that $h(z)$ is a analytic function inside $\mathfrak{A}$. Assuming $\mathcal{G}$ is a function that is analytic within $\mathfrak{A}$ and

$$\varpi\left(\mathcal{G}(z), z\,\mathcal{G}'(z), z^2\mathcal{G}''(z), z^3\mathcal{G}^{(3)}(z), z^4\mathcal{G}^{(4)}(z); z\right)$$

Fulfill requirements of designated univalent function in $\mathfrak{A}$& fourth-order differential superordination:

$$h(z) \prec \varpi\left(\mathcal{G}(z), z\,\mathcal{G}'(z), z^2\mathcal{G}''(z), z^3\mathcal{G}^{(3)}(z), z^4\mathcal{G}^{(4)}(z); z\right) \quad (4)$$

Consequently, $\mathcal{G}(z)$ represents a differential superordination solution (4). The univalent function $q(z)$ subordinate for solution (4), if for $\mathcal{G}(z)$ fulfill (4), $q(z) \prec \mathcal{G}(z)$. the univalent subordinates $\tilde{q}(z)$, having the condition $q(z) \prec \tilde{q}(z)$ for every subordinates in (4), is finest subordinate. It is observed that, the optimal subordinate is unique.

Definition 1.4: [13] Letting $q \in Q$, $n \in \mathbb{N} - \{2\}$, and take Ω belongs to $\mathfrak{C}$. thus q included in admissible functions class $\psi_n[\Omega, q]$ are functions $\varpi: \mathfrak{C}^5 \times \mathfrak{A} \to \mathfrak{C}$ that meet subsequent admissibility criteria: $\varpi(u, r, x, y, g; \xi) \notin \Omega$

where $u = q(\xi)$, $r = k\xi q'(\xi)$, $R\left(\frac{x}{r} + 1\right) \geq k\, R\left\{\frac{\xi q''(\xi)}{q'(\xi)} + 1\right\}$,

$$R\left(\frac{y}{r}\right) \geq k^3 R\left\{\frac{\xi^3 q^{(3)}(\xi)}{q'(\xi)}\right\},$$

and $R\left(\frac{g}{r}\right) \geq k^4 R\left\{\frac{\xi^3 q^{(4)}(\xi)}{q'(\xi)}\right\}$($z \in \mathfrak{A}, \xi \in \partial\mathfrak{A} \setminus E(q)$ and $k \geq 3$).

Definition 1.5: [13] letting $q \in \mathrm{H}[c, n]$, and take Ω belongs to $\mathfrak{C}$ and $q(z) \neq 0$, functions $\varpi: \mathfrak{C}^5 \times \mathfrak{A} \to \mathfrak{C}$ which satisfy admissibility criteria in class $\psi'_n[\Omega, q]$ of admissible functions: $\varpi(u, r, x, y, g; \xi) \in \Omega$

where $u = q(z)$, $r = \frac{\xi q'(z)}{m}$, $R\left(\frac{x}{r} + 1\right) \leq \frac{1}{m} R\left\{\frac{\xi q''(z)}{q'(z)} + 1\right\}$,

$$R\left(\frac{y}{r}\right) \leq \frac{1}{m^2} R\left\{\frac{\xi^3 q^{(4)}(z)}{q'(z)}\right\},$$

and $R\left(\frac{g}{r}\right) \leq \frac{1}{m^3} R\left\{\frac{\xi^3 q^{(4)}(z)}{q'(z)}\right\}$ ($z \in \mathfrak{A}, \xi \in \partial\mathfrak{A}$ and $m \geq n \geq 3$).

Lemma 1.6: [14] *Let $\mathcal{G}$ belong to $\mathrm{H}[c, n]$ with n in $\mathbb{N}$ excluding $\{2\}$, and let q be belong to $Q(c)$ that meets the subsequent conditions:*

$$R\left\{\frac{\xi^2 q^{(3)}(\xi)}{q'(\xi)}\right\} \geq 0, \left|\frac{z^2 \mathcal{G}''(z)}{q'(\xi)}\right| \leq k^2.$$

where $k \geq n, z \in \mathfrak{A}, \xi \in \partial\mathfrak{A} \setminus E(q)$. If Ω included in $\mathfrak{C}$, also $\varpi \in \psi_n[\Omega, q]$

and $\varpi\left(\mathcal{G}(z), z\,\mathcal{G}'(z), z^2\mathcal{G}''(z), z^3\mathcal{G}^{(3)}(z), z^4\mathcal{G}^{(4)}(z); z\right) \in \Omega$

then $\mathcal{G}(z) \prec q(z)$ $(z \in \mathfrak{A})$.

Lemma 1.7: [14] *Let $\mathcal{G}$ belong to $\mathrm{H}[c, n]$ with $m \geq n \geq 3$, and let q be belong to $Q(c)$ that meets the subsequent conditions:*

$$R\left\{\frac{\xi^2 q^{(3)}(z)}{q'(\xi)}\right\} \geq 0, \left|\frac{z^2 \mathcal{G}''(z)}{q'(\xi)}\right| \leq \frac{1}{m^2}$$

where $k \geq n, z \in \mathfrak{A}, \xi \in \partial\mathfrak{A}$. If Ω included in $\mathfrak{C}$, also $\varpi \in \psi'_n[\Omega, q]$

and $\Omega \in \varpi\left(\mathcal{G}(z), z\,\mathcal{G}'(z), z^2\mathcal{G}''(z), z^3\mathcal{G}^{(3)}(z), z^4\mathcal{G}^{(4)}(z); z\right)$

then $q(z) \prec \mathcal{G}(z)$ $(z \in \mathfrak{A})$.

