Journal of Interdisciplinary Mathematics
ISSN: 0972-0502 (Print), ISSN: 2169-012X (Online)
Vol. 28 (2025), No. 3-B, pp. 1043–1057
DOI : 10.47974/JIM-2077

On the fourth Hankel determinant introduced by using a new subclass

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Abstract

This article investigates the fourth Hankel determinant for $\mathcal{N}(\alpha, \gamma, \delta, \beta, \eta, \lambda, \mu)$, the novel subclass for analytic function characterized by subordination. Estimates on a coefficients $|a_i|$ with $i = 2, 3, 4, 5, 6$ and 7 are provided for the functions in this recently described class, as well as an upper limit on fourth Hankel determinant. Authors asserted and demonstrated through the constructing of a new subclass of upper limits for fourth Hankel determinant of subclass for analytic function. A corollaries for main theorems provide some interesting insights, and the accompanying paper highlights the importance of the findings. These results can be used in various applications in mathematics and engineering sciences. For example, system identification, orthogonal polynomials, signal processing, approximation Theory, and this study contributes to the ability to expand our knowledge and understanding for analytical function for unit and its interactive higher relation. These findings are believed to open the doors to explorations and many future applications, not only in mathematics but also in computer science and engineering.

Subject Classification (2010): *30C45.*

Keywords: *Analytic function, Coefficient bounds, Fekete-Szegő functional, Hankel determinant, Subordination.*

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1. Introduction

The geometric features for analytic function is branch in complex analysis field known as geometric function theory. An original inspiration of these idea came from Koebe's 1907 book writings[1], Gronwall in 1914, and Bieberbach in 1916. Their work have been regarded as foundation of other advancements of this subject, including Bieberbach's second coefficient estimate, Gronwall's Area theorem, and Koebe's seminal publication [2]. By then, univalent function theory had become separate field for work. Differential subordination proposed by Petru and Sanford. Mocanu and Miller initially presented differential subordination in their book, published in 2000 [3]. In 2003, they introduced differential superordination concept as a complementing [4].

The normalized class for functions f satisfy conditions $f(0) = f'(0) - 1 = 0$, denoted by $\mathcal{A}$, is represented by the following Taylor expansion:

$$f(\zeta) = \zeta + \sum_{j=2}^{\infty} \mathfrak{a}_j \zeta^j \ , \zeta \in \mathfrak{U} \tag{1.1}$$

Here, $\mathfrak{U} = \{\zeta \in \mathbb{C} \ and \ |\zeta| < 1\}$ denotes to open unit disk in complex plane, where $\mathbb{C}$ represents complex numbers, and function is analytic.

The principles of subordination are fundamental when comparing two analytic functions $\mathcal{F}(\zeta)$ and $r(\zeta)$ in $\mathfrak{U}$. Function $\mathcal{F}(\zeta)$ is subordinate to $r(\zeta)$, denoted $\mathcal{F}(\zeta) \prec r(\zeta)$, if there exists an analytic function $t(\zeta)$ in $\mathfrak{U}$, with $t(0) = 0$ and $|t(\zeta)| < 1$, such that $\mathcal{F}(\zeta) = r(t(\zeta))$. If r is univalent in $\mathfrak{U}$, then $\mathcal{F}(\zeta) \prec r(\zeta)$ implies $\mathcal{F}(0) = r(0)$ and $\mathcal{F}(\mathfrak{U}) \subset r(\mathfrak{U})$.

Mendiratta Nagpal, and Ravichandran [5] developed subclass S_1^* for analytic function associated with the exponential function . Using the subordination approach, Minda and Ma [6] created the class for starlike function and studied classes $S^*(\Theta)$ & $G^*(\Theta)$ given by:

$$S^*(\Theta) = \left\{ f \in \mathcal{A}: \frac{\zeta f'(\zeta)}{f(\zeta)} \prec \Theta(\zeta), \zeta \in \mathfrak{U} \right\}$$

and

$$G^*(\theta) = \left\{ f \in \mathcal{A}: 1 + \frac{\zeta f''(\zeta)}{f(\zeta)} \prec \Theta(\zeta), \zeta \in \mathfrak{U} \right\}.$$

Mendiratta Nagpal, and Ravichandran [5] introduced the family $S_e^* := S^*(e^{\zeta})$ as follows:

$$S_e^* = \left\{ f \in \mathcal{A}: \frac{\zeta f'(\zeta)}{f(\zeta)} \prec e^{\zeta}, \zeta \in \mathfrak{U} \right\}. \tag{1.2}$$

Similarly, the following class was likewise added in by use of an Alexander-type relation in [5]:

$$G_e^* = \left\{ f \in \mathcal{A}: 1 + \frac{\zeta f''(\zeta)}{f(\zeta)} \prec e^{\zeta}, \zeta \in \mathfrak{U} \right\}. \tag{1.3}$$

The families C_e^* and G_e^* provided in the investigation are symmetric about the real axis, based on those definitions.

Pommerenke [7, 8] defined the Hankel determinant $\mathcal{H}_q(i)$ for $f \in \mathcal{A}$ of the type (1.1) with specified parameters $q, i \in \mathbb{N} = \{1,2, \dots\}$, as follows:

$$\mathcal{H}_{q}(i) = \begin{vmatrix} \mathfrak{a}_{i} & \mathfrak{a}_{i+1} \cdots & \mathfrak{a}_{i+q-1} \\ \mathfrak{a}_{i+1} & \mathfrak{a}_{i+2} \cdots & \mathfrak{a}_{i+q} \\ \vdots & \vdots & \vdots \\ \mathfrak{a}_{i+q-1} & \mathfrak{a}_{i+q} \cdots & \mathfrak{a}_{i+2q-2} \end{vmatrix}, (\mathfrak{a}_1 = 1). \tag{1.4}$$

The study for a theory of singularities, & power series with integral coefficient both depends on an understanding of the Hankel determinant [9]. Readers are encouraged to read for in-depth understanding [10, 11]. The estimated values of $\mathcal{H}_{q}(i)$ for various univalent function subfamilies have been studied. The well-known Fekete-Szegő functional for $q = 2$ and $i = 1$, function $\mathcal{H}_2(1) = \mathfrak{a}_3 - \mathfrak{a}_2^2$. According to [12, 13], the $\mathcal{H}_2(2)$ second Hankel determinant, which is given as $\mathcal{H}_2(2) = \mathfrak{a}_2\mathfrak{a}_4 - \mathfrak{a}_3^2$, was examined for a bi-convex & bi-starlike function class. A precise estimate of $\mathcal{H}_2(2)$for the collection of Bazilevič functions was given by Krishna and RamReddy [14]. The second Hankel determinant estimate for bi-univalent functions using the symmetric q-derivative operator was recently found by Srivastava and his associates [15], and the Hankel & Toeplitz determinant of subcollections for q -starlike function associated with general form for conic domain were discussed by the authors in [16] for more literary works[17-21] .

The following is third Hankel determinant of function for type (1.1):

$$\mathcal{H}_3(1) = \begin{vmatrix} 1 & \mathfrak{a}_2 & \mathfrak{a}_3 \\ \mathfrak{a}_2 & \mathfrak{a}_3 & \mathfrak{a}_4 \\ \mathfrak{a}_3 & \mathfrak{a}_4 & \mathfrak{a}_5 \end{vmatrix} = -\mathfrak{a}_5\mathfrak{a}_2^2 + 2\mathfrak{a}_2\mathfrak{a}_3\mathfrak{a}_4 - \mathfrak{a}_3^3 + \mathfrak{a}_3\mathfrak{a}_5 - \mathfrak{a}_4^2.$$

In 2010 by Babalola [22] presented some intriguing results on $\mathcal{H}_3(1)$ that have motivated other scholars.

