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New accurate method for solving fourth order two-dimension evolution equations

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Abstract

In this article, a 4th order non-linear two-dimension evolution equation as a kind of partial differential equations and its reduction is studied and solved by efficient approach to get exact solution. This type of differential equations represents mathematical models of various physical phenomena in the real world. The authors reduced the order of partial differential equation by using inverse operator for the linear partial derivative with respect to the time independent variable. Then the unknown dependent function u can be expressed as infinite series. Two illustrated examples are introduced and the results show the proposed approach is reliable, effective and direct so can be used to solve any differential equation.

Subject Classification: *13C15, 14M07.*

Keywords: *Partial differential equations, Evolution equations, Iterative method, Decomposition method.*

1. Introduction

Partial differential equations (PDEs) theory plays a central role within the general advance meant of mathematics, since they help us to describe the evolution of many phenomena in various field of science, engineering, and numerous other applications [1, 2]. PDEs that contain unknown dependent variable with respect to the time derivatives are said to be evolution equations or nonlinear evolution [3]. Wave equations are PDEs involving 2nd derivatives with

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respect to the time independent variable (t). Such equations have been intensively studied for the past decades. Several new methods either analytically or numerically are available to solve in-homogeneous non-linear PDEs, [4, 5].

A mathematical model is a description of a complex issue that we may encounter in our daily lives and is difficult to solve. It may be in the field of medicine, the environment, biology, economics, and other different sciences. It is explained through this model, which includes linear or non-linear equation and mathematical symbols according to the type of system adopted [6, 7]. Its purpose is to understand the issue, find appropriate solution, and analyze its results. The primary aim of an ecosystem is to achieve ecological balance through a series of continuous interactions between living organisms and their surroundings, generating complex dynamics may be described as differential equations [8, 9].

There are many approaches can be used to solve fourth order PDEs specially evolution equation such as tanh-coth method [10], collocation method [11], $\left(\mathrm{G}/\mathrm{G}^2\right)$ expansion method [12], DTM [13], F-expansion method [14], VIM [15, 16], Darboux Transformation [17, 18] and parallel processing technique [19-22]. In this article, the researcher will be proposed a new method to solve 4th order non-linear, in-homogeneous PDEs and get reliable, efficient and analytical accurate results. The following section articulates the steps of the new suggested approach.

2. Suggested Method

Consider the non-linear, in-homogeneous 4th order PDE as follows

$$\mathfrak{l}(u(x,y,t)) + \mathcal{R}(u(x,y,t)) + \aleph(u(x,y,t)) = g(x,y,t) \tag{1}$$

For initial conditions:

$$\frac{\partial^k u(x,y,t)}{\partial t^k}\Big|_{t=0} = f_k(x,y), \quad k = 0,1,\ldots,3 \tag{2}$$

Where $\mathfrak{l}(.)$ is a linear operator for the partial derivative of time t, and defined as $\mathfrak{l}(.) = \frac{\partial^n(.)}{\partial t^n}$, $n = 1;2;3$, $g(x,y,t)$: inhomogeneous part, $\aleph(.)$: a nonlinear part, $\mathcal{R}(.)$: is the remained of the linear part, y and x are independent variables for space. $\mathcal{R}(.)$ and $\aleph(.)$ are free order of partial derivatives of t. In suggested approach the unidentified dependent variable $u(x,y,t)$ will be defined as infinite series has the form

$$u(\mathrm{x},\mathrm{y},t) = u_0(\mathrm{x},\mathrm{y}) + u_1(\mathrm{x},\mathrm{y})t + u_2(\mathrm{x},\mathrm{y})t^2 + \cdots = \sum_{k=0}^{\infty} u_k(x,y)t^k \tag{3}$$

where

$$u_k(x,y) = \frac{1}{k!}\frac{\partial^k u(x,y,t)}{\partial t^k}\Big|_{t=0} \tag{4}$$

The components u_n $(n = 0,1,2\ \ldots)$ are calculated in the next step.