2. Principal results of differential subordination using $D_{\gamma,\sigma}^{s,m,k} f(z)$

Prior to establishing the theorems of differential subordination employing the operator $D_{\gamma,\sigma}^{s,m,k} f(z)$ where $f(z)$ given by (2), we explain a new class of admissible functions.

Definition 2.1: Give Ω subset for $\mathfrak{C}$, & make q belong to intersection for $\mathcal{Q}_0$ and $\mathcal{J}_0$. Assume $\mathrm{M}_i[\Omega, q]$ a class function which admissible & includes functions $\varpi: \mathfrak{C}^5 \times \mathfrak{A} \to \mathfrak{C}$ which fulfill the specified admissibility conditions:

$$\varpi(u, r, x, y, g;\ \xi) \in \Omega$$

where $u = q(\xi)$, $r = k\,\gamma(1-\sigma)^m \xi q'(\xi) + (1-\gamma(1-\sigma)^m)q(\xi)$

$$R\left(\frac{x-2(1-\gamma(1-\sigma)^m)r+(1-\gamma(1-\sigma)^m)^2 u}{\gamma(1-\sigma)^m(r-(1-\gamma(1-\sigma)^m)u)}\right) \geq k\, R\left\{\frac{\xi q''(\xi)}{q'(\xi)}+1\right\},$$

$$R\left(\frac{y-3x+\left(3-\gamma^2(1-\sigma)^{2m}\right)r-\left(1-\gamma^2(1-\sigma)^{2m}\right)u}{\gamma^2(1-\sigma)^{2m}(r-(1-\gamma(1-\sigma)^m)u)}\right) \geq k^2 R\left\{\frac{\xi^2 q^{(3)}(\xi)}{q'(\xi)}\right\},$$

and $$R\left(\frac{\begin{array}{c} g-2(2+\gamma(1-\sigma)^m)y-\left(\gamma^2(1-\sigma)^{2m}-6\gamma(1-\sigma)^m-6\right)x \\ +2\left(\gamma^3(1-\sigma)^{3m}+\gamma^2(1-\sigma)^{2m}-3\gamma(1-\sigma)^m-2\right)r \\ +(1-\gamma(1-\sigma)^m)\left(1+3\gamma(1-\sigma)^m+2\gamma^2(1-\sigma)^{2m}\right)u \end{array}}{\gamma^3(1-\sigma)^{3m}(r-(1-\gamma(1-\sigma)^m)u)}\right) \geq k^3 R\left\{\frac{\xi^3 q^{(4)}(\xi)}{q'(\xi)}\right\},$$

for ($z \in \mathfrak{A}, \xi \in \partial\mathfrak{A} \setminus E(q)$ and $k \geq 3$).

Theorem 2.2: *Assume that* $\varpi \in \mathrm{M}_i[\Omega, q]$. *If function* $f(z) \in \tilde{\hat{\mathrm{A}}}$. *let* $q \in \mathcal{Q}_0$ *meet a following criteria:*

$$R\left(\frac{\xi^2 q^{(3)}(\xi)}{q'(\xi)}\right) \geq 0, \left|\frac{D_{\gamma,\sigma}^{s,m,k+2} f(z)}{q'(\xi)}\right| \leq k^2 \tag{5}$$

and $$\left\{\varpi\begin{pmatrix} D_{\gamma,\sigma}^{s,m,k} f(z), D_{\gamma,\sigma}^{s,m,k+1} f(z), D_{\gamma,\sigma}^{s,m,k+2} f(z), \\ D_{\gamma,\sigma}^{s,m,k+3} f(z), D_{\gamma,\sigma}^{s,m,k+4} f(z) \end{pmatrix}; z \in \mathfrak{A}\right\} \subset \Omega \tag{6}$$

then $D_{\gamma,\sigma}^{s,m,k} f(z) \prec q(z)$ $(z \in \mathfrak{A})$.

***Proof*:** Assume $p(z)$ denote the analytic function defined by

$$p(z) = D_{\gamma,\sigma}^{s,m,k} f(z)\ (z \in \mathfrak{A}) \tag{7}$$

Subsequently, after differentiating equation (7) with respect to z and putting equation (2), we yield

$$D_{\gamma,\sigma}^{s,m,k+1} f(z) = \gamma(1-\sigma)^m\, zp'(z) + (1-\gamma(1-\sigma)^m)p(z) \tag{8}$$

Subsequent calculations indicate that:

$$D_{\gamma,\sigma}^{s,m,k+2} f(z) = \gamma^2(1-\sigma)^{2m} z^2 p''(z) + \gamma(1-\sigma)^m(2-\gamma(1-\sigma)^m)zp'(z) \\ +(1-\gamma(1-\sigma)^m)^2 p(z) \tag{9}$$

$$D_{\gamma,\sigma}^{s,m,k+3} f(z) = \gamma^3(1-\sigma)^{3m} z^3 p^{(3)}(z) + 3\gamma^2(1-\sigma)^{2m} p''(z) + \gamma(1-\sigma)^m \\ (\gamma^2(1-\sigma)^{2m} - 3\gamma(1-\sigma)^m + 3)zp'(z) + (1-\gamma(1-\sigma)^m)^3 \tag{10}$$

and

$$D_{\gamma,\sigma}^{s,m,k+4}f(z) = \gamma^4(1-\sigma)^{4m}\, z^4p^{(4)}(z) + 2\gamma^3(1-\sigma)^{3m}(\gamma(1-\sigma)^m + 2)\, z^3p^{(3)}(z) + \gamma^2(1-\sigma)^{2m}(\gamma^2(1-\sigma)^{2m} + 6)z^2p''(z) - \gamma(1-\sigma)^m (\gamma^3(1-\sigma)^{3m} - 4\gamma^2(1-\sigma)^{2m} + 6\gamma(1-\sigma)^m - 4)zp'(z) + (1-\gamma(1-\sigma)^m)^4p(z) \tag{11}$$