A specific subclass of starlike functions third Hankel determinant was investigated in [23]. A third Hankel determinant is not employed in current work, as noted ofd this work [23]. In the following section of this research, a new definition of the fourth Hankel determinate-defined as the determinant of a matrix is presented. The univalent starlike functions' third Hankel determinant bound was determined by Zaprawa Obradović, and Tuneski [24] .

Mendiratta Nagpal, and Ravichandran have presented a subclass S_1^* of analytic function related to exponential function recently in [5].

The definition of the first and second class Chebyshev polynomials [25, 26] for a real variable v on the interval $(-1,1)$ is

$$\mathcal{G}_i(v) = \cos(i \arccos v)$$

and

$$\mathcal{G}_i(v) = \frac{\sin[(i+1)\arccos v]}{\sin(\arccos v)} = \frac{\sin[(i+1)\arccos v]}{\sqrt{1-v^2}},$$

respectively

It looks at the function

$$\mathcal{H}(\mu,\zeta)=\frac{1}{1-2\mu+\zeta^2},\mu\in\left(\frac{1}{2},1\right),\zeta\in\mathfrak{A}.$$

It's commonly understood that if $\mu=\cos\psi$, $\psi\in\left(0,\frac{\pi}{3}\right)$, then

$$\mathcal{H}(\mu,\zeta)=1+\sum_{i=1}^{\infty}\frac{\sin[(i+1)\psi]}{\sin\psi}\zeta^i=1+2\cos\psi\zeta+(3\cos^2\psi-\sin^2\psi)\zeta^2$$
$$+(8\cos^3\psi-4\cos\psi)\zeta^3+\cdots,\zeta\in\mathfrak{A},$$

that is

$$\mathcal{H}(\mu,\zeta)=1+\mathcal{G}_1(\mu)\zeta+\mathcal{G}_2(\mu)\zeta^2+\mathcal{G}_3(\mu)\zeta^3+\mathcal{G}_4(\mu)\zeta^4+\cdots,$$
$$\mu\in\left(\frac{1}{2},1\right),\zeta\in\mathfrak{A}$$

where

$$\mathcal{G}_i(\mu)=\frac{\sin[(i+1)\arccos\mu]}{\sqrt{1-\mu^2}},i\in\mathbb{N},$$

and $\mathcal{G}_i(\mu)$ are Chebyshev polynomials of the second sort. The concept of second-class Chebyshev polynomials makes it obvious that

$$\mathcal{G}_{i+1}(\mu)=2\mu\mathcal{G}_i(\mu)-\mathcal{G}_{i-2}(\mu).$$

Getting that

$\mathcal{G}_1(\mu)=2\mu,\mathcal{G}_2(\mu)=4\mu^2-1,,\mathcal{G}_3(\mu)=8\mu^3-4\mu,$ (for each $i\in\mathbb{N}$).

For a function f of type (1.1), the fourth Hankel determinant[27] is as follows.

$$\mathcal{H}_4(1)=\begin{vmatrix}1 & a_2 & a_3 & a_4\\ a_2 & a_3 & a_4 & a_5\\ a_3 & a_4 & a_5 & a_6\\ a_4 & a_5 & a_6 & a_7\end{vmatrix}=-a_4\mathcal{C}_1+a_5\mathcal{C}_2-a_6\mathcal{C}_3+a_7\mathcal{C}_4, \tag{1.5}$$

where

$$\begin{aligned}|\mathcal{C}_1|&=|a_2||a_4a_6-a_5^2|+|a_3||a_3a_6-a_4a_5|-|a_4||a_3a_5-a_4^2|,\\ |\mathcal{C}_2|&=|a_4a_6-a_5^2|-|a_2||a_3a_6-a_4a_5|+|a_3||a_3a_5-a_4^2|,\\ |\mathcal{C}_3|&=|a_3a_6-a_4a_5|+|a_2||a_2a_6-a_3a_5|-|a_4||a_2a_4-a_3^2|,\\ |\mathcal{C}_4|&=|a_3||a_2a_4-a_2^2|-|a_4||a_4-a_2a_3|+|a_5||a_3-a_2^2|.\end{aligned} \tag{1.6}$$

In a recent study, Arif Raza, and Zaprawa [28] effectively achieved the bound of $\mathcal{H}_4(1)$ for the class of limited turning functions by studying the problem of fourth Hankel determinant $\mathcal{H}_4(1)$ for the first time. Upper bounds for the third- and fourth-order Hankel determinants associated with certain kinds for function with limited turning that is linked to sine function were recently studied by Khan et al. [29].

In addition, estimates for fourth Hankel determinant's upper bound are presented of the novel class of analytic functions formed via subordination. The following lemmas are used to illustrate the research's original theorems.

Lemma 1.1: [30] *Let $\wp$ be the class containing every $\wp(\zeta)$ analytic functions of the following:*

$$\wp(\zeta) = 1 + \sum_{i=1}^{\infty} \wp_i \zeta^i, \tag{1.7}$$

with $\Re e\, \wp(\zeta) > 0$ and $\wp(0) = 1$ for every $\zeta \in \mathfrak{A}$. Then $|\wp_i| \leq 2$, for each $(i = 1,2,3, \dots)$. This difference is significant for every i.

Lemma 1.2: [5] *If function f that belongs to S_1^* is of the form*(1.1), *then*

$$|\mathfrak{a}_2| \leq 1, |\mathfrak{a}_3| \leq \frac{3}{4}, |\mathfrak{a}_4| \leq \frac{17}{36}, |\mathfrak{a}_5| \leq 1,$$

The purpose of this article is to examine the estimate of $|\mathcal{H}_4(1)|$for each of the subclass $\mathcal{N}(\alpha,\gamma,\delta,\beta,\eta,\lambda,\mu)$that are called in the introduction.

2. Main Results

Definition 2.1: If $f \in \mathcal{A}$ defined as in (1.1), then its named in a subclass $\mathcal{N}(\alpha,\gamma,\delta,\beta,\eta,\lambda,\mu)$ if the subsequent condition is true:

$$1 + \frac{1}{\alpha}\left[\frac{\delta\zeta f'(\zeta)}{f(\zeta)} + \frac{\gamma\zeta^2 f''(\zeta)}{f(\zeta)} + \frac{\zeta f'(\zeta)+\zeta^2 f''(\zeta)}{\zeta f'(\zeta)+\beta\zeta^2 f''(\zeta} + \eta\zeta^2 f'''(\zeta) + (1-\lambda)\zeta f''(\zeta) - 1\right] \prec \mathcal{H}(\mu,\zeta), \tag{2.1}$$

where $(0 \leq \delta \leq 1, 0 \leq \beta \leq 1, \gamma \geq 0, \lambda \geq 0, \eta \geq 0\ )$, $\alpha \in \mathbb{C} \setminus \{0\}$ and

$\mathcal{H}(\mu,\zeta) = 1 + \mathcal{G}_1(\mu)\,\zeta + \mathcal{G}_2(\mu)\zeta^2 + \mathcal{G}_3(\mu)\zeta^3 + \mathcal{G}_4(\mu)\zeta^4 + \cdots, \mu \in \left(\frac{1}{2}, 1\right), \zeta \in \mathfrak{A}.$

Remark 2.1: Some special cases for definition can be given as below

If $\alpha = 1\,, \delta = 0\,, \gamma = 0\,, \eta = 0, \beta = 0$ and $\lambda = 1$ then $\mathcal{N}(1,0,0,0,0,1,\mu)$ is called the class of convex function[30] .

We now report and validate the primary findings of our current study.

For functions in the class $\mathcal{N}(\alpha,\gamma,\delta,\beta,\eta,\lambda,\mu)$, we start by estimating the coefficients$|\mathfrak{a}_i|$ and $i = 2,3,4,5,6,7,$.

