

New results in fuzzy soft bitopological spaces via (1,2)-fuzzy soft pre-open sets

Ameer Al-Abayechi *

Haneen Al-Janabi [†]

Heyam Kh. Hassan Alkhayyat [§]

Department of Mathematics

Faculty of Computer Science and Mathematics

University of Kufa

Najaf

Iraq

Abstract

Makherjee and Park [1] introduced and studied a notion for a fuzzy soft bitopological space. This article introduce notions for fuzzy soft pre-open (closed) set of fuzzy soft bitopological space and studied their basic properties. we use these notions to characterize fundamental concepts of fuzzy soft bitopological spaces such as fuzzy soft pre-closures and fuzzy soft pre-interior of a fuzzy soft bitopological space and prove some of their axioms. Through the use of a notion for soft quasi coincidence, as well as we characterized the concept for a fuzzy soft quasi pre-separation axioms of bitopological space pre- T_i ($i = 0, 1, 2$) of bitopological space and prove some theorems related to the relation between them.

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1. Introduction

Molodtsov [2] was studied soft set in 1999. Shabir and Naz [3] introduced and investigated topological structures for soft set such as soft closed sets, &soft open sets lead to soft closure, &soft interior points and soft separation axiom.

In recent times, fuzzification of soft set theory is progress rapidly. Maji, Biswas and Roy [4], was Combined soft set & fuzzy set, and defined novel notion known as fuzzysoft set and prove some properties that related to it.

* E-mail: ameerm.hasan@uokufa.edu.iq (Corresponding Author)

[†] E-mail: haneenk.aljanabi@uokufa.edu.iq

[§] E-mail: hiyamk.hasan@uokufa.edu.iq

In 2011 a topological structure of fuzzy soft sets was started by Tanay and Burç kademir [5]. Then many researchers got interest to improve this study and gave new results [6, 7, 8, 9, 10, 11, 12].

In 2015 Mukherjee and Park presented concept for fuzzy-soft bi-top. Sp. [1], so studied some for its properties.

The manuscript will try to develop several concepts related to fuzzy soft bitopological spaces. We introduce a concept for fuzzy soft *pre-open* & *pre-closed* set in fuzzy soft bitopological space. Some related concepts were introduced such as a per-closure and pre-interior for fuzzy soft bi-top. Sp. and its properties are investigated. Furthermore, in order to support our results in our work, we reviewed and extend the results obtained in (cf. [13]) and define the concept of fuzzy soft quasi $Pre - T_i$ -space ($i = 0, 1, 2$) and prove that every fuzzy soft $Pre - T_2$ space implies to fuzzy soft $Pre - T_1$ space as well as every fuzzy soft $Pre - T_1$ space implies to fuzzy soft $Pre - T_0$.

2. Preliminaries

In this article X refers to universal set, $\mathbb{E}$ denotes a set for each possible parameter of X , F_s denote a fuzzy soft. While q_{F_s} means fuzzy soft quasi.

Definition 2.1 [2]: A soft set is f_A over X , where f is map from A into the power set of U .

Definition 2.2 [10]: A soft set f_A , named a F_s set over X , where $f: A \rightarrow I^X$ is map define as $f_A(e) = \begin{cases} 0 & \text{if } e \notin A \\ \text{otherwise,} & \text{if } e \in A \end{cases}$, and the complement of f_A described by $(f_A)^c$, where $(f_A)^c = 1 - f_A$.

Definition 2.3 [6]: A F_s set f_A is called F_s soft subset of g_B , when $A \leq B$ & $f(e) \leq g(e) \forall e \in A$ we write $f_A \lesssim g_B$, and equal if $f_A \lesssim g_B$ and $g_B \lesssim f_A$.

Definition 2.4 [1]: The intersection of f_A & g_B is a F_s set h_C , such that $C = A \cap B$, $h_C(e) = f_A(e) \cap g_B(e)$, for all $e \in C$ and denoted by $f_A \tilde{\wedge} g_B$, and the union of two F_s sets f_A, g_B is a F_s set j_D , such that $D = A \cup B$, $j_D(e) = f_A(e) \cup g_B(e)$, for all $e \in D$ and denoted by $f_A \tilde{\vee} g_B$, a F_s set f_A is null, respectively absolute F_s set if for all $e \in A$, $f_A(\ell) = \tilde{0}_E$, respectively $f_A(\ell) = \tilde{1}_E$.

Definition 2.5 [10]: A F_s set f_A is called F_s point in $(X, \tau, \mathbb{E})$ denoted by $\ell(f_A)$, if each element $\ell \in \mathbb{E}$, such that $f(\ell) \neq \tilde{0}_E$ and $f(\ell^c) = \tilde{0}_E$, for all $\ell^c \in \mathbb{E} \setminus \ell$.

Definition 2.6 [10]: A F_s point $\ell(f_A)$ is said to be belongs to the F_s set g_B , denoted by $\ell(f_A) \in g_B$, if for the element $\ell \in \mathbb{E}$, $f(\ell) \lesssim g(\ell)$. Evidently, every F_s set g_B can be describe as the union of all F_s points which belong to g_B .

Definition 2.7 [10]: A complement of a F_s point $\ell(f_A)$, is a F_s point $(\ell(f_A))^c$, such that, for all $\ell^c \in \mathbb{E} \setminus \ell$, $f(\ell^c) = \tilde{0}_E$ and $f(\ell) \neq \tilde{0}_E$, for any element $\ell \in \mathbb{E}$. If

the F_s point $\ell(f_A)$ is in the F_s set g_B , then not necessary a complement F_s point $(\ell(f_A))^c$ is in F_s set $(g_B)^c$.