Specify the transformation from $\mathfrak{C}^5$ onto $\mathfrak{C}$ by

$$u(r,s,t,w,b) = r,\ \mathscr{r}(r,s,t,w,b) = \gamma(1-\sigma)^m s + (1-\gamma(1-\sigma)^m)r,$$

$$x(r,s,t,w,b) = \gamma^2(1-\sigma)^{2m}t + \gamma(1-\sigma)^m(2-\gamma(1-\sigma)^m)s + (1-\gamma(1-\sigma)^m)^2 r$$

$$\mathscr{y}(r,s,t,w,b) = \gamma^3(1-\sigma)^{3m}w + 3\gamma^2(1-\sigma)^{2m}t + \gamma(1-\sigma)^m (\gamma^2(1-\sigma)^{2m} - 3\gamma(1-\sigma)^m + 3)s + (1-\gamma(1-\sigma)^m)^3 r\,. \tag{12}$$

$$\mathscr{g}(r,s,t,w,b) = \gamma^4(1-\sigma)^{4m}b + 2\gamma^3(1-\sigma)^{3m}(\gamma(1-\sigma)^m + 2)w\, (1-\sigma)^{2m}(\gamma^2(1-\sigma)^{2m} + 6)t - \gamma(1-\sigma)^m(\gamma^3(1-\sigma)^{3m} - 4\gamma^2(1-\sigma)^{2m} + 6\gamma(1-\sigma)^m - 4)s + (1-\gamma(1-\sigma)^m)^4 r\,. \tag{13}$$

Let $\mathcal{X}(r,s,t,w,b;z) = \varpi(u,\mathscr{r},x,\mathscr{y},\mathscr{g};z) =$

$$\varpi\begin{pmatrix} r, \gamma(1-\sigma)^m s + (1-\gamma(1-\sigma)^m)r, \gamma^2(1-\sigma)^{2m}t + \\ \gamma(1-\sigma)^m(2-\gamma(1-\sigma)^m)s + (1-\gamma(1-\sigma)^m)^2\, r, \gamma^3(1-\sigma)^{3m}w + \\ 3\gamma^2(1-\sigma)^{2m}t + \gamma(1-\sigma)^m(\gamma^2(1-\sigma)^{2m} - 3\gamma(1-\sigma)^m + 3)s \\ +(1-\gamma(1-\sigma)^m)^3 r, \gamma^4(1-\sigma)^{4m}b + \\ 2\gamma^3(1-\sigma)^{3m}(\gamma(1-\sigma)^m + 2)w + \gamma^2(1-\sigma)^{2m}(\gamma^2(1-\sigma)^{2m} + 6)t \\ -\gamma(1-\sigma)^m(\gamma^3(1-\sigma)^{3m} - 4\gamma^2(1-\sigma)^{2m} + 6\gamma(1-\sigma)^m - 4)s \\ +(1-\gamma(1-\sigma)^m)^4 r; z \end{pmatrix} \tag{14}$$

Also, lemma 1.6. will be applied in proof. From (14), utilizing equations (8)-(13) and, we derive:

$$\mathcal{X}\big(p(z), zp'(z), z^2p''(z), z^3p^{(3)}(z), z^4p^{(4)}(z); z\big) = \varpi\big(D_{\gamma,\sigma}^{s,m,k}f(z), D_{\gamma,\sigma}^{s,m,k+1}f(z), D_{\gamma,\sigma}^{s,m,k+2}f(z), D_{\gamma,\sigma}^{s,m,k+3}f(z), D_{\gamma,\sigma}^{s,m,k+4}f(z); z\big) \tag{15}$$

Consequently, equation (6) is reformulated as:

$$\mathcal{X}\big(p(z), zp'(z), z^2p''(z), z^3p^{(3)}(z), z^4p^{(4)}(z); z\big) \in \Omega,$$

We note that:

$$\frac{t}{s} + 1 = \frac{x - 2(1-\gamma(1-\sigma)^m)\mathscr{r} + (1-\gamma(1-\sigma)^m)u}{\gamma(1-\sigma)^m(\mathscr{r} - (1-\gamma(1-\sigma)^m)u)}$$

$$\frac{w}{s} = \frac{y - 3x + \big(3-\gamma^2(1-\sigma)^{2m}\big)\mathscr{r} - \big(1-\gamma^2(1-\sigma)^{2m}\big)u}{\gamma^2(1-\sigma)^{2m}(\mathscr{r} - (1-\gamma(1-\sigma)^m)u)}$$

$$\frac{b}{s} = \frac{\begin{pmatrix} \mathscr{g} - 2(\gamma(1-\sigma)^m + 2)\mathscr{y} - \big(\gamma^2(1-\sigma)^{2m} - 6\gamma(1-\sigma)^m - 6\big)x + \\ 2\big(\gamma^3(1-\sigma)^{3m} + \gamma^2(1-\sigma)^{2m} - 3\gamma(1-\sigma)^m - 2\big)\mathscr{r} - \\ (\gamma(1-\sigma)^m - 1)\big(1 + 3\gamma(1-\sigma)^m + 2\gamma^2(1-\sigma)^{2m}\big)u \end{pmatrix}}{\gamma^3(1-\sigma)^{3m}(\mathscr{r} - (1-\gamma(1-\sigma)^m)u)}$$