Theorem 2.1: *Let $f(\zeta)$ be a function defined by* (1.1) *that belongs to the subclass $\mathcal{N}(\alpha,\gamma,\delta,\beta,\lambda,\mu)$. Then*

$$|\mathfrak{a}_2| \leq \frac{4\alpha}{(\delta+2\gamma+2\beta-2\lambda)}, |\mathfrak{a}_3| \leq \frac{6\alpha}{(\delta+3\gamma+3\beta+3\eta-3\lambda)},$$

$$|\mathfrak{a}_4| \leq \frac{12\alpha}{(\delta+4\gamma+4\beta+8\eta-4\lambda)},$$

$$|\mathfrak{a}_5| \leq \frac{27\alpha}{(\delta+5\gamma+5\beta+12\eta-5\lambda)}, |\mathfrak{a}_6| \leq \frac{324\alpha}{(\delta+6\gamma+6\beta+20\eta-6\lambda)},$$

$$|\mathfrak{a}_7| \leq \frac{162\alpha}{(\delta+7\gamma+7\beta+35\eta-7\lambda)}.$$

Proof: If $f \in \mathcal{N}(\alpha,\gamma,\delta,\beta,\eta,\lambda,\mu)$, then $\mathfrak{A}$ has an analytic function ϑ with $|\vartheta(\zeta)| \leq 1$. It may write

$$1+\frac{1}{\alpha}\left[\frac{\delta\zeta f'(\zeta)}{f(\zeta)}+\frac{\gamma\zeta^2 f''(\zeta)}{f(\zeta)}+\frac{\zeta f'(\zeta)+\zeta^2 f''(\zeta)}{\zeta f'(\zeta)+\beta\zeta^2 f''(\zeta}+\eta\zeta^2 f'''(\zeta)+(1-\lambda)\zeta f''(\zeta)-1\right]$$
$$=\mathcal{H}\big(\mu,\mathcal{T}(\zeta)\big).$$

The Schwarz function $\mathcal{T}$ has the following form in accordance with the subordination idea:

$$\mathcal{T}(\zeta)=\sum_{i=1}^{\infty}\wp_i\,\zeta^i\,,\zeta\in\mathfrak{A},$$

and the subsequent text is writable:

$$F(\zeta)=\frac{1+\mathcal{T}(\zeta)}{1-\mathcal{T}(\zeta)}=1+2\,\wp_1\zeta+2(\,\wp_2+\wp_1^2)\zeta^2+2[\,\wp_3+\wp_1(2\,\wp_2+\wp_1^2)]\zeta^3+$$
$$2[\,\wp_4+\wp_2^2+\wp_1^2(3\,\wp_2+\wp_1^2)+2\,\wp_1\,\wp_3]\zeta^4+2[\,\wp_5+2\,\wp_2(\,\wp_3+2\wp_1^3)+$$
$$3\wp_1(\wp_1\wp_3+\wp_2^2)+\big(\wp_1(2\wp_4+\wp_1^4)\big)]\zeta^5+2[\,\wp_6+\wp_1^3(4\,\wp_3+\wp_1^3)+$$
$$\wp_1^2(3\wp_4+5\wp_2\wp_1^2)+2\wp_1(\wp_5+3\wp_2\wp_3)+\wp_3^2+\wp_2^2(\,\wp_2+6\wp_1^2)+$$
$$2\,\wp_2\,\wp_4]\zeta^6+\cdots. \tag{2.2}$$

Given that $f\in\mathcal{A}$ is expressed in the manner specified by (1.1), then

$$1+\frac{1}{\alpha}\left[\frac{\delta\zeta f'(\zeta)}{f(\zeta)}+\frac{\gamma\zeta^2 f''(\zeta)}{f(\zeta)}+\frac{\zeta f'(\zeta)+\zeta^2 f''(\zeta)}{\zeta f'(\zeta)+\beta\zeta^2 f''(\zeta}+\eta\zeta^2 f'''(\zeta)+(1-\lambda)\zeta f''(\zeta)-1\right]$$
$$=\frac{1}{\alpha}[(\delta+2\gamma+2\beta-2\lambda)\mathfrak{a}_2\zeta+(\delta+3\gamma+3\beta+3\eta-3\lambda)\mathfrak{a}_3\zeta^2+(\delta+4\gamma+$$
$$4\beta+8\eta-4\lambda)\mathfrak{a}_4\zeta^3+(\delta+5\gamma+5\beta+12\eta-5\lambda)\mathfrak{a}_5\zeta^4+(\delta+6\gamma+6\beta+$$
$$20\eta-6\lambda)\mathfrak{a}_6\zeta^5+(\delta+7\gamma+7\beta+35\eta-7\lambda)\mathfrak{a}_7\zeta^6+.\,...]. \tag{2.3}$$

contrasting the coefficients of ζ^i for $i\;=1,2,3,4,5,6,7$ using (2.2) and (2.3), we obtain

$$\frac{1}{\alpha}\big((\delta+2\gamma+2\beta-2\lambda)\mathfrak{a}_2\big)=2\,\wp_1,$$
$$\frac{1}{\alpha}\big((\delta+3\gamma+3\beta+3\eta-3\lambda)\mathfrak{a}_3\big)=2(\,\wp_2+\wp_1^2),$$
$$\frac{1}{\alpha}\big((\delta+4\gamma+4\beta+8\eta-4\lambda)\mathfrak{a}_4\big)=2[\,\wp_3+\wp_1(2\,\wp_2+\wp_1^2)],$$
$$\frac{1}{\alpha}\big((\delta+5\gamma+5\beta+12\eta-5\lambda)\mathfrak{a}_5\big)=2[\,\wp_4+\wp_2^2+\wp_1^2(3\,\wp_2+\wp_1^2)+2\,\wp_1\,\wp_3],$$
$$\frac{1}{\alpha}\big((\delta+6\gamma+6\beta+20\eta-6\lambda)\mathfrak{a}_6\big)$$
$$=2\big[\,\wp_5+2\,\wp_2(\,\wp_3+2\wp_1^3)+3\wp_1(\wp_1\wp_3+\wp_2^2)+\big(\wp_1(2\wp_4+\wp_1^4)\big)\big],$$
$$\frac{1}{\alpha}\big((\delta+7\gamma+7\beta+35\eta-7\lambda)\mathfrak{a}_7\big)=2[\,\wp_6+\wp_1^3(4\,\wp_3+\wp_1^3)+\wp_1^2$$
$$(3\wp_4+5\wp_2\wp_1^2)+2\wp_1(\wp_5+3\wp_2\wp_3)+\wp_3^2+\wp_2^2(\,\wp_2+6\wp_1^2)+2\,\wp_2\,\wp_4].$$

By Lemma 1.1, it getting

$$|\mathfrak{a}_2|\leq\frac{4\alpha}{(\delta+2\gamma+2\beta-2\lambda)}, \tag{2.4}$$

$$|\mathfrak{a}_3| \leq \frac{6\alpha}{(\delta+3\gamma+3\beta+3\eta-3\lambda)}, \tag{2.5}$$

$$|\mathfrak{a}_4| \leq \frac{12\alpha}{(\delta+4\gamma+4\beta+8\eta-4\lambda)}, \tag{2.6}$$

$$|\mathfrak{a}_5| \leq \frac{27\alpha}{(\delta+5\gamma+5\beta+12\eta-5\lambda)}, \tag{2.7}$$

$$|\mathfrak{a}_6| \leq \frac{324\alpha}{(\delta+6\gamma+6\beta+20\eta-6\lambda)}, \tag{2.8}$$

$$|\mathfrak{a}_7| \leq \frac{162\alpha}{(\delta+7\gamma+7\beta+35\eta-7\lambda)}. \tag{2.9}$$

This theorem has a comprehensive proof.

