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Operational calculus of the expanded SEE transform technique

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Abstract

In this paper, we showed the theorem of convergence and existence for the transformation by the SEE transform technique. We have structured the appropriate space of pseudo quotients for the SEE technique. The expanded SEE technique has been defined as the expanded SEE transformation on the space of pseudo quotients. We derived several propositions of the expanded SEE integral technique and advanced its operational calculus. This paper is considered a complementary paper to all studies on SEE transform.

Subject Classification: *15A04, 20M20.*

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1. Introduction

Recently an integral transformation called The SEE transformation has been introduced by Mansour, Kuffi, and Mehdi, [2, 3]. The SEE transformation

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technique is naught, however, a consolidation of Laplace's method, Sumudu's technique, El-Zaki's technique, and Sadik's transformation, those transformations whose kernels are exponential kind or like (modify) to Laplace's kernel transformation. SEE technique is more critical in properties, theorems, and complicated natural sciences and engineering applications.

The pseudo-quotient space is denoted as P(A, G). As previously mentioned, this area has been organized using algebra and applied elementary calculus [8]. L. Schwartz introduced the notion of divisions, which gives Dirac division and their differentiation in 1950. The structure of Schwartz divisions is relatively high, albeit all are propagated functions. Several writers are interested in the idea of the structure of pseudo quotients since they are patronized as generalized functions. In [6], scholars have extended Laplace's transformation on P(A, G). Use a space for pseudo quotients P(A, G), academics has developed operational calculus for a enlarged Sumudu's approach in [4]. In this article, we are restoring the structure of the suitable space for pseudo quotients P(A, G) when a SEE transformation technique, and advancing the operational calculus for the enlarged SEE technique. Laplace, and Sumudu procedures are special cases for a SEE transformation.

You can find the article's structure below. We address many theorems of pseudo quotients and elaborate on the structure of their space. Additionally, we present the SEE method, establish a connection between the SEE and Laplace transformations, and demonstrate and validate the SEE transformation's convergence and existence theorems. In addition, we broaden the scope of the SEE integral method to include the space of pseudo quotients. Our work advances the operational calculus of SEE integral transformation, defines convolution product, and propagates differentiation and multiplication via " t ". real analysis play in algebra see, [10] and [11].

2. Helpful results

In this part of the paper, we structure the space of pseudo quotients. Ago, our aim is to expand the SEE technique over a space pseudo quotient. So, we want a set for continuous functions that has mainly exponential growth. Khosravil in [9] calculates the structure of pseud quotients on detail. When a structure for pseudo quotients, we want the pair for set ($\mathcal{X}$, G).

Definitions 2.1: [1]

1. a pair for sets ($\mathcal{X}$, G) called Σ-pair when $\mathcal{X}$ represent nonempty set, and G in abelian semi-group expressible on$\mathcal{X}$.
2. the Σ–pair ($\mathcal{X}$, G) named injective if element for semi-group G acts over $\mathcal{X}$ injectively. Consider an injective Σ-pair ($\mathcal{X}$, G), we from a product space $\mathcal{X} \times G$.

3. relationship ~ on element $\mathcal{X}$ ×G known as:
   ($\varkappa$,g) ~ (y, h) if h $\varkappa$ = gy, ∀ ($\varkappa$, g), (y, h) ∈ $\mathcal{X} \times G$.
4. **4.** he space for every equivalence classes [($\varkappa$, g)] for all $\varkappa \in \mathcal{X}$ &g ϵ G called a space for pseudo quotients. Its symbolized via P ($\mathcal{X}$,G)

Theorem 2.2: [1] *A relationship ~ over $\mathcal{X} \times G$ an equivalence relationship*

To prove this theorem:

Suppose that ($\varkappa$, g), (e,g) € $\mathcal{X} \times G$

(1) Let g $\varkappa$ = g $\varkappa$, then ($\varkappa$, g) ~ ($\varkappa$, g), so ~ is a reflexive.

(2) Let ($\varkappa$, g)~ (e,g), (e,g) ~ (z, h), then gx = g e , h e =gz
Suppose that h ∈ G so hg $\varkappa$ = h g e.
Via commutatively in G, we get gh $\varkappa$ = hfy, and h e = gz
so gh $\varkappa$ = g gz , gh $\varkappa$ = ggz it implice that h $\varkappa$ = gz.
Then ($\varkappa$, g) ~ (z,h).