The admissibility criterion for ϖ , in Definition. 1.5. fundamentally equal to admissibility criteria of χ in Def. 1.4. for $\chi \in \psi_3[\Omega, q]$, for Consequently, by employing (5) and Lemma. 1.6. We obtain:

$$D_{\gamma,\sigma}^{s,m,k} f(z) \prec q(z)$$

With this, theorem 2.2 is fully proved. We extend theorem 2.2 to the case when $q(z) \in \partial\mathfrak{A}$ has undefinable behavior in our following corollary.

Corollary 2.3: *Assume Ω a subset of $\mathfrak{C}$, and q univalent function belongs to $\mathfrak{A}$ such that $q(0) = 1$. Letting ϖ belongs to $M_i[\Omega, q_\rho]$, $(0 < \rho < 1)$, and $q_\rho(z) = q(\rho z)$. If $f(z)$ function belongs to $\tilde{\hat{A}}$ and $q_\rho(z)$ meets the next criteria:*

$$R\left\{\frac{\xi^2 q_\rho^{(3)}(z)}{q_\rho'(z)}\right\} \geq 0, \left|\frac{D_{\gamma,\sigma}^{s,m,k+2} f(z)}{q_\rho'(z)}\right| \leq k^2 \tag{16}$$

$\left(z \in \mathfrak{A}, \xi \in \partial\mathfrak{A} \setminus E(q_\rho) \text{ and } k \geq n\right)$, and

$$\varpi\left(D_{\gamma,\sigma}^{s,m,k} f(z), D_{\gamma,\sigma}^{s,m,k+1} f(z), D_{\gamma,\sigma}^{s,m,k+2} f(z), D_{\gamma,\sigma}^{s,m,k+3} f(z), D_{\gamma,\sigma}^{s,m,k+4} f(z)\right) \prec h(z) \tag{17}$$

then $D_{\gamma,\sigma}^{s,m,k} f(z) \prec q(z)$ $(z \in \mathfrak{A})$

Proof: Utilizing theorem 2.2, we obtain $D_{\gamma,\sigma}^{s,m,k} f(z) \prec q_\rho(z)$

Subsequently, the outcome of the subordination connection is now known $q_\rho(z) \prec q(z)$ $(z \in \mathfrak{A})$.

Ω equal $h(\mathfrak{A}))$ of some conformal map $h(z)$ from $\mathfrak{A}$ onto Ω, whenever $\Omega \neq \mathfrak{C}$ simply connected domain. In this particular situation, $M_i[h(\mathfrak{A}), q]$ be a class symbolized as $M_i[h, q]$.

A subsequent two outcomes the direct ramifications for theorem 2.2 & corollary 2.3.

Theorem 2.4: *Suppose h is univalent function that included in $\mathfrak{A}$. Assume $\varpi: \mathfrak{C}^5 \times \mathfrak{A} \to \mathfrak{C}$ be a function and let the differential equation*

$$\begin{aligned}&\varpi\big(q(z), \gamma(1-\sigma)^m z q'(z) + (1-\gamma(1-\sigma)^m) q(z), \gamma^2(1-\sigma)^{2m} z^2 q''(z) + \\ &\gamma(1-\sigma)^m(2-\gamma(1-\sigma)^m) z q'(z) + (1-\gamma(1-\sigma)^m)^2 q(z), \gamma^3(1- \\ &\sigma)^{3m} z^3 q^{(3)}(z) + 3\gamma^2(1-\sigma)^{2m} z^2 q''(z) + \gamma(1-\sigma)^m(\gamma^2(1-\sigma)^{2m} - \\ &3\gamma(1-\sigma)^m + 3) z q'(z) + (1-\gamma(1-\sigma)^m)^3 q(z), \gamma^4(1-\sigma)^{4m} z^4 q^{(4)}(z) + \\ &2\gamma^3(1-\sigma)^{3m}(\gamma(1-\sigma)^m + 2) z^3 q^{(3)}(z) + \gamma^2(1-\sigma)^{2m}(\gamma^2(1-\sigma)^{2m} + \\ &6) z^2 q''(z) + \gamma(1-\sigma)^m(\gamma^3(1-\sigma)^{3m} - 4\gamma^2(1-\sigma)^{2m} + 6\gamma(1-\sigma)^m - \\ &4) z q'(z) + (1-\gamma(1-\sigma)^m)^4 q(z); z\big) = h(z)\end{aligned} \tag{18}$$

possesses a solution $q(z)$ such that $q(0) = 1$ and fulfill condition (5). If the function f belongs to $\tilde{\hat{A}}$ meets criterion (16), and

$$\varpi\left(D_{\gamma,\sigma}^{s,m,k} f(z), D_{\gamma,\sigma}^{s,m,k+1} f(z), D_{\gamma,\sigma}^{s,m,k+2} f(z), D_{\gamma,\sigma}^{s,m,k+3} f(z), D_{\gamma,\sigma}^{s,m,k+4} f(z); z\right)$$

analytic in $\mathfrak{A}$, then

$D_{\gamma,\sigma}^{s,m,k} f(z) \prec q(z)$ $(z \in \mathfrak{U})$ *and* $q(z)$ *best dominant.*

Proof: From theorem 2.2., $q(z)$ is dominant over (17). since $q(z)$ satisfies (18) so that $q(z)$ solution for (17). thus $q(z)$ will dominant for every dominant; hence, is optimal dominant.