Rewrite (1) as the form:

$$\mathfrak{l}(u(x,\mathrm{y},t)) = -\mathcal{R}(u(\mathrm{x},y,t)) - \aleph(u(\mathrm{x},y,t)) + g(\mathrm{x},y,t) \tag{5}$$

Taking the inverse operator $\mathfrak{l}^{-1}$ for each sides of (5), to get:

$$\mathfrak{l}^{-1}\big(\mathfrak{l}(u(\mathrm{y},\mathrm{x},t))\big) = -\mathfrak{l}^{-1}[\mathcal{R}(u) + \aleph(u)] + \mathfrak{l}^{-1}[g(\mathrm{y},\mathrm{x},t)] \tag{6}$$

$$u(\mathrm{y},\mathrm{x},t)-\sum_{k=0}^{n-1}\frac{t^k}{k!}\frac{\partial^k u(x,y,t)}{\partial t^k}\Big|_{t=0}=-\mathcal{L}^{-1}[\mathcal{R}(u)+\tilde{\mathrm{N}}(u)]+\mathcal{L}^{-1}[g(\mathrm{y},\mathrm{x},t)] \quad (7)$$

Depending on (2) , can be get:

$$u(y,x;\,t)\;=\sum_{k=0}^{n-1}t^k f_k(\mathrm{y},\mathrm{x})-\mathcal{L}^{-1}[\mathcal{R}(u)+\tilde{\mathrm{N}}(u)]+\mathcal{L}^{-1}[g(\mathrm{y},\mathrm{x},t)] \quad (8)$$

In the follow we substitute (3) in (8), to get:

$$\mathcal{L}^{-1}(\mathcal{R}(u))=\mathcal{L}^{-1}\big(\mathcal{R}(\textstyle\sum_{k=0}^{\infty}u_k(y,x)t^k)\big)=\sum_{k=0}^{\infty}\mathcal{R}(u_k(y,x))\frac{k!}{(n+k)!}t^{n+k} \quad (9)$$

Rewrite the nonlinear part $\tilde{\mathrm{N}}(u)$ in (8), as follows:

$$\tilde{\mathrm{N}}(u)=\sum_{k=0}^{\infty}N_k t^k \quad (10)$$

Such that

$$N_k=\frac{1}{k!}\frac{\partial^k N(u(\mathrm{y},\mathrm{x},t))}{\partial t^k}\Big|_{t=0} \quad (11)$$

Hence

$$\mathcal{L}^{-1}(\tilde{\mathrm{N}}(u))=\mathcal{L}^{-1}[\textstyle\sum_{k=0}^{\infty}N_k t^k]=\sum_{k=0}^{\infty}N_k\mathcal{L}^{-1}(t^k)=\sum_{k=0}^{\infty}N_k\frac{k!}{(n+k)!}t^{n+k} \quad (12)$$

Rewrite the inhomogeneous part as:

$$G(y,x,t)=\mathcal{L}^{-1}[g(\mathrm{y},\mathrm{x},t)]=\sum_{k=0}^{\infty}g_k\frac{t^k}{k!} \quad (13)$$

Where,

$$g_k=\frac{1}{k!}\frac{\partial^k G(\mathrm{x},\mathrm{y},t)}{\partial t^k}\Big|_{t=0} \quad (14)$$

In the following step we substitute (9), (12) and (13) in (8) to obtain:

$$\begin{aligned}u(y,x,t)=&\sum_{k=0}^{n-1}\frac{t^k}{k!}f_k(y,x)-\sum_{k=0}^{\infty}\mathcal{R}(u_k(y,x))\frac{k!}{(n+k)!}t^{n+k}\\&-\sum_{k=0}^{\infty}\mathcal{N}_k\frac{k!}{(n+k)!}t^{n+k}+\sum_{k=0}^{\infty}g_k\frac{t^k}{k!}\end{aligned} \quad (15)$$

Substituting (15) in (4), $\forall\ j\ \geq\ n,$ to get:

$$u_j(x,y)=\frac{1}{j!}\frac{\partial^j u(x,y,t)}{\partial t^j}\Big|_{t=0}=$$
$$\frac{1}{j!}\frac{\partial^j}{\partial t^j}\Big[\sum_{k=0}^{n-1}f_k(x,y)\frac{t^k}{k!}-\sum_{k=0}^{\infty}\frac{k!}{(n+k)!}(\mathcal{R}(u_k(x,y))+N_k)t^{n+k}+\sum_{k=0}^{\infty}g_k t^k\Big]_{t=0} \quad (16)$$

Since $j\ \geq n$, then

$$\frac{\partial^j}{\partial t^j}\Big[\sum_{k=0}^{n-1}f_k(x,y)\frac{t^k}{k!}\Big]=0 \text{ and } \frac{\partial^j}{\partial t^j}t^k=\begin{cases}0 & k<j\\ \frac{k!}{(k-j)!}t^{k-j} & k\geq j\end{cases}$$