(3) the prove of symmetry for ~ is observe.
Then ~ equivalence relationship

3. SEE Integral Technique

The Laplace technique has formerly shown its significance in natural sciences by finding the solution to various problems of differential and integral operator equations and integro-differential operator equations. Laplace's integral technique is useful with respect to it is domain & operational calculus. Laplace technique for operational calculus is fundamental to research. There are different integral techniques, like Laplace's technique, and all they displayed their might to calculate the solution of applications in branches of sciences via applying their operational calculus. An integral technique, "SEE Transform," has been inserted in (2021) via Eman A., Emad A., and Sadik A for more details [2, 3, 5-8]. It is a union of Laplace's technique and every one of those integral techniques, which are like a kind of Laplace's transformation.

Definitions 3.1:

1. A continuous piecewise function on ℝ (the set of real numbers) which disappear on the open interval (-∞,0) which has exponential order W, for all W∈ ℝ is said to be an SEE function Let β be a set of all SEE function
2. Let f(x) ∈ β then SEE denoted as S { f(x) } and is defined as :
   $$S\{ f(x) \} = \frac{1}{V^n}\int_0^\infty e^{-vx}\ f(x)dx = T(v, n) \tag{3.1}$$
   If the integral of eq. (3.1) exists the SEE technique of f(x) exists, and it is symbolized as T(v, n), for all n ∈ ℤ (the set of integer numbers), provided that v is a complex parameter (v = $v_1 + iv_2$), i ∈ ℂ .

3.1 *Relationship between Laplace's Transform with the SEE Transform*

If F(v) is a Laplace's integral technique with T (v, n) is a SEE Transform of function f(x) then:

$$\text{T (v, n)} = \frac{1}{v^n}\text{T(v)} \tag{3.2}$$

Definition 3.2: [1] Let f(x) and g(x) be functions , so the convolution product, denoted as f(x) $*$ g(x) and:

$$\text{F(x)} * \text{g(x)} = \int_0^x f(x-t)g(t)dt \tag{3.3}$$

Theorem 3.3: [1] *If f(x), g(x) ∈ β and f(x) $*$ g(x) convolution product for f(x) & g(x), hence a Laplace's integral technique for f(x) * g(x) defined as:*

$$\mathcal{L}\{\text{ f(x)} * \text{g(x) }\} = \text{T(v). G(v)} \tag{3.4}$$

For all T (v) with G (v) are the Laplace's transforms of the functions f(x) with g(x).

Theorem 3.4*: Let T (v, n), G(v , n) be the SEE techniques of the functions f(x) and g(x) respectively with f(x) $*$ g(x) is a product of a convolution so*

$$\text{S } \{\text{ f(x)} * \text{g(x) }\} = v^n\text{T (v) . G(v).} \tag{3.5}$$

Proof: Since

$$\mathcal{L}\{\text{ f(x)} * \text{g(x) }\} = \text{T (v) . G(v)} \tag{3.6}$$

$$\text{T (v, n)} = \frac{1}{v^n}\text{T(v) and G (v, n)} = \frac{1}{v^n}\text{G(v).}$$

Also,

$$\begin{aligned}\text{S } \{\text{ f(x)} * \text{g(x) }\} &= \frac{1}{v^n}\text{T(v, n) . G (v, n),}\\ &= v^n\,\frac{1}{v^n}\text{T(v).}\ \frac{1}{v^n}\text{G(v),}\\ &= v^n\text{ T (v, n). G (v, n).}\end{aligned}$$

Theorem 3.5: [1] *Let $T_1(v), T_2(v), T_3(v)$,..., $T_m(v)$ are Laplace's transforms of $f_1(x)$, f_2 (x), f_3 (X) ,...,f_m (x), then*

$$\text{L } F_1(x) * F_2(x) * F_3(x) * \ldots * F_m(x)]= T_1 (V) \text{ . ... . } T_m (V) \tag{3.7}$$

Theorem 3.6: *Let T_1 (v,n) ,T_2(v,n) , ... , T_m (v,n) are SEE transforms of $f_1(x)$,$f_2(x)$, ... ,f_m (x), then*

$$\text{S}\{ f_1(x) * \ldots * f_m (x) \} = v^{(m-1)n}\ T_1 (v,n) \text{ . ... . } T_m (v,n) \tag{3.8}$$