Definition 2.8 [14]: Two F_s points $\ell_1(f_A)$ and $\ell_2(f_A)$ are said to be distinct if and only if $\ell_1 \neq \ell_2$.

Definition 2.9 [6]: If f_A and g_B are F_s subsets over X , we say that the F_s set f_A has the property quasi-coincident with g_B signified by $f_A q g_B$ if and only if there exists $\ell \in \mathbb{E}$, $h \in X$ where $f_A(\ell)(h) + g_B(\ell)(h) \geq 1$. Otherwise f_A is not quasi-coincident with g_B and we write $f_A \hat{q} g_B$, that is $f_A \hat{q} g_B$ iff $f_A(\ell)(h) + g_B(\ell)(h) \leq 1$, i.e., $f_A(\ell)(h) \leq (g_B(\ell)(h))^c$ for all $\ell \in \mathbb{E}$ and $h \in X$. A F_s point $\ell(f_A)$ is quasi-coincident with a F_s set g_B , denoted by $\ell(f_A) q g_B$ if and only if there exists $\ell \in \mathbb{E}$, $h \in X$ such that $\ell(f_A)(h) + g_B(h) \geq 1$.

Theorem 2.10 [6]: Let f_A and g_B are F_s subsets over X . Then the following are hold:

- (1) If $f_A \tilde{\wedge} g_B = \tilde{0}_E$, then $f_A \hat{q} g_B$.
- (2) If $f_A q g_B$, then $f_A \tilde{\nless} (g_B)^c$.

Definition 2.11 [15]: A F_s topological space is write as triplet $(X, \sigma, \mathbb{E})$ where X non-empty set and σ is class for F_s sets over X hold next conditions:

- (1) $\tilde{0}_E, \tilde{1}_E \in \sigma$.
- (2) $f_A \tilde{\wedge} g_B \in \sigma$, for $f_A, g_B \in \sigma$.
- (3) If $f_{A_i} \in \sigma$, $\forall i \in J$, then $\tilde{\vee}_{i \in J} f_{A_i} \in \sigma$.

Then σ is topology for F_s set on X . each member in σ called F_s open. A F_s closed set is the complement of F_s open set.

Definition 2.12: For $(X, \sigma, \mathbb{E})$ if $Y \subseteq X$ and $h_{\mathbb{E}}^Y$ is F_s set over $(Y, \mathbb{E})$ s.t $h_{\mathbb{E}}^Y: \mathbb{E} \rightarrow I^Y$ and $h_{\mathbb{E}}^Y(\ell) = \mu_{h_{\mathbb{E}}^Y}^{\ell}$,

$$h_{\mathbb{E}}^Y(x) = \begin{cases} 1 & x \in Y \\ 0 & x \notin Y \end{cases}$$

Let $\sigma_Y = \{h_{\mathbb{E}}^Y \tilde{\wedge} f_A: f_A \in \sigma\}$, then the F_s topology σ_Y on $(Y, \mathbb{E})$ is called F_s sub space of $(Y, \mathbb{E})$ & $(Y, \sigma_Y, \mathbb{E})$ called F_s subspace of $(X, \sigma, \mathbb{E})$. If $h_{\mathbb{E}}^Y \in \sigma$ (resp. $h_{\mathbb{E}}^Y \in \sigma^c$), so $(Y, \sigma_Y, \mathbb{E})$ is F_s open (resp. closed) subspace for $(X, \sigma, \mathbb{E})$.

Definition 2.13 [10]: If f_A is F_s subset for $(X, \sigma, \mathbb{E})$. The F_s closure of f_A is symbolized by $cl(f_A)$ is the intersection of all F_s closed of f_A . Evidently, $cl(f_A)$ smallest F_s closed in X include f_A .

Theorem 2.14 [10]: For $(X, \sigma, \mathbb{E})$ if f_A, g_B is F_s subsets for $(X, \mathbb{E})$. Then,

- (1) $cl(\tilde{0}_E) = \tilde{0}_E$ and $cl(\tilde{1}_E) = \tilde{1}_E$.
- (2) $f_A \tilde{\leq} cl(f_A)$.

- (3) $cl(cl(f_A)) = cl(f_A)$.
- (4) If $f_A \lesssim g_B$, then $cl(f_A) \lesssim cl(g_B)$.
- (5) f_A is a F_s closed set if and only if $f_A = cl(f_A)$.
- (6) $cl(f_A \tilde{\vee} g_B) = cl(f_A) \tilde{\vee} cl(g_B)$.
- (7) $cl(f_A \tilde{\wedge} g_B) \lesssim cl(f_A) \tilde{\wedge} cl(g_B)$.

Definition 2.15 [10]: If f_A is F_s subset for $(X, \sigma, \mathbb{E})$. The F_s interior of f_A is $int(f_A) = \bigcup_{g_B \lesssim f_A} g_B$. Note that, $int(f_A)$ will be the largest F_s open over X contained in f_A .