In the specific instance, where $q(z) = \mathcal{M}z$ $(\mathcal{M} > 0)$, and using definition 2.1. Class of admissible functions $\mathrm{M}_i[\Omega, q]$ represented as $\mathrm{M}_i[\Omega, \mathcal{M}]$, can be characterized as follow

Definition 2.5: Suppose Ω belong to $\mathfrak{C}$ & $(\mathcal{M} > 0)$. $\mathrm{M}_i[\Omega, \mathcal{M}]$ class for admissible function that contain function $\varpi: \mathfrak{C}^5 \times \mathfrak{U} \to \mathfrak{C}$, where ϖ fulfill the admissibility condition:

$$\varpi\big(\mathcal{M}e^{i\theta}, \big(k\gamma(1-\sigma)^m + (1-\gamma(1-\sigma)^m)\big)\mathcal{M}e^{i\theta}, \gamma^2(1-\sigma)^{2m}L + (\gamma(1-\sigma)^m(2-\gamma(1-\sigma)^m)k + (1-\gamma(1-\sigma)^m)^2)\mathcal{M}e^{i\theta}, \gamma^3(1-\sigma)^{3m}N + 3\gamma^2(1-\sigma)^{2m}L + (\gamma(1-\sigma)^m(\gamma^2(1-\sigma)^{2m} - 3\gamma(1-\sigma)^m + 3)k + (1-\gamma(1-\sigma)^m)^3)\mathcal{M}e^{i\theta}, \gamma^4(1-\sigma)^{4m}A + 2\gamma^3(1-\sigma)^{3m}(\gamma(1-\sigma)^m + 2)N + \gamma^2(1-\sigma)^{2m}(\gamma^2(1-\sigma)^{2m} + 6)L - (\gamma(1-\sigma)^m(\gamma^3(1-\sigma)^{3m} - 4\gamma^2(1-\sigma)^{2m} + 6\gamma(1-\sigma)^m - 4)k + (1-\gamma(1-\sigma)^m)^4)\mathcal{M}e^{i\theta}; z\big) \notin \Omega \quad (19)$$

whenever $\mathrm{Re}\big(Le^{-i\theta}\big) \geq (k-1)kM$, $\mathrm{Re}\,\big(Ne^{-i\theta}\big) \geq 0$ and, $\mathrm{Re}\big(Ae^{-i\theta}\big) \geq 0$

$$(z \in \mathfrak{U}, \theta \in \mathbb{R}, \rho > -\gamma, k \geq 3\,).$$

Theorem 2.6: *Suppose* ϖ *belongs to* $\mathrm{M}_i[\Omega, \mathcal{M}]$. *when analytic function* $f \in \tilde{\hat{\mathrm{A}}}$ *meets next criteria:*

$\left|D_{\gamma,\sigma}^{s,m,k+2} f(z)\right| \leq k^2\mathcal{M}$ $(k \geq 3\,; \mathcal{M} > 0)$, and

$$\varpi\big(D_{\gamma,\sigma}^{s,m,k} f(z), D_{\gamma,\sigma}^{s,m,k+1} f(z), D_{\gamma,\sigma}^{s,m,k+2} f(z), D_{\gamma,\sigma}^{s,m,k+3} f(z), D_{\gamma,\sigma}^{s,m,k+4} f(z); z\big) \in \Omega$$

then $\left|D_{\gamma,\sigma}^{s,m,k} f(z)\right| < \mathcal{M}$.

In a particular situation, $\Omega = \{w: |w| < \mathcal{M}\}$. $\mathrm{M}_i[\Omega, \mathcal{M}]$ simply connected, also represented by $\mathrm{M}_i[\mathcal{M}]$.

Corollary 2.7: *Assume* $k \geq 3, \gamma > -\rho$ *(* $\mathcal{M}$ *greater than 0). If the function* $f \in \tilde{\hat{\mathrm{A}}}$ *fulfill* $\left|D_{\gamma,\sigma}^{s,m,k+2} f(z)\right| \leq k^2\mathcal{M}$ *, and*

$\left|D_{\gamma,\sigma}^{s,m,k+2} f(z) - (1-\gamma(1-\sigma)^m)D_{\gamma,\sigma}^{s,m,k+1} f(z)\right|$, then $\left|D_{\gamma,\sigma}^{s,m,k} f(z)\right| < \mathcal{M}$

Proof: Assume $\varpi(u, r, x, y, g; z) = x - (1-\gamma(1-\sigma)^m)r$, $\Omega = h(\mathfrak{U})$

Let $h(z) = (|\gamma(1-\sigma)^m| + 2|\gamma^2(1-\sigma)^{2m}|)3\mathcal{M}z, \mathcal{M} > 0$