Proof, we will show this theorem via applying eq. (3.2) and eq. (3.6):

$$\text{S}\{ f_1 (x) * \ldots * f_m (x) \} = \frac{1}{v^n}\mathcal{L}\{ f_1(x) * f_2(x) * \ldots * f_m (x) \}\ v$$

$$= \frac{1}{v^n} \text{T}_1 \text{(V)} \ldots . \text{T}_m \text{(v)},$$
$$= \text{v}^{(m-1)n} \left[\frac{1}{v^n} \text{T}_1 \text{(V)} \right] \ldots \left[\frac{1}{v^n} \text{T}_m \text{(v)} \right],$$
$$= \text{v}^{(m-1)n} \text{T}_1\text{(v)} \ldots . \text{T}_m \text{(V)}.$$

Theorem 3.7: *Assume $\mathcal{H}$ (X) is unit-step function, so a SEE technique for $\mathcal{H}$(x) defined as: S { $\mathcal{H}$ (x) } = $\frac{1}{v^{n+1}}$, n ∈ $\mathbb{Z}$ With the fundamental part of (v) is more significant than zero.*

Proof: Since $\mathcal{H}$(x) =1 in the open interval (0,∞) and $\mathcal{H}$(x) = 0 otherwise, then S{ $\mathcal{H}$(x)} = $\frac{1}{v^n} \int_0^\infty e^{-vt}$ $\mathcal{H}$ (x) dx= $\frac{1}{v^n} \int e^{-vt}$ (1) dt = $\frac{1}{v^n} \int_0^\infty e^{-vt}$ (1) dt = $\frac{1}{v^n}$ $\left[\frac{e^{-vt}}{-v}\right]_0^\infty = \frac{1}{v^{n+1}}$. Re (v) > 0.

Theorem 3.8: [7] *Suppose that T (v,n) is a SEE technique of f (x) hence*

S { t f (t) } = $\frac{-1}{v}$ [v $\acute{T}$(v,n)+(v,n)] , real part of v is greater than zero .

$$= -\acute{T} \text{ (v,n)} - \frac{1}{v} \text{T(v,n)} \tag{3.9}$$

Theorem 3.9: *Suppose that f(x) ∈ β so the SEE technique of f(x) exists and converges absolutely $for\ each$ The real part of (v) is more significant than zero.*

To prove this theorem:

For some real w, we have

$$|f(x)| \leq P_1 e^{wx}, \ \text{x} \geq x_0.$$

Also, since f(x) is a continuous piecewise function in the close internal [0,x_0], therefore f(x)is limited on the close interval [0,x_0]

$$|f(x)| \leq P_{2,} for\ x \in [0, x_0].$$

Therefore, for all x> 0, an arbitrary constant P can be chosen sufficiently large, so that

$$|f(x)| \leq Pe^{wt}$$

Now, by substituting

$$\text{v} = v_1 + iv_2 \rightarrow v^{-n} = \alpha + i\beta \rightarrow |v^{-n}| = \sqrt{\alpha^2 + \beta^2}, |e^{-vx}| = \left|e^{-(v_1+iv)x}\right|$$
$$= e^{-v_1 x}. \text{ We have}$$

$$\int_o^{\mathcal{T}} |v^{-n}| \ |e^{-vx}| \ |f(x)| \ dx \leq \frac{P.(\sqrt{\alpha^2+\beta^2}}{v_1 - w}\left[1 - e^{-(v_1-w)\mathcal{T}}\right].$$

Taking Limit as $\mathcal{T} \rightarrow \infty, we\ get$:

$$\int_o^{\mathcal{T}} |v^{-n}| \ |e^{-vx}| \ |f(x)| \ dx \leq \frac{P(\alpha^2+\beta^2)^{\frac{1}{2}}}{v_1 - w}.$$

Then, the SEE transform converges absolutely for all $v_1 - w > 0, that\ is$ $Re(v) > w$.