Theorem 2.16 [10]: If $(X, \sigma, \mathbb{E})$ is τ_{F_s} space over X & f_A, g_B are F_s subsets of $(X, \mathbb{E})$. Then,

- (1) $int(\tilde{0}_{\mathbb{E}}) = \tilde{0}_{\mathbb{E}}$ and $int(\tilde{1}_{\mathbb{E}}) = \tilde{1}_{\mathbb{E}}$.
- (2) $int(f_A) \lesssim (f_A)$.
- (3) $int(int(f_A)) = int(f_A)$.
- (4) If $f_A \lesssim g_B$, then $int(f_A) \lesssim int(g_B)$.
- (5) f_A is a F_s open set if and only if $f_A = int(f_A)$.
- (6) $int(f_A \tilde{\wedge} g_B) = int(f_A) \tilde{\wedge} int(g_B)$.
- (7) $int(f_A) \tilde{\vee} int(g_B) \lesssim int(f_A \tilde{\vee} g_B)$.

Definition 2.17: Suppose $(X, \mathbb{E}, \tau_1)$ & $(X, \mathbb{E}, \tau_2)$ two different fuzzy soft top. on $(X, \mathbb{E})$. so $(X, \mathbb{E}, \sigma_1, \sigma_2)$ called F_s bitopological space members for σ_1 & σ_2 are known as $\sigma_i (i = 1, 2)$ - F_s open sets. A set X equipped with two different F_s topologies $\sigma_i (i = 1, 2)$, named F_s bitopological space, denotes as $(X, \mathbb{E}, \sigma_1, \sigma_2)$ where $(X, \sigma_1, \mathbb{E}), (X, \sigma_2, \mathbb{E})$ two F_s top. Sp. defined on $(X, \mathbb{E})$. The members of σ_1 and σ_2 are known as $\sigma_i (i = 1, 2)$ - F_s open sets.

Definition 2.18: Given $(X, \sigma_1, \sigma_2, \mathbb{E})$ F_s bitopological space & $Y \subseteq X$. A relative bitopological spaces for $(Y, \mathbb{E})$ is denote as $(Y, (\sigma_1)_Y, (\sigma_2)_Y, \mathbb{E})$, such that $(\sigma_1)_Y = \{h_{\mathbb{E}}^Y \tilde{\wedge} f_A : f_A \in \sigma_1\}$ and $(\sigma_2)_Y = \{h_{\mathbb{E}}^Y \tilde{\wedge} g_B : g_B \in \sigma_2\}$, where $h_{\mathbb{E}}^Y$ is F_s set of Y . Then $(Y, (\sigma_1)_Y, (\sigma_2)_Y, \mathbb{E})$ called F_s subspace for F_s bitopological space $(X, \sigma_1, \sigma_2, \mathbb{E})$. A family for every open set with respect to $(Y, \mathbb{E})$ is denoted by $O(X)_Y$.

3. Results

(1,2)-Fuzzy Soft Pre-open Set for Fuzzy Soft Bitopological Space

This part, introduces several definitions for F_s bitopological space and some propositions.

Definition 3.1.1: Consider $(X, \sigma_1, \sigma_2, \mathbb{E})$ be a F_s bitopological space. A F_s set f_A over X is called (1,2)- F_s pre-open (pre-open for short) set with respect to the

two F_s topological spaces $(X, \sigma_1, \mathbb{E})$ and $(X, \sigma_2, \mathbb{E})$ if $f_A \preceq \text{int}_1(\text{cl}_2(f_A))$. The family of all F_s *pre-open* sets relative to, two F_s topologies denote as $\text{Pre-O}(X)$.

Definition 3.1.2: Consider $(X, \sigma_1, \sigma_2, \mathbb{E})$ be a F_s bitopological space. A F_s set f_A over X is called (1,2)- F_s *pre-closed* (*pre-closed* for short) set with respect to the two F_s topological spaces $(X, \sigma_1, \mathbb{E})$ and $(X, \sigma_2, \mathbb{E})$ if $\text{cl}_1(\text{int}_2(f_A)) \preceq f_A$, i.e. A class for every F_s *pre-closed* set with respect to two F_s topologies is denoted by $\text{Pre-C}(X)$.

Example 3.1.3: Let $X = \{h_1, h_2, h_3\}$, $\mathbb{E} = \{\ell_1, \ell_2\}$, $\sigma_1 = \{\tilde{0}_{\mathbb{E}}, \tilde{1}_{\mathbb{E}}, f_{A_1}, f_{A_2}, f_{A_3}, f_{A_4}\}$ and $\sigma_2 = \{\tilde{0}_{\mathbb{E}}, \tilde{1}_{\mathbb{E}}, f_{A_1}, f_{A_3}\}$ where $f_{A_1}, f_{A_2}, f_{A_3}, f_{A_4}$ are F_s , describe by:

$$\begin{aligned} f_1(\ell_1)(h_1) &= 0.5 & f_1(\ell_1)(h_2) &= 0.3 & f_1(\ell_1)(h_3) &= 0.2 \\ f_1(\ell_2)(h_1) &= 0.3 & f_1(\ell_2)(h_2) &= 0.5 & f_1(\ell_2)(h_3) &= 0.2 \\ f_2(\ell_1)(h_1) &= 1 & f_2(\ell_1)(h_2) &= 0 & f_2(\ell_1)(h_3) &= 0.5 \\ f_2(\ell_2)(h_1) &= 0.5 & f_2(\ell_2)(h_2) &= 0.3 & f_2(\ell_2)(h_3) &= 1 \\ f_3(\ell_1)(h_1) &= 0.5 & f_3(\ell_1)(h_2) &= 0 & f_3(\ell_1)(h_3) &= 0.2 \\ f_3(\ell_2)(h_1) &= 0.3 & f_3(\ell_2)(h_2) &= 0.3 & f_3(\ell_2)(h_3) &= 0.2 \\ f_4(\ell_1)(h_1) &= 1 & f_4(\ell_1)(h_2) &= 0.3 & f_4(\ell_1)(h_3) &= 0.5 \\ f_4(\ell_2)(h_1) &= 0.5 & f_4(\ell_2)(h_2) &= 0.5 & f_4(\ell_2)(h_3) &= 1 \end{aligned}$$