The estimations on $|\mathcal{H}_4(1)|$ are found in the following theorem.

Theorem 2.2: *Let* $f(\zeta)$ *defined in* (1.1) *belongs to the subclass* $\mathcal{N}(\alpha, \gamma, \delta, \beta, \eta, \lambda, \mu)$, *then*

$$|\mathcal{H}_4(1)| \leq \frac{-1728\alpha^4(\delta+2\gamma+2\beta-2\lambda)(\delta+5\gamma+5\beta+12\eta-5\lambda)(\delta+6\gamma+6\beta+20\eta-6\lambda)(\delta+7\gamma+7\beta+35\eta-7\lambda)[(\mathcal{K}_1(\mu,w))]}{\mathcal{K}(\mu,w)} + \frac{6561\alpha^3(\delta+2\gamma+2\beta-2\lambda)(\delta+6\gamma+6\beta+20\eta-6\lambda)(\delta+7\gamma+7\beta+35\eta-7\lambda)(\delta+4\gamma+4\beta+8\eta-4\lambda)^2[(\mathcal{K}_2(\mu,w))]}{\mathcal{K}(\mu,w)} - \frac{10497\alpha^3(\delta+4\gamma+4\beta+8\eta-4\lambda)^2(\delta+5\gamma+\beta+12\eta-5\lambda)^2(\delta+7\gamma+7\beta+35\eta-7\lambda)[(\mathcal{K}_3(\mu,w))]}{\mathcal{K}(\mu,w)} + \frac{1555\alpha^4(\delta+3\gamma+3\beta+3\eta-3\lambda)(\delta+4\gamma+4\beta+8\eta-4\lambda)^2(\delta+5\gamma+5\beta+12\eta-5\lambda)(\delta+6\gamma+6\beta+20\eta-6\lambda)^2[(\mathcal{K}_4(\mu,w))]}{\mathcal{K}(\mu,w)}$$

where

$$\mathcal{K}(\mu,w) = 25\,(\delta+3\gamma+3\beta+3\eta-3\lambda)^2(\delta+4\gamma+4\beta+8\eta-4\lambda)^4(\delta+6\gamma+6\beta+20\eta-6\lambda)^2\,(\delta+2\gamma+2\beta-2\lambda)^2(\delta+5\gamma+5\beta+12\eta-5\lambda)^3(\delta+7\gamma+7\beta+35\eta-7\lambda),$$

$$\mathcal{K}_1(\mu,w) = \alpha^3\{144(\delta+5\gamma+5\beta+12\eta-5\lambda)^2[108+81(\delta+4\gamma+4\beta+8\eta-4\lambda)] - 90(\delta+3\gamma+3\beta+3\eta-3\lambda)^2(\delta+6\gamma+6\beta+20\eta-6\lambda)\,\langle 162(\delta+4\gamma+4\beta+8\eta-4\lambda)^3 - 96(\delta+5\gamma+5\beta+12\eta-5\lambda)^2(\delta+2\gamma+2\beta-2\lambda)\rangle - 1720(\delta+2\gamma+2\beta-2\lambda)(\delta+3\gamma+3\beta+3\eta-3\lambda)(\delta+5\gamma+5\beta+12\eta-5\lambda)(\delta+6\gamma+\beta+20\eta-6\lambda)[1+(\delta+4\gamma+4\beta+8\eta-4\lambda)^2]\},$$

$$\mathcal{K}_2(\mu,w) = \alpha^2\{243(\delta+4\gamma+4\beta+8\eta-4\lambda)(\delta+3\gamma+3\beta+3\eta-3\lambda)(\delta+2\gamma+2\beta-2\lambda)[16(\delta+5\gamma+5\beta+12\eta-5\lambda)^2(\delta+3\gamma+3\beta+3\eta-3\lambda) - 15(\delta+4\gamma+4\beta+8\eta-4\lambda)(\delta+6\gamma+\beta+20\eta-6\lambda)] - 1296\alpha(\delta+4\gamma+4\beta+8\eta-4\lambda)(\delta+3\gamma+3\beta+3\eta-3\lambda)(\delta+5\gamma+5\beta+12\eta-5\lambda)[6(\delta+4\gamma+4\beta+8\eta-4\lambda)+(\delta+3\gamma+3\beta+3\eta-3\lambda)(\delta+6\gamma+6\beta+20\eta-6\lambda)] + 45(\delta+6\gamma+\beta+20\eta-6\lambda)(\delta+5\gamma+\beta+12\eta-5\lambda)(\delta+2\gamma+2\beta-2\lambda)[103(\delta+4\gamma+4\beta+8\eta-4\lambda)^2 - 96(\delta+3\gamma+3\beta+3\eta-3\lambda)(\delta+5\gamma+5\beta+12\eta-5\lambda)\},$$

$\mathcal{K}_3(\mu,w) = \alpha^2\{324(\delta+3\gamma+3\beta+3\eta-3\lambda)(\delta+4\gamma+4\beta+8\eta-4\lambda)(\delta+2\gamma+2\beta-2\lambda)^2[6(\delta+4\gamma+4\beta+8\eta-4\lambda)(\delta+5\gamma+5\beta+12\eta-5\lambda)-5(\delta+3\gamma+3\beta+3\eta-3\lambda)(\delta+6\gamma+\beta+20\eta-6\lambda)]+5184\alpha(\delta+3\gamma+3\beta+3\eta-3\lambda)^2(\delta+5\gamma+\beta+12\eta-5\lambda)(\delta+4\gamma+4\beta+8\eta-4\lambda)-3240\alpha(\delta+3\gamma+3\beta+3\eta-3\lambda)(\delta+6\gamma+6\beta+20\eta-6\lambda)(\delta+2\gamma+2\beta-2\lambda)-720\alpha(\delta+6\gamma+6\beta+20\eta-6\lambda)(\delta+2\gamma+2\beta-2\lambda)(\delta+5\gamma+5\beta+12\eta-5\lambda)[4(\delta+3\gamma+3\beta+3\eta-3\lambda)^2+3(\delta+2\gamma+2\beta-2\lambda)(\delta+4\gamma+4\beta+8\eta-4\lambda)]\}$,

$\mathcal{K}_4(\mu,w) = \alpha^2\{96(\delta+2\gamma+2\beta-2\lambda)(\delta+4\gamma+4\beta+8\eta-4\lambda)(\delta+5\gamma+\beta+12\eta-5\lambda)[3(\delta+2\gamma+2\beta-2\lambda)-(\delta+4\gamma+4\beta+8\eta-4\lambda)]-144(\delta+2\gamma+2\beta-2\lambda)(\delta+5\gamma+5\beta+12\eta-5\lambda)[(\delta+2\gamma+2\beta-2\lambda)(\delta+3\gamma+3\beta+3\eta-3\lambda)+2\alpha(\delta+4\gamma+4\beta+8\eta-4\lambda)]+54(\delta+4\gamma+4\beta+8\eta-4\lambda)^2[3(\delta+2\gamma+2\beta-2\lambda)-8\alpha(\delta+3\gamma+3\beta+3\eta-3\lambda)]\}$.