4. SEE Integral Technique on The Space of Pseudo quotients

In this part of the paper, we will expand the SEE integral technique on the appropriate space of pseudo quotients. Consider exponential function g(x)=$e^{-\frac{1}{x}}$, for all x>0 with g(x)= 0 for all x=0 , Since g(x) is a continuous limited function in the open interval (0,∞), then via the above theorem g(x) ϵ β let G(v , n) be the SEE integral technique of g(x), and suppose that

$$\mathbb{G} = \{g^m \mid m\in N\}. \tag{4.1}$$

As well as, here g^m = g(x) ∗ g(x) ∗ ... ∗g(x) (n-times) and g(x)=$e^{-\frac{1}{x}}$.

Notation 4.1: [1] An algebra structure ($\mathbb{G}$, ∗) is abelian semi-group.

Definition 4.2: [1] An action for $\mathbb{G}$ in (β) symbolized as (f ∗ $g^m)_{(x)}$ & its defined by:

$$(f * g^m)\,(t) = \int_0^x g^m\,(t-x)\, f\,(t)\, dt,\ \forall\, f \in \beta \tag{4.2}$$

Lemma 4.3: *The action for* $\mathbb{G}$ *in* β *an injective.*

Proof: Assume that f∈ β, g^m ∈ $\mathbb{G}$ & m∈ ℝ.

Our goal is to show if f∗ g^m = 0 therefore f =0·

Since f∗ g^m=0, using the SEE integral technique on both sides, we get

$$v^n \mathrm{T(v,n)}.\frac{1}{v^{m+n}} = 0\,, \forall\, Re\,(v) > 0.$$

Therefore

$$\frac{T(v,n)}{v^m} = 0\,, so\ T(v,n) = 0 \text{ so } S\{f(x)\}=0, \text{ so}$$

f(x) = 0. Then action for $\mathbb{G}$ over β is injective.

Definitions 4.4: [1]

1. the set for product β × $\mathbb{G}$ given as:

β× $\mathbb{G}$ = {(f,g^m) / f ∈ β , g^m ∈ G for some m∈ ℕ}

2. Assume that m , l ∈ ℕ and f_1 , f_2 ∈ β then (f_1 , g^m) and (f_2 , g^l) are called equivalent if and only if $f_1 \times g^m = f_1 \times g^l$ so, (β , $\mathbb{G}$) is a Σ- pair.

3. The set for pseudo quotients is a set for P(β , $\mathbb{G}$) for equivalence classes [(f, g^m)] , that is p (β , $\mathbb{G}$) = { $\frac{f}{g^m}$ $/f \in \beta$, g^m ∈ $\mathbb{G}$, m ∈ ℕ}

We consider (+) addition with, (∗) convolution & (.) scalar multiplication of every element for P(β , 𝔾) as below approach :

Assume that $\frac{f_1}{g^m}$,$\frac{f2}{g^l}$ $\in P(\beta , \mathbb{G})$ and $\alpha \in \mathbb{C}$, then

(1) $\frac{f_1}{g^m} + \frac{f_2}{g^l} = \frac{g^m * f_1 + g^l * f_2}{g^{l+m}}$

(2) $\frac{f_1}{g^m} * \frac{f_2}{g^l} = \frac{f_1 * f_2}{g^{m+l}}$

(3) $\alpha \frac{f}{g^m} = \frac{\alpha f}{g^m}$, $\frac{f}{g^m} \in P(\beta, \mathbb{G})$

Theorem 4.5: [1] *P (β, 𝔾) commutative linear algebra + , ∗ , . and unity.*

Definition 4.6: Suppose that $p \in P(\beta, \mathbb{G}) \ni P = \frac{f}{g^m}\ \forall\ f \epsilon \beta$ and $m \in \mathbb{N}$ (the set of natural numbers). Then its expanded SEE integral transform is defined as:

$$S\{\rho(x)\} = \frac{1}{V^{mn}} \cdot \frac{T_{(v,n)}}{G^m(v,n)} \cdot \forall\ Re(v) > 0 \tag{4.3}$$

S{f(x)} = T(v,n), & S{ g(x)} = G(v,n) with G^m multiplication power.