Then σ_1, σ_2 are a F_s topologies on X , we get $(X, \sigma_1, \sigma_2, \mathbb{E})$ is a F_s bitopological space over X . A class for every F_s *pre-open* set in X is $\text{Pre-O}(X) = \sigma_1 = \{\tilde{0}_{\mathbb{E}}, \tilde{1}_{\mathbb{E}}, f_{A_1}, f_{A_2}, f_{A_3}, f_{A_4}\}$.

Remark 3.1.4:

- (1) Every F_s σ_1 -open (σ_1 -closed) set is F_s *pre-open* (*pre-closed*).
- (2) A F_s bitopological space $(X, \sigma_1, \sigma_2, A)$ is called discrete F_s bitopological space over *pre-open* if $\text{Pre-O}(X)$ includes each subsets of X & we named indiscrete F_s bitopological space with respect to *pre-open* if $\text{Pre-O}(X) = \{\tilde{0}_A, \tilde{1}_A\}$.
- (3) intersection for any *pre-open* (*pre-closed*) sets is *pre-open* (*pre-closed*) set with respect to F_s bitopological space.
- (4) A union for any *pre-open* sets is *pre-open* set relative to F_s bitop. sp
- (5) A union for any *pre-closed* sets is not necessary to be *pre-closed* set with respect to F_s bitopological space. For instance: Let $X = \{h_1, h_2, h_3, h_4\}$, $\mathbb{E} = \{\ell_1, \ell_2\}$, $\sigma_1 = \{\tilde{0}_{\mathbb{E}}, \tilde{1}_{\mathbb{E}}, f_A\}$ and $\sigma_2 = \{\tilde{0}_{\mathbb{E}}, \tilde{1}_{\mathbb{E}}, f_{A_1}, f_{A_2}, f_{A_3}\}$ where $f_A, f_{A_1}, f_{A_2}, f_{A_3}$ are F_s sets, describe as follows:

$$\begin{aligned} f(\ell_1)(h_1) &= 0.5 & f(\ell_1)(h_2) &= 0 & f(\ell_1)(h_3) &= 0.7 & f(\ell_1)(h_4) &= 0 \\ f(\ell_2)(h_1) &= 0 & f(\ell_2)(h_2) &= 0 & f(\ell_2)(h_3) &= 0.3 & f(\ell_2)(h_4) &= 0 \\ f_1(\ell_1)(h_1) &= 0 & f_1(\ell_1)(h_2) &= 0 & f_1(\ell_1)(h_3) &= 0.3 & f_1(\ell_1)(h_4) &= 0.5 \\ f_1(\ell_2)(h_1) &= 0.3 & f_1(\ell_2)(h_2) &= 0 & f_1(\ell_2)(h_3) &= 0 & f_1(\ell_2)(h_4) &= 0.6 \end{aligned}$$

$$\begin{aligned}
f_2(\ell_1)(h_1) &= 0 & f_2(\ell_1)(h_2) &= 0.4 & f_2(\ell_1)(h_3) &= 0 & f_2(\ell_1)(h_4) &= 0.5 \\
f_2(\ell_2)(h_1) &= 0 & f_2(\ell_2)(h_2) &= 0.1 & f_2(\ell_2)(h_3) &= 0 & f_2(\ell_2)(h_4) &= 0 \\
f_3(\ell_1)(h_1) &= 0 & f_3(\ell_1)(h_2) &= 0.4 & f_3(\ell_1)(h_3) &= 0.3 & f_3(\ell_1)(h_4) &= 0.5 \\
f_3(\ell_2)(h_1) &= 0.3 & f_3(\ell_2)(h_2) &= 0.1 & f_3(\ell_2)(h_3) &= 0 & f_3(\ell_2)(h_4) &= 0.6
\end{aligned}$$

Then σ_1, σ_2 are a F_s topologies and hence $(X, \sigma_1, \sigma_2, \mathbb{E})$ is a F_s bitopological space. Observe that f_A, f_{A_1} are two F_s *pre*-closed sets and the union those sets is not necessary to be *pre*-closed.

Theorem 3.1.5: Let f_A and g_B be F_s subsets of fuzzy bitopological space $(X, \sigma_1, \sigma_2, \mathbb{E})$ over X . Then:

- (1) f_A is a F_s *pre*-open iff there exists a F_s open set g_B in σ_1 where $f_A \lesssim g_B \lesssim cl_2(f_A)$.
- (2) f_A is a F_s *pre*-closed iff there is a F_s closed set g_B in σ_1 where $int_2(f_A) \lesssim g_B \lesssim (f_A)$.

Proof:

- (1) Suppose that f_A is a F_s *pre*-open subset of $(X, \sigma_1, \sigma_2, \mathbb{E})$. This implies that $f_A \lesssim int_1(cl_2(f_A))$. Take $g_B = int_1(cl_2(f_A))$. Then g_B is a F_s open set in σ_1 and $f_A \lesssim g_B \lesssim cl_2(f_A)$.