Utilizing theorem 2.6. we must demonstrate that $\varpi \in \mathrm{M}_{i,1}[\Omega, \mathcal{M}]$. Given that it is observed that

$$\varpi\big|\big(\mathcal{M}e^{i\theta}, \big(k\gamma(1-\sigma)^m + (1-\gamma(1-\sigma)^m)\big)\mathcal{M}e^{i\theta}, \gamma^2(1-\sigma)^{2m}L + (\gamma(1-\sigma)^m(2-\gamma(1-\sigma)^m)k + (1-\gamma(1-\sigma)^m)^2)\mathcal{M}e^{i\theta}, \gamma^3(1-\sigma)^{3m}N + 3\gamma^2(1-\sigma)^{2m}L + (\gamma(1-\sigma)^m[\gamma^2(1-\sigma)^{2m} - 3\gamma(1-\sigma)^m + 3]k +$$

$(1-\gamma(1-\sigma)^m)^3)\mathcal{M}e^{i\theta}, \gamma^4(1-\sigma)^{4m}A + 2\gamma^3(1-\sigma)^{3m}(\gamma(1-\sigma)^m + 2)N + \gamma^2(1-\sigma)^{2m}(\gamma^2(1-\sigma)^{2m} + 6)L - (\gamma(1-\sigma)^m(\gamma^3(1-\sigma)^{3m} - 4\gamma^2(1-\sigma)^{2m} + 6\gamma(1-\sigma)^m - 4)k + (1-\gamma(1-\sigma)^m)^4)\mathcal{M}e^{i\theta}; z)\big| =$

$$\left|\begin{matrix}\gamma^2(1-\sigma)^{2m}L + (\gamma(1-\sigma)^m(2-\gamma(1-\sigma)^m)k + (1-\gamma(1-\sigma)^m)^2)\mathcal{M}e^{i\theta} \\ -(1-\gamma(1-\sigma)^m)\big(k\gamma(1-\sigma)^m + (1-\gamma(1-\sigma)^m)\big)\mathcal{M}e^{i\theta}\end{matrix}\right|$$

$$= \left|\begin{matrix}\gamma^2(1-\sigma)^{2m}Le^{-i\theta} + \\ \big(\gamma(1-\sigma)^m(2-\gamma(1-\sigma)^m) - (1-\gamma(1-\sigma)^m)(\gamma(1-\sigma)^m)\big) \\ k\mathcal{M}e^{i\theta} + ((1-\gamma(1-\sigma)^m)^2 - (1-\gamma(1-\sigma)^m)^2)\mathcal{M}e^{i\theta}\end{matrix}\right|$$

$$\geq |\gamma^2(1-\sigma)^2|R\big(Le^{-i\theta}\big) + |\gamma(1-\sigma)^m|k\mathcal{M}$$

$$\geq (|\gamma(1-\sigma)^m| + 2|\gamma^2(1-\sigma)^{2m}|)3\mathcal{M}$$

whenever $z \in \mathfrak{A}$, $\mathrm{Re}\big(Le^{-i\theta}\big) \geq (k-1)kM$, $\mathrm{Re}\ \big(Ne^{-i\theta}\big) \geq 0$ and, $\mathrm{Re}\big(Ae^{-i\theta}\big) \geq 0$ $(\theta \in \mathbb{R}, \rho > -\gamma\ and\ k \geq 3)$. So, proof is finished.

3. Principal results of differential superordination related to $D_{\gamma,\sigma}^{s,m,k}f(z)$

We establish specific fourth-order differential superordination for univalent functions related to $D_{\gamma,\sigma}^{s,m,k}f(z)$. The following explanation specifies a set of admissible functions for this study.

Definition 3.1: Suppose Ω based in $\mathfrak{C}$, $q(z) \in H_0$ & $q'(z) \neq 0$. A class for admissible function $\mathrm{M}_i[\Omega, q]$ is made up for function $\varpi: \mathfrak{C}^5 \times \mathfrak{A} \to \mathfrak{C}$ that satisfy the admissibility requirement listed below:

$\varpi(u, r, x, y, g; \xi) \in \Omega$, whenever

$$u = q(z)\ , r = k\gamma(1-\sigma)^m\xi q'(z) + (1-\gamma(1-\sigma)^m)q(z),$$

$$R\left(\frac{x - 2(1-\gamma(1-\sigma)^m)r + (1-\gamma(1-\sigma)^m)^2u}{\gamma(1-\sigma)^m\big(r - (1-\gamma(1-\sigma)^m)\big)u}\right) \leq \frac{1}{m}\ R\left\{\frac{zq''(z)}{q'(z)} + 1\right\},$$

$$R = \left(\frac{y - 3x + \big(3-\gamma^2(1-\sigma)^{2m}\big)r - \big(1-\gamma^2(1-\sigma)^{2m}\big)u}{\gamma^2(1-\sigma)^{2m}\big(r - (1-\gamma(1-\sigma)^m)\big)u}\right) \leq \frac{1}{m^2}\ R\left\{\frac{z^2q^{(3)}(z)}{q'(z)}\right\}$$

$$\text{and } R\left(\frac{\begin{matrix}g - 2(2+\gamma(1-\sigma)^m)y - \big(\gamma^2(1-\sigma)^{2m} - 6\gamma(1-\sigma)^m - 6\big)x \\ +2\big(\gamma^3(1-\sigma)^{3m} + \gamma^2(1-\sigma)^{2m} - 3\lambda(1-\sigma)^m - 2\big)r \\ +(1-\gamma(1-\sigma)^m)\big(1 + 3\gamma(1-\sigma)^m + 2\gamma^2(1-\sigma)^{2m}\big)u\end{matrix}}{\gamma^3(1-\sigma)^{3m}\big(r - (1-\gamma(1-\sigma)^m)\big)u}\right) \leq \frac{1}{m^3}\ R\left\{\frac{z^3q^{(4)}(z)}{q'(z)}\right\}$$

where $(z \in \mathfrak{A}, \xi \in \partial\mathfrak{A}, k \geq 3)$.