Proof: Fourth Hankel determinant may be expressed as follows using (1.1):

$$|\mathcal{H}_4(1)| = |-\mathfrak{a}_4\mathcal{C}_1+\mathfrak{a}_5\mathcal{C}_2-\mathfrak{a}_6\mathcal{C}_3+\mathfrak{a}_7\mathcal{C}_4|,$$

where

$$|\mathcal{C}_1| = |\mathfrak{a}_2||\mathfrak{a}_4\mathfrak{a}_6-\mathfrak{a}_5^2|+|\mathfrak{a}_3||\mathfrak{a}_3\mathfrak{a}_6-\mathfrak{a}_4\mathfrak{a}_5|-|\mathfrak{a}_4||\mathfrak{a}_3\mathfrak{a}_5-\mathfrak{a}_4^2|,$$
$$|\mathcal{C}_2| = |\mathfrak{a}_4\mathfrak{a}_6-\mathfrak{a}_5^2|-|\mathfrak{a}_2||\mathfrak{a}_3\mathfrak{a}_6-\mathfrak{a}_4\mathfrak{a}_5|+|\mathfrak{a}_3||\mathfrak{a}_3\mathfrak{a}_5-\mathfrak{a}_4^2|,$$
$$|\mathcal{C}_3| = |\mathfrak{a}_3\mathfrak{a}_6-\mathfrak{a}_4\mathfrak{a}_5|+|\mathfrak{a}_2||\mathfrak{a}_2\mathfrak{a}_6-\mathfrak{a}_3\mathfrak{a}_5|-|\mathfrak{a}_4||\mathfrak{a}_2\mathfrak{a}_4-\mathfrak{a}_3^2|,$$
$$|\mathcal{C}_4| = |\mathfrak{a}_3||\mathfrak{a}_2\mathfrak{a}_4-\mathfrak{a}_2^2|-|\mathfrak{a}_4||\mathfrak{a}_4-\mathfrak{a}_2\mathfrak{a}_3|+|\mathfrak{a}_5||\mathfrak{a}_3-\mathfrak{a}_2^2|.$$

When we insert (2.4) - (2.9) in (1.6), we obtain

$$|\mathcal{C}_1| = \frac{12\alpha^3\mathcal{K}_1(\mu,w)}{B_1(\mu,w)}, \tag{2.10}$$

$$|\mathcal{C}_2| = \frac{27\alpha^4\mathcal{K}_2(\mu,w)}{B_2(\mu,w)}, \tag{2.11}$$

$$|\mathcal{C}_3| = \frac{324\alpha^3\mathcal{K}_3(\mu,w)}{B_3(\mu,w)}, \tag{2.12}$$

$$|\mathcal{C}_4| = \frac{162\alpha^3\mathcal{K}_4(\mu,w)}{B_4(\mu,w)}, \tag{2.13}$$

where

$B_1(\mu,w) = 5(\delta+2\gamma+2\beta-2\lambda)(\delta+4\gamma+4\beta+8\eta-4\lambda)^3(\delta+6\gamma+6\beta+20\eta-6\lambda)(\delta+5\gamma+5\beta+12\eta-5\lambda)^2(\delta+3\gamma+3\beta+3\eta-3\lambda)^2$,

$B_2(\mu,w) = 5(\delta+4\gamma+4\beta+8\eta-4\lambda)^2(\delta+6\gamma+6\beta+20\eta-6\lambda)(\delta+5\gamma+5\beta+12\eta-5\lambda)^2(\delta+3\gamma+3\beta+3\eta-3\lambda)^2(\delta+2\gamma+2\beta-2\lambda)$,

$B_3(\mu,w) = 5(\delta+3\gamma+3\beta+3\eta-3\lambda)^2(\delta+4\gamma+4\beta+8\eta-4\lambda)^2(\delta+6\gamma+6\beta+20\eta-6\lambda)(\delta+2\gamma+2\beta-2\lambda)^2(\delta+5\gamma+5\beta+12\eta-5\lambda)$,

$B_4(\mu, w) = (\delta+2\gamma+2\beta-2\lambda)^2(\delta+4\gamma+4\beta+8\eta-4\lambda)^2(\delta+3\gamma+3\beta+3\eta-3\lambda)(\delta+5\gamma+\beta+12\eta-5\lambda)$.

From (2.10) to (2.13) in (1.5), can obtain

$$|\mathcal{H}_4(1)| \leq \frac{-1728\alpha^4(\delta+2\gamma+2\beta-2\lambda)(\delta+5\gamma+5\beta+12\eta-5\lambda)(\delta+6\gamma+6\beta+20\eta-6\lambda)(\delta+7\gamma+7\beta+35\eta-7\lambda)[(\mathcal{K}_1(\mu,w))]}{\mathcal{K}(\mu,w)} + \frac{6561\alpha^3(\delta+2\gamma+2\beta-2\lambda)(\delta+6\gamma+6\beta+20\eta-6\lambda)(\delta+7\gamma+7\beta+35\eta-7\lambda)(\delta+4\gamma+4\beta+8\eta-4\lambda)^2[(\mathcal{K}_2(\mu,w))]}{\mathcal{K}(\mu,w)} - \frac{10497\alpha^3(\delta+4\gamma+4\beta+8\eta-4\lambda)^2(\delta+5\gamma+5\beta+12\eta-5\lambda)^2(\delta+7\gamma+7\beta+35\eta-7\lambda)[(\mathcal{K}_3(\mu,w))]}{\mathcal{K}(\mu,w)} + \frac{1555\alpha^4(\delta+3\gamma+3\beta+3\eta-3\lambda)(\delta+4\gamma+4\beta+8\eta-4\lambda)^2(\delta+5\gamma+5\beta+12\eta-5\lambda)(\delta+6\gamma+6\beta+20\eta-6\lambda)^2[(\mathcal{K}_4(\mu,w))]}{\mathcal{K}(\mu,w)},$$

where

$$\mathcal{K}(\mu, w) = 25\,(\delta+3\gamma+3\beta+3\eta-3\lambda)^2(\delta+4\gamma+4\beta+8\eta-4\lambda)^4(\delta+6\gamma+6\beta+20\eta-6\lambda)^2\,(\delta+2\gamma+2\beta-2\lambda)^2(\delta+5\gamma+5\beta+12\eta-5\lambda)^3(\delta+7\gamma+7\beta+35\eta-7\lambda),$$

$\mathcal{K}_1(\mu, w) = \alpha^3\{144(\delta+5\gamma+5\beta+12\eta-5\lambda)^2[108+81(\delta+4\gamma+4\beta+8\eta-4\lambda)] - 90(\delta+3\gamma+3\beta+3\eta-3\lambda)^2(\delta+6\gamma+6\beta+20\eta-6\lambda)\,\langle 162(\delta+4\gamma+4\beta+8\eta-4\lambda)^3 - 96(\delta+5\gamma+5\beta+12\eta-5\lambda)^2(\delta+2\gamma+2\beta-2\lambda)\rangle - 1720(\delta+2\gamma+2\beta-2\lambda)(\delta+3\gamma+3\beta+3\eta-3\lambda)(\delta+5\gamma+5\beta+12\eta-5\lambda)(\delta+6\gamma+\beta+20\eta-6\lambda)[1+(\delta+4\gamma+4\beta+8\eta-4\lambda)^2]\}$,

$\mathcal{K}_2(\mu, w) = \alpha^2\{243(\delta+4\gamma+4\beta+8\eta-4\lambda)(\delta+3\gamma+3\beta+3\eta-3\lambda)(\delta+2\gamma+2\beta-2\lambda)[16(\delta+5\gamma+5\beta+12\eta-5\lambda)^2(\delta+3\gamma+3\beta+3\eta-3\lambda) - 15(\delta+4\gamma+4\beta+8\eta-4\lambda)(\delta+6\gamma+6\beta+20\eta-6\lambda)] - 1296\alpha(\delta+4\gamma+4\beta+8\eta-4\lambda)(\delta+3\gamma+3\beta+3\eta-3\lambda)(\delta+5\gamma+5\beta+12\eta-5\lambda)[6(\delta+4\gamma+4\beta+8\eta-4\lambda)+(\delta+3\gamma+3\beta+3\eta-3\lambda)(\delta+6\gamma+6\beta+20\eta-6\lambda)] + 45(\delta+6\gamma+6\beta+20\eta-6\lambda)(\delta+5\gamma+5\beta+12\eta-5\lambda)(\delta+2\gamma+2\beta-2\lambda)[103(\delta+4\gamma+4\beta+8\eta-4\lambda)^2 - 96(\delta+3\gamma+3\beta+3\eta-3\lambda)(\delta+5\gamma+5\beta+12\eta-5\lambda)\}$,