Conversely, consider that g_B is a F_s open subset of σ_1 with $f_A \lesssim g_B \lesssim cl_2(f_A)$. We must prove that f_A is a F_s *pre*-open subset of $(X, \sigma_1, \sigma_2, \mathbb{E})$. Our supposition follows that $f_A \lesssim int_1(g_B) \lesssim int_1(cl_2(f_A))$. This implies that f_A is a F_s *pre*-open subset of $(X, \sigma_1, \sigma_2, \mathbb{E})$.

- (2) The second part is clear.

Definition 3.1.6: Suppose $(X, \sigma_1, \sigma_2, \mathbb{E})$ is F_s bitopological space & f_A is F_s set in X , then intersection for each fuzzy soft *pre*-closed sets contain f_A called F_s *pre*-closure for f_A & denote as $Pre-cl(f_A)$ & the $Pre-cl(f_A)$ is F_s *pre*-closed, and it is smallest F_s *pre*-closed which containing f_A .

Theorem 3.1.7: If $(X, \sigma_1, \sigma_2, \mathbb{E})$ be a F_s bitopological space and f_A, g_B are F_s subsets of $(X, \mathbb{E})$. Then,

- (1) $Pre-cl(\tilde{0}_{\mathbb{E}}) = \tilde{0}_{\mathbb{E}}$ and $Pre-cl(\tilde{1}_{\mathbb{E}}) = \tilde{1}_{\mathbb{E}}$.
- (2) $f_A \lesssim Pre-cl(f_A)$.
- (3) The F_s set f_A is *pre*-closed if and only if $f_A = Pre-cl(f_A)$.
- (4) If $f_A \lesssim g_B$, then $Pre-cl(f_A) \lesssim Pre-cl(g_B)$.
- (5) $Pre-cl(Pre-cl(f_A)) = Pre-cl(f_A)$.
- (6) $Pre-cl(f_A \tilde{\vee} g_B) = Pre-cl(f_A) \tilde{\vee} Pre-cl(g_B)$.
- (7) $Pre-cl(f_A \tilde{\wedge} g_B) \lesssim Pre-cl(f_A) \tilde{\wedge} Pre-cl(g_B)$.

Proof:

- (1) Obviously.
- (2) Directly from the definition.
- (3) If f_A is a F_s pre-closed set, then f_A itself a F_s pre-closed set which contains f_A . So f_A is the smallest F_s pre-closed set containing f_A and $f_A = \text{Pre} - \text{cl}(f_A)$. Conversely, suppose that $f_A = \text{Pre} - \text{cl}(f_A)$. Since $\text{Pre} - \text{cl}(f_A)$ is a F_s pre-closed set, so f_A is a F_s pre-closed set.
- (4) Let $f_A \lesssim g_B$. Since $g_B \lesssim \text{Pre} - \text{cl}(g_B)$, then $f_A \lesssim \text{Pre} - \text{cl}(g_B)$. The smallest F_s pre-closed set containing f_A is $\text{Pre} - \text{cl}(f_A)$, hence we obtain the result.
- (5) Let $g_A = \text{Pre} - \text{cl}(f_A)$. So g_A is a F_s pre-closed set, then $g_A = \text{Pre} - \text{cl}(g_A)$. Hence $\text{Pre} - \text{cl}(f_A) = \text{Pre} - \text{cl}(\text{Pre} - \text{cl}(f_A))$.
- (6) Since $f_A \lesssim (f_A \tilde{\vee} g_B)$ and $g_B \lesssim (f_A \tilde{\vee} g_B)$. By (3), $\text{Pre} - \text{cl}(f_A) \lesssim \text{Pre} - \text{cl}(f_A \tilde{\vee} g_B)$ and $\text{Pre} - \text{cl}(g_B) \lesssim \text{Pre} - \text{cl}(f_A \tilde{\vee} g_B)$. Hence $\text{Pre} - \text{cl}(f_A) \tilde{\vee} \text{Pre} - \text{cl}(g_B) \lesssim \text{Pre} - \text{cl}(f_A \tilde{\vee} g_B)$. Conversely, since $f_A \tilde{\vee} g_B \lesssim \text{Pre} - \text{cl}(f_A) \tilde{\vee} \text{Pre} - \text{cl}(g_B)$, and $\text{Pre} - \text{cl}(f_A) \tilde{\vee} \text{Pre} - \text{cl}(g_B)$ is a F_s pre-closed set. Since $\text{Pre} - \text{cl}(f_A \tilde{\vee} g_B)$ is the smallest F_s pre-closed set containing $(f_A \tilde{\vee} g_B)$, we have $\text{Pre} - \text{cl}(f_A \tilde{\vee} g_B) \lesssim \text{Pre} - \text{cl}(f_A) \tilde{\vee} \text{Pre} - \text{cl}(g_B)$.
- (7) Since $f_A \lesssim \text{Pre} - \text{cl}(f_A)$ and $g_B \lesssim \text{Pre} - \text{cl}(g_B)$. Thus $(f_A \tilde{\wedge} g_B) \lesssim \text{Pre} - \text{cl}(f_A) \tilde{\wedge} \text{Pre} - \text{cl}(g_B)$ and $\text{Pre} - \text{cl}(f_A) \tilde{\wedge} \text{Pre} - \text{cl}(g_B)$ is a F_s pre-closed set because of the intersection of two F_s pre-closed sets is F_s pre-closed. Also, we have the smallest F_s pre-closed set containing $(f_A \tilde{\wedge} g_B)$ is $\text{Pre} - \text{cl}(f_A \tilde{\wedge} g_B)$. Hence, we get the result.