Theorem 3.2: *Suppose $\varpi \in \mathrm{M}_i[\Omega, q]$. If the function $f \in \tilde{\hat{A}}$ and $D_{\gamma,\sigma}^{s,m,k}f(z) \in \mathcal{Q}_0$ fulfill the following requirements:*

$$\mathrm{Re}\left\{\frac{z^2q^{(3)}(z)}{q'(z)}\right\} \geq 0\ , \left|\frac{D_{\gamma,\sigma}^{s,m,k+2}f(z)}{q'(z)}\right| \leq \frac{1}{m^2} \tag{20}$$

and $\varpi\begin{pmatrix} D_{\gamma,\sigma}^{s,m,k}f(z), D_{\gamma,\sigma}^{s,m,k+1}f(z), D_{\gamma,\sigma}^{s,m,k+2}f(z), \\ D_{\gamma,\sigma}^{s,m,k+3}f(z), D_{\gamma,\sigma}^{s,m,k+4}f(z); z \end{pmatrix}$ is univalent in $\mathfrak{A}$ with

$$\Omega \subset \left\{\varpi\begin{pmatrix} D_{\gamma,\sigma}^{s,m,k}f(z), D_{\gamma,\sigma}^{s,m,k+1}f(z), D_{\gamma,\sigma}^{s,m,k+2}f(z), \\ D_{\gamma,\sigma}^{s,m,k+3}f(z), D_{\gamma,\sigma}^{s,m,k+4}f(z); z \end{pmatrix}; \ z \in \mathfrak{A}\right\} \tag{21}$$

then $q(z) \prec D_{\gamma,\sigma}^{s,m,k}f(z)$ $(z \in \mathfrak{A})$

***Proof*:** Consider the function $p(z)$ as described by the equation (7) and χ by (14). We have $\varpi \in \mathrm{M}_i[\Omega, q]$. As a result, using equations (15) and (21). We get

$$\Omega \subset \{\chi(p(z), zp'(z), z^2p''(z), z^3p^{(3)}(z), z^4p^{(4)}(z); z); z \in \mathfrak{A}\}.$$

Using equations (13) and (14), the admissibility condition for χ in definition 1.5 is established for k=3. Utilizing lemma 1.7 and equation (21), and given that $\chi \in \psi_3[\Omega, q]$, we obtain:

$$q(z) \prec D_{\gamma,\sigma}^{s,m,k}f(z)$$

Ω is equal $h(\mathfrak{A})$) for conformal mapping $h(z)$: $\mathfrak{A} \to \Omega$, whenever $\Omega \neq \mathfrak{C}$ is a simply connected domain. For this specific case, $\mathrm{M}_i[h(\mathfrak{A}), q]$ be a class symbolized as $\mathrm{M}_i[h, q]$.

The next theorem is as a result of the preceding theorem.

Theorem 3.3: *For* $f \in \tilde{\hat{\mathrm{A}}}$, $D_{\gamma,\sigma}^{s,m,k}f(z) \in \mathcal{Q}_0$ *and* $q \in \mathcal{J}_0$ *with* $q'(z) \neq 0$. *let* $\varpi: \mathfrak{C}^5 \times \mathfrak{A} \to \mathfrak{C}$. *Suppose* h *be analytic function in* $\mathfrak{A}$.

If the function $\varpi \in \mathrm{M}_i[\,\Omega, q]$ satisfy the condition (20), also $\left\{\varpi\begin{pmatrix} D_{\gamma,\sigma}^{s,m,k}f(z), D_{\gamma,\sigma}^{s,m,k+1}f(z), D_{\gamma,\sigma}^{s,m,k+2}f(z), \\ D_{\gamma,\sigma}^{s,m,k+3}f(z), D_{\gamma,\sigma}^{s,m,k+4}f(z); z \end{pmatrix} : z \in \mathfrak{A}\right\}$ is univalent in $\mathfrak{A}$, and

$$h(z) \prec \varpi\begin{pmatrix} D_{\gamma,\sigma}^{s,m,k}f(z), D_{\gamma,\sigma}^{s,m,k+1}f(z), D_{\gamma,\sigma}^{s,m,k+2}f(z), \\ D_{\gamma,\sigma}^{s,m,k+3}f(z), D_{\gamma,\sigma}^{s,m,k+4}f(z); z \end{pmatrix} \tag{22}$$

then $q(z) \prec D_{\gamma,\sigma}^{s,m,k}f(z)$ $(z \in \mathfrak{A})$.

***Proof*:** The proving of the theorem resembles the one in theorem 2.4 and is therefore removed here.

4. Sandwich- Type Theorems:

Putting theorems 2.4. and 3.3 together. We derive the subsequent sandwich-type theorem.