$\mathcal{K}_3(\mu, w) = \alpha^2\{324(\delta+3\gamma+3\beta+3\eta-3\lambda)(\delta+4\gamma+4\beta+8\eta-4\lambda)(\delta+2\gamma+2\beta-2\lambda)^2[6(\delta+4\gamma+4\beta+8\eta-4\lambda)(\delta+5\gamma+5\beta+12\eta-5\lambda) - 5(\delta+3\gamma+3\beta+3\eta-3\lambda)(\delta+6\gamma+6\beta+20\eta-6\lambda)] + 5184\alpha(\delta+3\gamma+3\beta+3\eta-3\lambda)^2(\delta+5\gamma+5\beta+12\eta-5\lambda)(\delta+4\gamma+4\beta+8\eta-4\lambda) - 3240\alpha(\delta+3\gamma+3\beta+3\eta-3\lambda)(\delta+6\gamma+6\beta+20\eta-6\lambda)(\delta+2\gamma+2\beta-2\lambda) - 720\alpha(\delta+6\gamma+6\beta+20\eta-6\lambda)(\delta+2\gamma+2\beta-2\lambda)(\delta+5\gamma+5\beta+12\eta-5\lambda)[4(\delta+3\gamma+3\beta+3\eta-3\lambda)^2+3(\delta+2\gamma+2\beta-2\lambda)(\delta+4\gamma+4\beta+8\eta-4\lambda)]\}$,

$\mathcal{K}_4(\mu, w) = \alpha^2\{96(\delta+2\gamma+2\beta-2\lambda)(\delta+4\gamma+4\beta+8\eta-4\lambda)(\delta+5\gamma+5\beta+12\eta-5\lambda)[3(\delta+2\gamma+2\beta-2\lambda)-(\delta+4\gamma+4\beta+8\eta-4\lambda)] - 144(\delta+2\gamma+2\beta-2\lambda)(\delta+5\gamma+\beta+12\eta-5\lambda)[(\delta+2\gamma+2\beta-2\lambda)(\delta+$

$3\gamma + 3\beta + 3\eta - 3\lambda) + 2\alpha(\delta + 4\gamma + 4\beta + 8\eta - 4\lambda)] + 54(\delta + 4\gamma + 4\beta + 8\eta - 4\lambda)^2[3(\delta + 2\gamma + 2\beta - 2\lambda) - 8\alpha(\delta + 3\gamma + 3\beta + 3\eta - 3\lambda)]\}.$

The following Corollary results when $\delta = 0$.

Corollary 2.1: *Let f defined by* (1.1) *belongs to the subclass $\mathcal{N}(\alpha,\gamma,0,\beta,\eta,\lambda,\mu)$, then*

$$|\mathcal{H}_4(1)| \leq \frac{-1728\alpha^4(2\gamma+2\beta-2\lambda)(5\gamma+5\beta+12\eta-5\lambda)(6\gamma+6\beta+20\eta-6\lambda)(7\gamma+7\beta+35\eta-7\lambda)[(\mathcal{K}_1(\mu,w))]}{\mathcal{K}(\mu,w)} + \frac{6561\alpha^3(2\gamma+2\beta-2\lambda)(6\gamma+6\beta+20\eta-6\lambda)(7\gamma+7\beta+35\eta-7\lambda)(4\gamma+4\beta+8\eta-4\lambda)^2[(\mathcal{K}_2(\mu,w))]}{\mathcal{K}(\mu,w)} - \frac{10497\alpha^3(4\gamma+4\beta+8\eta-4\lambda)^2(5\gamma+5\beta+12\eta-5\lambda)^2(7\gamma+7\beta+35\eta-7\lambda)[(\mathcal{K}_3(\mu,w))]}{\mathcal{K}(\mu,w)} + \frac{1555\alpha^4(3\gamma+3\beta+3\eta-3\lambda)(4\gamma+4\beta+8\eta-4\lambda)^2(5\gamma+5\beta+12\eta-5\lambda)(6\gamma+\beta+20\eta-6\lambda)^2[(\mathcal{K}_4(\mu,w))]}{\mathcal{K}(\mu,w)}$$

where

$$\mathcal{K}(\mu,w) = 25\,(3\gamma + 3\beta + 3\eta - 3\lambda)^2(4\gamma + 4\beta + 8\eta - 4\lambda)^4(6\gamma + \beta + 20\eta - 6\lambda)^2\,(2\gamma + 2\beta - 2\lambda)^2(5\gamma + 5\beta + 12\eta - 5\lambda)^3(7\gamma + 7\beta + 35\eta - 7\lambda),$$

$\mathcal{K}_1(\mu,w) = \alpha^3\{144(5\gamma + 5\beta + 12\eta - 5\lambda)^2[108 + 81(4\gamma + 4\beta + 8\eta - 4\lambda)] - 90(3\gamma + 3\beta + 3\eta - 3\lambda)^2(6\gamma + 6\beta + 20\eta - 6\lambda)\,\langle 162(4\gamma + 4\beta + 8\eta - 4\lambda)^3 - 96(5\gamma + \beta + 12\eta - 5\lambda)^2(2\gamma + 2\beta - 2\lambda)\rangle - 1720(2\gamma + 2\beta - 2\lambda)(\delta + 3\gamma + 3\beta + 3\eta - 3\lambda)(5\gamma + 5\beta + 12\eta - 5\lambda)(6\gamma + \beta + 20\eta - 6\lambda)[1 + (4\gamma + 4\beta + 8\eta - 4\lambda)^2]\},$

$\mathcal{K}_2(\mu,w) = \alpha^2\{243(4\gamma + 4\beta + 8\eta - 4\lambda)(3\gamma + 3\beta + 3\eta - 3\lambda)(2\gamma + 2\beta - 2\lambda)[16(5\gamma + 5\beta + 12\eta - 5\lambda)^2(3\gamma + 3\beta + 3\eta - 3\lambda) - 15(4\gamma + 4\beta + 8\eta - 4\lambda)(6\gamma + 6\beta + 20\eta - 6\lambda)] - 1296\alpha(4\gamma + 4\beta + 8\eta - 4\lambda)(3\gamma + 3\beta + 3\eta - 3\lambda)(5\gamma + 5\beta + 12\eta - 5\lambda)[6(4\gamma + 4\beta + 8\eta - 4\lambda) + (3\gamma + 3\beta + 3\eta - 3\lambda)(6\gamma + \beta + 20\eta - 6\lambda)] + 45(6\gamma + 6\beta + 20\eta - 6\lambda)(5\gamma + 5\beta + 12\eta - 5\lambda)(2\gamma + 2\beta - 2\lambda)[103(4\gamma + 4\beta + 8\eta - 4\lambda)^2 - 96(3\gamma + 3\beta + 3\eta - 3\lambda)(5\gamma + \beta + 12\eta - 5\lambda)\},$