Definition 3.1.8: Suppose $(X, \sigma_1, \sigma_2, \mathbb{E})$ is F_s bitopology & f_A is F_s set of X , A union for all F_s pre-open set containing f_A is called F_s pre-interior of f_A and is denoted by $\text{Pre} - \text{int}(f_A)$.

Theorem 3.1.9: If f_A is F_s subset for F_s bitopology $(X, \sigma_1, \sigma_2, \mathbb{E})$. Then:

- (1) $\text{Pre} - \text{int}(f_A)$ is F_s pre-open, and it is largest F_s pre-open over X that included in f_A .
- (2) F_s set f_A is pre-open if and only if $f_A = \text{Pre} - \text{int}(f_A)$

Proof:

- (1) Directly from the above definition.
- (2) Let f_A be a F_s pre-open set. Since $\text{Pre} - \text{int}(f_A)$ is the largest F_s pre-interior of f_A contained in f_A , we have $f_A = \text{Pre} - \text{int}(f_A)$. Conversely, consider that $f_A = \text{Pre} - \text{int}(f_A)$, then f_A is obviously F_s pre-open set.

Theorem 3.1.10: If $(X, \sigma_1, \sigma_2, \mathbb{E})$ be a F_s bitopological space and f_A, g_B are F_s subsets of $(X, \mathbb{E})$. Then,

- (1) $\text{Pre} - \text{int}(\tilde{0}_{\mathbb{E}}) = \tilde{0}_{\mathbb{E}}$ and $\text{Pre} - \text{int}(\tilde{1}_{\mathbb{E}}) = \tilde{1}_{\mathbb{E}}$.

- (2) $Pre - int(f_A) \preceq (f_A)$.
- (3) $Pre - int(Pre - int(f_A)) = Pre - int(f_A)$.
- (4) If $f_A \preceq g_B$, then $Pre - int(f_A) \preceq Pre - int(g_B)$.
- (5) $Pre - int(f_A \tilde{\wedge} g_B) = Pre - int(f_A) \tilde{\wedge} Pre - int(g_B)$.
- (6) $Pre - int(f_A) \tilde{\vee} Pre - int(g_B) \preceq Pre - int(f_A \tilde{\vee} g_B)$.

Proof:

- (1) Obviously.
- (2) Directly from the definition.
- (3) Let $Pre - int(f_A) = g_B$. Since g_B is a F_s pre-open set, then $Pre - int(g_B) = g_B$, so $Pre - int(Pre - int(f_A)) = Pre - int(f_A)$.
- (4) Let $f_A \preceq g_B$. Since $Pre - int(f_A) \preceq f_A$, then $Pre - int(f_A) \preceq g_B$, also $Pre - int(g_B)$ largest F_s pre-open included in g_B , so $Pre - int(f_A) \preceq Pre - int(g_B)$.
- (5) Since $Pre - int(f_A) \preceq f_A$ and $Pre - int(g_B) \preceq g_B$. Hence $(Pre - int(f_A) \tilde{\wedge} Pre - int(g_B)) \preceq (f_A \tilde{\wedge} g_B)$. Since the largest F_s pre-closed set contained in $f_A \tilde{\wedge} g_B$ is $Pre - int(f_A \tilde{\wedge} g_B)$, then $Pre - int(f_A) \tilde{\wedge} Pre - int(g_B) \preceq Pre - int(f_A \tilde{\wedge} g_B)$.
Conversely, $Pre - int(f_A \tilde{\wedge} g_B) \preceq Pre - int(f_A)$ and $Pre - int(f_A \tilde{\wedge} g_B) \preceq Pre - int(g_B)$. Hence $Pre - int(f_A \tilde{\wedge} g_B) \preceq Pre - int(f_A) \tilde{\wedge} Pre - int(g_B)$.
- (6) Since $Pre - int(f_A) \preceq f_A$ and $Pre - int(g_B) \preceq g_B$. Hence $(Pre - int(f_A) \tilde{\vee} Pre - int(g_B)) \preceq (f_A \tilde{\vee} g_B)$. Since the largest F_s pre-open set contained in $f_A \tilde{\vee} g_B$ is $Pre - int(f_A \tilde{\vee} g_B)$ and so $Pre - int(f_A) \tilde{\vee} Pre - int(g_B) \preceq Pre - int(f_A \tilde{\vee} g_B)$.

Fuzzy Soft Quasi Pre-Separation Axiom for Fuzzy Soft Bitopological Space

Definition 3.2.1: A F_s bi-topological space $(X, \sigma_1, \sigma_2, \mathbb{E})$ claimed to the quasi F_s pre - T_o space (q_{F_s} Pre - T_o , for short). If for every distinct pair of F_s points $\ell_1(f_A)$ and $\ell_2(f_A)$, there exists a F_s pre-open subset g_A of X such that $\ell_1(f_A)qg_A \preceq (\ell_2(f_A))^c$ or $\ell_2(f_A)qg_A \preceq (\ell_1(f_A))^c$.

Theorem 3.2.2: Every F_s subspace of q_{F_s} Pre - T_o -space is a q_{F_s} Pre - T_o -space.