Theorem 4.1: *Suppose* $h_1(z)$ *and* $q_1(z)$ *are analytic functions in* $\mathfrak{A}$, *and let the function* $h_2(z)$ *be univalent function in* $\mathfrak{A}$. *Assume that* $q_2(z) \in \mathcal{Q}_0$ *with* $q_1(0) = q_2(0) = 1$ *and* $\varpi \in \mathrm{M}_i[h_1, q_1] \cap \mathrm{M}_i[h_2, q_2]$ *.If* $f \in \tilde{\hat{\mathrm{A}}}$, $D_{\gamma,\sigma}^{s,m,k}f(z) \in \mathcal{Q}_0 \cap \mathcal{J}_0$ *and if the function* ϖ *given by*

$$\varpi\left(D_{\gamma,\sigma}^{s,m,k} f(z), D_{\gamma,\sigma}^{s,m,k+1} f(z), D_{\gamma,\sigma}^{s,m,k+2} f(z) D_{\gamma,\sigma}^{s,m,k+3} f(z), D_{\gamma,\sigma}^{s,m,k+4} f(z); z\right)$$

is univalent in $\mathfrak{U}$ *, then the following set inclusion:*

$$h_1(z) \prec \varpi \begin{pmatrix} D_{\gamma,\sigma}^{s,m,k} f(z), D_{\gamma,\sigma}^{s,m,k+1} f(z), D_{\gamma,\sigma}^{s,m,k+2} f(z), \\ D_{\gamma,\sigma}^{s,m,k+3} f(z), D_{\gamma,\sigma}^{s,m,k+4} f(z); z \end{pmatrix} \prec h_2(z)$$

implies that: $q_1(z) \prec D_{\gamma,\sigma}^{s,m,k} f(z) \prec q_2(z)$

5. Conclusion

The operator $D_{\gamma,\sigma}^{s,m,k} f(z)$ is utilized to yield conclusions that involve fourth-order differential subordination for analytical functions. Examining the appropriate sets of admissible functions yields the results. Additionally, specific instances of those types of acceptable functions and some significant inequalities have been established. Our findings are associated with those in various prior studies pertaining to notion of differential subordination and superordination within geometrically function notion

References

[1] M. R. R. Omar and Shaharuddin Cik Soh, "New differential operator for analytic univalent functions associated with binomial series," in AIP Conference Proceedings (2024).

[2] M. S. Mocanu and P. T. Mocanu, Differential Subordinations: Theory and Applications, Series on Monographs and Textbooks in Pure and Applied Mathematics, vol. 225, New York: Marcel Dekker (2000).

[3] J. Abdulhalim Abdulhalim and Sanford S. Miller, "Third-order differential inequalities and subordinations in the complex plane," *Complex Variables and Elliptic Equations*, vol. 56, pp. 439–454 (2011).

[4] D. A. Salman, A. R. S. Jassim, and Ali Al-Fayadh, "Fuzzy differential for subclass of analytic functions defined by linear operator," *Journal of Interdisciplinary Mathematics*, vol. 26, no. 4, pp. 585–593 (2022).

[5] S. A. Al-Abbasi, W. G. Al-Rahmani, and Faez Ali Al-Maamori, "On sandwich results of univalent functions defined by a linear operator," *Journal of Interdisciplinary Mathematics*, vol. 23, no. 2, pp. 803–809 (2020).

[6] K. A. Jassim, Z. H. Mohammed, and Buthyna N. Shihab, "On the characteristic properties for subclasses of the higher derivative of analytic functions," *Iraqi Journal of Science*, vol. 61, no. 12, pp. 3372–3378 (2020).

[7] A. Kareem and A. H. Majeed, "New strong differential subordinations and superordinations of symmetric analytic functions," *International Journal of Mathematical Analysis*, vol. 11, no. 11, pp. 543–549 (2017).

[8] M. K. Rasheed and Abdulrahman H. Majeed, "Seven-parameter Mittag-Leffler operator with second-order differential subordination results," *Nonlinear Functional Analysis and Applications*, vol. 28, no. 4, pp. 903–917 (2023).

[9] A. A. Al-Tamimi, M. S. Saleem, and Abdelhamid Albaid, "Third-order differential subordination for meromorphic functions associated with generalized Mittag-Leffler function," *Fractal and Fractional*, vol. 7, p. 175, pp. 1–12 (2023).

[10] L. T. Kareem, H. F. Al-Jubouri, and Ahmed M. Ali, "Differential sandwich subordination of convolution operator acting on complex meromorphic functions," *Journal of Interdisciplinary Mathematics,* vol. 26, no. 2, pp. 245–260 (2023).

[11] M. I. Hamid and Buthyna Najad Shihab, "Several subclasses of r-fold symmetric bi-univalent functions possess coefficient bounds," *Iraqi Journal of Science*, vol. 63, no. 12, pp. 5438–5446 (2022).

[12] L. T. Ahmed and Abdul Rahman S. Juma, "Majorization and quasi-subordination problems for certain subclasses of holomorphic functions utilizing differential operator," *Iraqi Journal of Science*, vol. 65, no. 7, pp. 3860–3865 (2024).

[13] Alina Alb Lupaş and Firas Ghanim, "Strong differential subordination and superordination results for extended q-analogue of multiplier transformation," *Symmetry*, vol. 15, pp. 1–16 (2023).

[14] A. M. Dawood, W. G. Al-Rahmani, and Ali Hussein Battor, "Fourth-order differential subordination and superordination results of meromorphic multivalent functions defined by multiplier transformation," *International Journal of Nonlinear Analysis and Applications*, vol. 12, special issue, pp. 2297–2313 (2021).

[15] S. S. Miller and P. T. Mocanu, "Subordinants of differential superordinations," *Complex Variables, Theory and Application: An International Journal*, vol. 48, pp. 815–826 (2003).

Received October, 2024