$\mathcal{K}_3(\mu,w) = \alpha^2\{324(3\gamma + 3\beta + 3\eta - 3\lambda)(4\gamma + 4\beta + 8\eta - 4\lambda)(2\gamma + 2\beta - 2\lambda)^2[6(4\gamma + 4\beta + 8\eta - 4\lambda)(5\gamma + 5\beta + 12\eta - 5\lambda) - 5(3\gamma + 3\beta + 3\eta - 3\lambda)(6\gamma + 6\beta + 20\eta - 6\lambda)] + 5184\alpha(3\gamma + 3\beta + 3\eta - 3\lambda)^2(5\gamma + 5\beta + 12\eta - 5\lambda)(4\gamma + 4\beta + 8\eta - 4\lambda) - 3240\alpha(3\gamma + 3\beta + 3\eta - 3\lambda)(6\gamma + 6\beta + 20\eta - 6\lambda)(2\gamma + 2\beta - 2\lambda) - 720\alpha(6\gamma + 6\beta + 20\eta - 6\lambda)(2\gamma + 2\beta - 2\lambda)(5\gamma + \beta + 12\eta - 5\lambda)[4(3\gamma + 3\beta + 3\eta - 3\lambda)^2 + 3(2\gamma + 2\beta - 2\lambda)(4\gamma + 4\beta + 8\eta - 4\lambda)]\},$

$\mathcal{K}_4(\mu,w) = \alpha^2\{96(2\gamma + 2\beta - 2\lambda)(4\gamma + 4\beta + 8\eta - 4\lambda)(5\gamma + \beta + 12\eta - 5\lambda)[3(2\gamma + 2\beta - 2\lambda) - (4\gamma + 4\beta + 8\eta - 4\lambda)] - 144(2\gamma + 2\beta - 2\lambda)(5\gamma +$

$5\beta + 12\eta - 5\lambda)[(2\gamma + 2\beta - 2\lambda)(3\gamma + 3\beta + 3\eta - 3\lambda) + 2\alpha(4\gamma + 4\beta + 8\eta -$
$4\lambda)] + 54(4\gamma + 4\beta + 8\eta - 4\lambda)^2[3(2\gamma + 2\beta - 2\lambda) - 8\alpha(3\gamma + 3\beta + 3\eta -$
$3\lambda)]\}$.

In case $\delta = 1$, $\alpha = 1$, we obtain the subsequent Corollary.

Corollary 2.2: Let f defined by (1.1) belongs to the subclass $\mathcal{N}(1, \gamma, 1, \beta, \eta, \lambda, \mu)$ then

$$|\mathcal{H}_4(1)| \leq \frac{-1728(1+2\gamma+2\beta-2\lambda)(1+5\gamma+\beta+12\eta-5\lambda)(1+6\gamma+6\beta+20\eta-6\lambda)(1+7\gamma+7\beta+35\eta-7\lambda)[(\mathcal{K}_1(\mu,w))]}{\mathcal{K}(\mu,w)} + \frac{6561(\delta+2\gamma+2\beta-2\lambda)(\delta+6\gamma+6\beta+20\eta-6\lambda)(\delta+7\gamma+7\beta+35\eta-7\lambda)(\delta+4\gamma+4\beta+8\eta-4\lambda)^2[(\mathcal{K}_2(\mu,w))]}{\mathcal{K}(\mu,w)} - \frac{10497(1+4\gamma+4\beta+8\eta-4\lambda)^2(1+5\gamma+5\beta+12\eta-5\lambda)^2(1+7\gamma+7\beta+35\eta-7\lambda)[(\mathcal{K}_3(\mu,w))]}{\mathcal{K}(\mu,w)} + \frac{1555(1+3\gamma+3\beta+3\eta-3\lambda)(1+4\gamma+4\beta+8\eta-4\lambda)^2(1+5\gamma+5\beta+12\eta-5\lambda)(1+6\gamma+6\beta+20\eta-6\lambda)^2[(\mathcal{K}_4(\mu,w))]}{\mathcal{K}(\mu,w)}$$

where

$$\mathcal{K}(\mu, w) = 25\,(1 + 3\gamma + 3\beta + 3\eta - 3\lambda)^2(1 + 4\gamma + 4\beta + 8\eta - 4\lambda)^4(1 + 6\gamma + 6\beta + 20\eta - 6\lambda)^2\,(1 + 2\gamma + 2\beta - 2\lambda)^2(1 + 5\gamma + 5\beta + 12\eta - 5\lambda)^3(1 + 7\gamma + 7\beta + 35\eta - 7\lambda),$$

$\mathcal{K}_1(\mu, w) = \{144(1 + 5\gamma + 5\beta + 12\eta - 5\lambda)^2[108 + 81(1 + 4\gamma + 4\beta + 8\eta -$
$4\lambda)] - 90(1 + 3\gamma + 3\beta + 3\eta - 3\lambda)^2(1 + 6\gamma + 6\beta + 20\eta - 6\lambda)\,\langle 162(1 +$
$4\gamma + 4\beta + 8\eta - 4\lambda)^3 - 96(1 + 5\gamma + 5\beta + 12\eta - 5\lambda)^2(1 + 2\gamma + 2\beta -$
$2\lambda)\rangle - 1720(1 + 2\gamma + 2\beta - 2\lambda)(1 + 3\gamma + 3\beta + 3\eta - 3\lambda)(1 + 5\gamma + 5\beta +$
$12\eta - 5\lambda)(1 + 6\gamma + 6\beta + 20\eta - 6\lambda)[1 + (1 + 4\gamma + 4\beta + 8\eta - 4\lambda)^2]\}$,

$\mathcal{K}_2(\mu, w) = \{243(1 + 4\gamma + 4\beta + 8\eta - 4\lambda)(1 + 3\gamma + 3\beta + 3\eta - 3\lambda)(1 +$
$2\gamma + 2\beta - 2\lambda)[16(1 + 5\gamma + 5\beta + 12\eta - 5\lambda)^2(1 + 3\gamma + 3\beta + 3\eta - 3\lambda) -$
$15(1 + 4\gamma + 4\beta + 8\eta - 4\lambda)(1 + 6\gamma + 6\beta + 20\eta - 6\lambda)] - 1296(1 + 4\gamma +$
$4\beta + 8\eta - 4\lambda)(1 + 3\gamma + 3\beta + 3\eta - 3\lambda)(1 + 5\gamma + 5\beta + 12\eta - 5\lambda)[6(1 +$
$4\gamma + 4\beta + 8\eta - 4\lambda) + (1 + 3\gamma + 3\beta + 3\eta - 3\lambda)(1 + 6\gamma + 6\beta + 20\eta -$
$6\lambda)] + 45(1 + 6\gamma + 6\beta + 20\eta - 6\lambda)(1 + 5\gamma + 5\beta + 12\eta - 5\lambda)(1 + 2\gamma +$
$2\beta - 2\lambda)[103(1 + 4\gamma + 4\beta + 8\eta - 4\lambda)^2 - 96(1 + 3\gamma + 3\beta + 3\eta - 3\lambda)(1 +$
$5\gamma + 5\beta + 12\eta - 5\lambda)\}$,

$\mathcal{K}_3(\mu, w) = \{324(1 + 3\gamma + 3\beta + 3\eta - 3\lambda)(1 + 4\gamma + 4\beta + 8\eta -$
$4\lambda)(1 + 2\gamma + 2\beta - 2\lambda)^2[6(1 + 4\gamma + 4\beta + 8\eta - 4\lambda)(1 + 5\gamma + 5\beta + 12\eta -$
$5\lambda) - 5(1 + 3\gamma + 3\beta + 3\eta - 3\lambda)(1 + 6\gamma + 6\beta + 20\eta - 6\lambda)] + 5184(1 +$
$3\gamma + 3\beta + 3\eta - 3\lambda)^2(1 + 5\gamma + 5\beta + 12\eta - 5\lambda)(1 + 4\gamma + 4\beta + 8\eta - 4\lambda) -$
$3240(1 + 3\gamma + 3\beta + 3\eta - 3\lambda)(1 + 6\gamma + 6\beta + 20\eta - 6\lambda)(1 + 2\gamma + 2\beta -$
$2\lambda) - 720(1 + 6\gamma + 6\beta + 20\eta - 6\lambda)(1 + 2\gamma + 2\beta - 2\lambda)(1 + 5\gamma + 5\beta +$