Proof: Let $(Y, (\sigma_1)_Y, (\sigma_2)_Y, \mathbb{E})$ be a F_s sub space of a q_{F_s} Pre - T_o -space $(X, \sigma_1, \sigma_2, \mathbb{E})$. To prove that the F_s subspace is a q_{F_s} Pre - T_o -space. Let $\ell_1(f_A) = \ell_2(f_A) \notin Y$, since $Y \subseteq X$, then $\ell(f_A) = \ell_2(f_A) \notin X$ and $(X, \sigma_1, \sigma_2, \mathbb{E})$ is a q_{F_s} Pre - T_o -space, therefore there exists a F_s pre-open subset g_A such that $\ell_1(f_A)qg_A \preceq (\ell_2(f_A))^c$ or $\ell_2(f_A)qg_A \preceq (\ell_1(f_A))^c$. So we get $\ell_1(f_A)q$

$(h_{\mathbb{E}}^Y \tilde{\wedge} g_A) \preceq (\ell_2(f_A))^c$ or $\ell_2(f_A)q(h_{\mathbb{E}}^Y \tilde{\wedge} g_A) \preceq (\ell_1(f_A))^c$. Hence $(Y, (\sigma_1)_Y, (\sigma_2)_Y, \mathbb{E})$ is a q_{F_s} Pre – T_o -space.

Theorem 3.2.3: *A F_s bitopological space $(X, \sigma_1, \sigma_2, \mathbb{E})$ is quasi Pre – T_o -space iff all distinct F_s points $\ell_1(f_A), \ell_2(f_A)$, $\text{Pre} - cl(\ell_1(f_A)) = \text{Pre} - cl(\ell_2(f_A))$.*

Proof: Suppose $(X, \sigma_1, \sigma_2, \mathbb{E})$ is a q_{F_s} Pre – T_o space, then for every distinct pair of F_s points $\ell_1(f_A)$ and $\ell_2(f_A)$, $\exists F_s$ pre-open subset g_A of X such that $\ell_1(f_A)qg_A \preceq (\ell_2(f_A))^c$ or $\ell_2(f_A)qg_A \preceq (\ell_1(f_A))^c$. If $\ell_1(f_A)qg_A \preceq (\ell_2(f_A))^c$, then $\ell_1(f_A) \not\in (g_A)^c$ and $g_A \preceq (\ell_2(f_A))^c$, that is, $\ell_1(f_A) \not\in (g_A)^c$ and $\ell_2(f_A) \in (g_A)^c$. Since $(g_A)^c$ is a F_s pre-closed set and $\text{Pre} - cl(\ell_2(f_A))$ is the smallest F_s pre-closed set containing $\ell_2(f_A)$, then $\text{Pre} - cl(\ell_2(f_A)) \preceq (g_A)^c$. Since $\ell_1(f_A) \not\in (g_A)^c$ and $\ell_1(f_A) \in \text{Pre} - cl(\ell_1(f_A))$, then $\text{Pre} - cl(\ell_1(f_A)) \neq \text{Pre} - cl(\ell_2(f_A))$.

Conversely, consider that $\ell_1(f_A)$ and $\ell_2(f_A)$ are distinct pair of F_s points in X such that $\text{Pre} - cl(\ell_1(f_A)) \neq \text{Pre} - cl(\ell_2(f_A))$. Let $\ell_3(f_A)$ be a F_s point in X such that $\ell_3(f_A) \in \text{Pre} - cl(\ell_1(f_A))$ and $\ell_3(f_A) \not\in \text{Pre} - cl(\ell_2(f_A))$. We claim that $\ell_1(f_A) \not\in \text{Pre} - cl(\ell_2(f_A))$. For, if $\ell_1(f_A) \in \text{Pre} - cl(\ell_2(f_A))$, then $\text{Pre} - cl(\ell_1(f_A)) \preceq \text{Pre} - cl(\ell_2(f_A))$. This contradicts the fact that $\ell_3(f_A) \not\in \text{Pre} - cl(\ell_2(f_A))$. Hence $\ell_1(f_A) \not\in \text{Pre} - cl(\ell_2(f_A))$, that is, $\ell_1(f_A)q(\text{Pre} - cl(\ell_2(f_A)))^c$, and since $g_A = (\text{Pre} - cl(\ell_2(f_A)))^c$ is a F_s pre-open subset of X and $g_A = (\text{Pre} - cl(\ell_2(f_A)))^c \preceq (\ell_2(f_A))^c$, then $\ell_1(f_A)qg_A \preceq (\ell_2(f_A))^c$. That is, $(X, \sigma_1, \sigma_2, \mathbb{E})$ is F_s Pre – T_o space.

Definition 3.2.4: A F_s bi-top. sp. $(X, \sigma_1, \sigma_2, \mathbb{E})$ claimed to the quasi F_s pre – T_1 space (q_{F_s} Pre – T_1 , for short). If for every two distinct F_s points $\ell_1(f_A)$ and $\ell_2(f_A)$, there exists g_A and j_A are F_s pre-open subsets of X such that $\ell_1(f_A)qg_A \preceq (\ell_2(f_A))^c$ and $\ell_2(f_A)qj_A \preceq (\ell_1(f_A))^c$.