Definition 4.7: [1] Generalization of Differentiate Assume that $p \in P(\beta, \mathbb{G}) \ni p = \frac{f}{g^m}$,the generalized differentiate of P(x) is defined as:

$$D\{p(x)\} = \frac{\acute{f}}{g^m} + \frac{f_{(0)}\ \delta_{(x)}}{g^m} = \frac{\acute{f} + f_{(0)}\ \delta_{(x)}}{g^m}$$

Definition 4.8:

1. Suppose that $p \in P(\beta, \mathbb{G}) \ni P = \frac{f}{g^m}$ with D is the generalized differentiate then the SEE technique of D(p(x)) is knew as:

$$S\{D(p(x))\} = \frac{1}{V^n G^m_{(V,n)}} [S\{\acute{f}(x)\} + f_{(0)} \cdot S\{\delta(x)\}]$$

For Re (V) > 0.

2. Assume that $p \in P(\beta, \mathbb{G}) \ni P = \frac{f}{g^m}$,we define $(F*p)(x) = \frac{F \cdot f}{g^m} - \frac{m(f * F_g)}{g^{m+1}}$, $\forall\ m > 1$, and F is a function defined as T(x)= x $\mathcal{H}$(x), $x \in \mathbb{R}$. one above definition provide the law of multiplication of pseudo quotient via "X".

Theorem 4.9: *Let* $p \in P(\beta , \mathbb{G}) \ni P = \frac{f}{g^m}$ *and* $(F.p)_{(x)} = \frac{F_t}{g^m} - \frac{m\,(f \times F_g)}{g^{m+1}}$, $\forall\ m > 1$*, where F is an ordinary multiplication for f as x then*

$$S\{F \cdot P\} = \frac{-1}{V} [v \frac{d}{dV} S\{P\} + n\,S\{P\}] \tag{4.4}$$

$$= \frac{-n}{v} S\{P\} - v \frac{d}{dV} S\{P\}$$

Proof: It is given that

$$(F \cdot P)_{(X)} = \frac{F_f}{g^m} - \frac{m(f * F_g)}{g^{(m+1)}}, \text{ for m} > 1.$$

Using the SEE transform on both sided, we obtain

$$S\{F \cdot p\} = \frac{1}{V^{n(m+1)} \cdot G^{m+1}} \left[\frac{-v\acute{T}}{V} - \frac{n\,T}{V} + \frac{mTv\acute{G}}{G\,.\,V} + \frac{mT_n\,G}{G\,.V}\right]$$

$$= \frac{-1}{V}\left[\frac{V\acute{T}}{V^{mn}\;G^m} + \frac{n\,T}{V^{mn}\quad G^m} - \frac{m\,T_V\acute{G}}{V^{mn}\,G^{m+1}} - \frac{m\,T_n}{V^{mn}G^m}\right]$$

$$= \frac{-1}{V}\left[\frac{V\acute{T}}{V^{mn}G^m} - \frac{n.T.G.V}{V^{mn}G^{m+1}} + \frac{n.T}{V^{mn}G^m} - \frac{nm\,T}{V^{mn}G^m}\right]$$

$$= \frac{-1}{V}\left[v\left(\frac{\acute{T}}{V^{mn}G^m} - \frac{N.T.G}{V^{mn}G^{m+1}} - \frac{M.n.T}{V^{mn}G^m}\right) + \frac{n\,T}{V^{mn}G^m}\right]$$

$$= \frac{-n}{V}\,S\{p\} - \frac{d}{dV}\,S\{p\}.$$

Hence proved.

5. Conclusion

In this paper, we have generalized the SEE ((Sadiq - Emad - Eman)) method to the space of obtained pseudo quotients. In order to compute integral operator equations (D.O.$E.^S$) and differential operator equations (D.O.$E.^S$), the properties of the SEE technique's advanced differentiation and propagated multiplication via "x" are crucial. The solutions to these equations will be propagated functions. This article presents a study of the operational calculus of the enlarged SEE transform, even if the Sadik transform technique is more generic. Every transform has its own unique set of challenges, which is why there is no universally preferred transform. In this paper, we have extended the SEE ((Sadiq – Emad – Eman) technique on the space of pseudo quotients achieved. Properties on the SEE technique of advanced differentiate and propagated multiplication via "x" is important to calculate differential operator equations and integral operator equations whose solutions will be propagated functions. In this article, although the Sadik transform technique is more general than the SEE transforms, we have presented a study of the operational calculus of expanded SEE transform. The reason is that there is no preference between any transform in general, but rather every transform has a special problem.

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