$12\eta-5\lambda)[4(1+3\gamma+3\beta+3\eta-3\lambda)^2+3(1+2\gamma+2\beta-2\lambda)(1+4\gamma+4\beta+8\eta-4\lambda)]\big\},$

$\mathcal{K}_4(\mu,w)=\{96(1+2\gamma+2\beta-2\lambda)(1+4\gamma+4\beta+8\eta-4\lambda)(1+5\gamma+5\beta+12\eta-5\lambda)[3(1+2\gamma+2\beta-2\lambda)-(1+4\gamma+4\beta+8\eta-4\lambda)]-144(1+2\gamma+2\beta-2\lambda)(1+5\gamma+5\beta+12\eta-5\lambda)[(1+2\gamma+2\beta-2\lambda)(1+3\gamma+3\beta+3\eta-3\lambda)+2(1+4\gamma+4\beta+8\eta-4\lambda)]+54(1+4\gamma+4\beta+8\eta-4\lambda)^2[3(1+2\gamma+2\beta-2\lambda)-8(1+3\gamma+3\beta+3\eta-3\lambda)]\}.$

In case $\alpha=1$, $\delta=1$ and $\beta=0$ we obtain the subsequent Corollary.

Corollary 2.3: *Let $f(\zeta)$ is defined by (1.1) belongs to the subclass $\mathcal{N}(1,\gamma,1,\lambda,\mu)$, then*

$$|\mathcal{H}_4(1)|\leq\frac{-1728(1+2\gamma-2\lambda)(1+5\gamma+12\eta-5\lambda)(1+6\gamma+20\eta-6\lambda)(1+7\gamma++35\eta-7\lambda)[(\mathcal{K}_1(\mu,w))]}{\mathcal{K}(\mu,w)}+\frac{6561(\delta+2\gamma+-2\lambda)(\delta+6\gamma+20\eta-6\lambda)(\delta+7\gamma+35\eta-7\lambda)(\delta+4\gamma++8\eta-4\lambda)^2[(\mathcal{K}_2(\mu,w))]}{\mathcal{K}(\mu,w)}-\frac{10497(1+4\gamma+8\eta-4\lambda)^2(1+5\gamma+12\eta-5\lambda)^2(1+7\gamma+35\eta-7\lambda)[(\mathcal{K}_3(\mu,w))]}{\mathcal{K}(\mu,w)}+\frac{1555(1+3\gamma+3\eta-3\lambda)(1+4\gamma+4\beta+8\eta-4\lambda)^2(1+5\gamma+12\eta-5\lambda)(1+6\gamma+20\eta-6\lambda)^2[(\mathcal{K}_4(\mu,w))]}{\mathcal{K}(\mu,w)},$$

where

$$\mathcal{K}(\mu,w)=25\,(1+3\gamma+3\eta-3\lambda)^2(1+4\gamma+8\eta-4\lambda)^4(1+6\gamma+20\eta-6\lambda)^2\,(1+2\gamma-2\lambda)^2(1+5\gamma+\beta+12\eta-5\lambda)^3(1+7\gamma+35\eta-7\lambda),$$

$\mathcal{K}_1(\mu,w)=\big\{144(1+5\gamma+12\eta-5\lambda)^2[108+81(1+4\gamma+8\eta-4\lambda)]-90(1+3\gamma+3\eta-3\lambda)^2(1+6\gamma+20\eta-6\lambda)\,\langle162(1+4\gamma+8\eta-4\lambda)^3-96(1+5\gamma+12\eta-5\lambda)^2(1+2\gamma-2\lambda)\rangle-1720(1+2\gamma-2\lambda)(1+3\gamma+3\eta-3\lambda)(1+5\gamma+12\eta-5\lambda)(1+6\gamma+20\eta-6\lambda)[1+(1+4\gamma+8\eta-4\lambda)^2]\big\},$

$\mathcal{K}_2(\mu,w)=\{243(1+4\gamma++8\eta-4\lambda)(1+3\gamma++3\eta-3\lambda)(1+2\gamma-2\lambda)[16(1+5\gamma+12\eta-5\lambda)^2(1+3\gamma+3\eta-3\lambda)-15(1+4\gamma+8\eta-4\lambda)(1+6\gamma20\eta-6\lambda)]-1296(1+4\gamma+8\eta-4\lambda)(1+3\gamma+3\eta-3\lambda)(1+5\gamma+12\eta-5\lambda)[6(1+4\gamma+4\beta+8\eta-4\lambda)+(1+3\gamma+3\eta-3\lambda)(1+6\gamma+20\eta-6\lambda)]+45(1+6\gamma+20\eta-6\lambda)(1+5\gamma+12\eta-5\lambda)(1+2\gamma-2\lambda)[103(1+4\gamma+8\eta-4\lambda)^2-96(1+3\gamma+3\eta-3\lambda)(1+5\gamma+12\eta-5\lambda)\},$

$\mathcal{K}_3(\mu,w)=\big\{324(1+3\gamma+3\eta-3\lambda)(1+4\gamma+8\eta-4\lambda)(1+2\gamma-2\lambda)^2[6(1+4\gamma+8\eta-4\lambda)(1+5\gamma+12\eta-5\lambda)-5(1+3\gamma+3\eta-3\lambda)(1+6\gamma+20\eta-6\lambda)]+5184(1+3\gamma+3\eta-3\lambda)^2(1+5\gamma+12\eta-5\lambda)(1+4\gamma+8\eta-4\lambda)-3240(1+3\gamma+3\eta-3\lambda)(1+6\gamma+20\eta-6\lambda)(1+2\gamma+-2\lambda)-$

$720(1+6\gamma+20\eta-6\lambda)(1+2\gamma-2\lambda)(1+5\gamma+12\eta-5\lambda)[4(1+3\gamma+3\eta-3\lambda)^2+3(1+2\gamma-2\lambda)(1+4\gamma+8\eta-4\lambda)]\Big\},$

$\mathcal{K}_4(\mu,w)=\{96(1+2\gamma-2\lambda)(1+4\gamma+8\eta-4\lambda)(1+5\gamma+12\eta-5\lambda)[3(1+2\gamma-2\lambda)-(1+4\gamma+8\eta-4\lambda)]-144(1+2\gamma-2\lambda)(1+5\gamma+12\eta-5\lambda)[(1+2\gamma-2\lambda)(1+3\gamma+3\eta-3\lambda)+2(1+4\gamma++8\eta-4\lambda)]+54(1+4\gamma+8\eta-4\lambda)^2[3(1+2\gamma-2\lambda)-8(1+3\gamma++3\eta-3\lambda)]\}.$

In case $\alpha=1\,,\delta=1\,,\gamma=0,\lambda=0\;\beta=1aid\;\eta=0$ we obtain the subsequent Corollary.

Corollary 2.3: *Let $f(\zeta)$ is defined by (1.1) be in a class $\mathcal{N}(1,0,1,0,0,0,\mu)$, then*

$$|\mathcal{H}_4(1)|\le 1480789.56$$

3. Conclusion

The fourth Hankel determinant was given a new form, a notion for subordination is used to establish a new subclass $\mathcal{N}(\alpha,\gamma,\delta,\beta,\eta,\lambda,\mu)$. Coefficient estimates are provided, the class is analyzed, upper bound for recently proposed fourth Hankel determinant for this subclass is found. Since we are utilizing a new, very hard subclass, the findings shown here are better than those in previous studies. Moreover, these results have applications that go outside mathematics and help disciplines like computer science, data analytics, and engineering. Improved knowledge of the fourth Hankel determinant and its applications can help in approximation theory, signal processing, orthogonal polynomials, system identification, and other related fields. The acquired discoveries not only broaden our understanding but also provide new avenues for investigation and useful applications across several scientific fields.

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Received October, 2024