Theorem 3.2.5: *Every F_s subspace of a q_{F_s} Pre – T_1 space is a q_{F_s} Pre – T_1 space.*

Theorem 3.2.6: *A F_s bitopological space $(X, \sigma_1, \sigma_2, \mathbb{E})$ is quasi Pre – T_1 space if and only if every F_s singleton point $\ell_1(f_A)$ in X is a F_s pre-closed set.*

Proof: Suppose $(X, \sigma_1, \sigma_2, \mathbb{E})$ is a F_s Pre – T_1 space. Given $\ell_2(f_A) \leq 1 - \ell_1(f_A)$ such that $\ell_1 \neq \ell_2$. Then there exists two F_s subset g_A and j_A of X such that $\ell_1(f_A)qg_A \preceq (\ell_2(f_A))^c$ and $\ell_2(f_A)qj_A \preceq (\ell_1(f_A))^c$. In part $\ell_2(f_A)qj_A \preceq (\ell_1(f_A))^c$, we have $j_A \preceq (\ell_1(f_A))^c$. Let $k_A = \bigvee \{j_A : \ell_2(f_A)qj_A \preceq (\ell_1(f_A))^c\}$. One may easily verify that $k_A = 1 - \ell_1(f_A)$. Hence $1 - \ell_1(f_A)$ is a F_s pre-open set, that is, $1 - \ell_1(f_A)$ is a F_s pre-closed set.

Conversely, let $\ell_1(f_A), \ell_2(f_A)$ be a pair of distinct F_s points in X . Then $\ell_1(f_A)$ and $\ell_2(f_A)$ are F_s pre-closed sets. Consequently, $1 - \ell_1(f_A)$ and $1 - \ell_2(f_A)$ are F_s pre-open sets. Hence $\ell_2(f_A)q\ell_1(f_A) \leq 1 - \ell_1(f_A)$ and $\ell_1(f_A)q\ell_2(f_A) \leq 1 - \ell_2(f_A)$. Therefore, $(X, \sigma_1, \sigma_2, \mathbb{E})$ is a q_{F_s} Pre - T_1 space.

Definition 3.2.7: A F_s bitopological space $(X, \sigma_1, \sigma_2, \mathbb{E})$ is claimed to the quasi fuzzysoft pre - T_2 space (q_{F_s} Pre - T_2 , for short). If for every two distinct F_s points $\ell_1(f_A)$ and $\ell_2(f_A)$, there exists g_A and J_A are F_s pre-open subsets of X such that $\ell_1(f_A)qg_A \lesssim (\ell_2(f_A))^c$, $\ell_2(f_A)qJ_A \lesssim (\ell_1(f_A))^c$ and $g_A \hat{q} J_A$.

Theorem 3.2.8: Every F_s subspace of a F_s Pre - T_2 space is a F_s Pre - T_2 space.

Proof: The prove is look like to prove Theorem 3.2.2.

Definition 3.2.9: A F_s bitopological space $(X, \sigma_1, \sigma_2, \mathbb{E})$ is called quasi Pre-regular space if and only if for each pre-closed F_s set g_A and each $\ell(f_A)q(g_A)^c$, there exist pre-open F_s sets s_A, h_A such that $\ell(f_A)qs_A, g_A \lesssim h_A$ and $s_A \hat{q} (h_A)^c$.

Theorem 3.2.10: A F_s bitopological space $(X, \sigma_1, \sigma_2, E)$ is a quasi-Pre-regular if and only if for each F_s pre-open set f_A and $\ell(f_A)qf_A$, there exists F_s pr-open set s_A such that $\ell(f_A)qs_A, Pr - Cl(s_A) \lesssim f_A$.

Proof: Let $(X, \sigma_1, \sigma_2, E)$ be a quasi Pre-regular and let $\ell(f_A)qf_A$ where f_A is a F_s pr-open set. Suppose $h_A = 1 - f_A$, then h_A is a F_s pre-closed set with $\ell(f_A)q(h_A)^c$. Then there is F_s pre-open sets g_A, s_A such that $\ell(f_A)qs_A, h_A \lesssim g_A$ and $g_A \hat{q} s_A$. Moreover, $s_A < 1 - g_A$ with $Pre - Cl(s_A) \lesssim Pre - Cl(1 - g_A) = 1 - g_A$, $h_A \lesssim g_A$, then $1 - g_A < 1 - h_A = f_A$, then $1 - g_A < f_A$, so we get $Pre - Cl(s_A) \lesssim f_A$.

Conversely, let h_A be a F_s pre-closed set and $\ell(f_A)q(h_A)^c$. Let $f_A = 1 - h_A$, then f_A is a F_s pre-open set where $\ell(f_A)qf_A$. so, there is F_s pre-open set s_A s.t $\ell(f_A)qs_A, Pre - Cl(s_A) \lesssim f_A$ and $h_A \lesssim (Pre - Cl(s_A))^c$. Since $\ell(f_A)qs_A, s_A \hat{q} Pre - Cl(s_A)$. Hence $(X, \sigma_1, \sigma_2, E)$ is quasi Pre-regular space.

Conclusions

The issue of pattern discrimination appears to share a specific similarity with the separation qualities for F_s sets. Soft set theory's fuzzification has become a hot issue in recent years. F_s sets from imprecise multi-observer data are being employed in the literature to build new techniques daily for solving decision-making issues. In the present work, we have continued to study the properties of F_s pre-open (pre-closed) on F_s bitopological space. We also, introduce F_s closure and F_s interior, moreover, we defined and studied some properties of F_s separation axioms ($Pre - T_0$, $Pre - T_1$, $Pre - T_2$) on F_s bitopological space.

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