Thus (16) becomes:

$$u_j(x,y) = \frac{1}{j!}\Big[-\sum_{k=j-n}^{\infty} \frac{k!}{(n+k)!}\big(\mathcal{R}\big(u_k(x,y)\big) + N_k\big)\ \frac{(n+k)!}{(n+k-j)!}t^{n+k-j} + \sum_{k=0}^{\infty} g_k \frac{k!}{(k-j)!}t^{k-j}\Big]_{t=0} \quad (17)$$

$$u_j(x,y) = \frac{1}{j!}\left[-\frac{(j-n)!}{0!}\big(\mathcal{R}\big(u_{j-n}\big) + N_{j-n}\big) + g_j\,\frac{j!}{0!}t^{k-j}\right] \quad (18)$$

Hence

$$u_j(x,y) = g_j - \frac{(j-n)!}{j!}\Big(\mathcal{R}\left(u_{j-n}(x,y)\right) + N_{j-n}\Big),\quad j \geq n \quad (19)$$

Final step, substitute (19) in (3) to get the result $u(x,y\,,t)$.

3. Explanation Examples

Herein, two clearing examples for solving 4th order non-linear, in-homogeneous two-dimension evolution equation are presented by using suggested method.

3.1 *Example1*

Consider 4th order non-linear, in-homogeneous PDE [20]:

$$u_{xt} - 6uu_{xx} - 6(u_x)^2 + u_{xxxx} + 3u_{yy} = 0\ ,$$

With IC: $u_x(y,x,0) = -0.5\ csc^2\left(\frac{(y+x)}{2}\right)\coth\left(\frac{(y+x)}{2}\right)$

It is clear that $\mathfrak{l}(u) = \frac{\partial}{\partial t}$, $\mathcal{R}(u) = \big(u_{xxxx} + 3u_{yy}\big)$, $\aleph(u) = -(6uu_{xx} + 6(u_x)^2)$, $g(x,y,t) = 0 \Longrightarrow u_{xt} = 6uu_{xx} + 6(u_x)^2 - u_{xxxx} - 3u_{yy}$

From IC: $\mathrm{u_{0x}(y,x,0) = -0.5csc^2 0.5\,(y+x)coth 0.5(y+x) \Longrightarrow u_0 = \frac{csch^2 0.5(y+x)}{2}}$

$u_{0xx} = u_{0yy} = \frac{1}{4}csch^4 0.5\,(y+x) + \frac{1}{2}csch^2 0.5\,(y+x)coth^2 0.5\,(y+x)$

$u_{0xxx} = -csch^4 0.5(y+x)coth 0.5(y+x) - \frac{1}{2}csch^2 0.5(y+x)coth^3 0.5\,(y+x)$

$u_{0xxxx} = \frac{1}{2}csch^6\big(0.5\,(y+x)\big) + \frac{11}{4}csch^4\big(0.5\,(y+x)\big)coth^2\big(0.5\,(y+x)\big) + \frac{1}{2}csch^2\big(0.5\,(y+x)\big)coth^4\big(0.5\,(y+x)\big)$

from (11), to get: $-N_0 = 6(u_0u_{0xx} + (u_{0x})^2)$

$N_0 = 6\left[\frac{1}{8}csch^6\big(0.5\,(y+x)\big) + \frac{1}{2}csch^4\big(0.5\,(y+x)\big)coth^2\big(0.5\,(y+x)\big)\right]$

$\mathcal{R}(u_0) = -\frac{1}{2}csch^6 0.5\,(y+x) - \frac{11}{4}csch^4 0.5\,(y+x)coth^2 0.5\,(y+x) - \frac{1}{2}csch^2 0.5\,(y+x)coth^4 0.5\,(y+x) - \frac{3}{4}csch^4 0.5\,(y+x) - \frac{3}{2}csch^2 0.5\,(y+x)coth^2 0.5\,(y+x)$

By (19), we have: $u_{1x}(y+x) = -\frac{0!}{1!}\big(\mathcal{R}\big(u_0(y+x)\big) + N_0\big)$

$u_{1x}(\mathrm{y}+\mathrm{x}) = -csch^4\big(0.5\ (\mathrm{y}+\mathrm{x})\big) - 2csch^2\big(0.5\ (\mathrm{y}+\mathrm{x})\big)coth^2\big(0.5\ (\mathrm{y}+\mathrm{x})\big)$
$= d\big[2csch^2\big(0.5\ (\mathrm{y}+\mathrm{x})\big)\ coth\big(0.5\ (\mathrm{y}+\mathrm{x})\big)\big]$; $u_1 = 2csch^2 0.5\ (\mathrm{y}+$
$\mathrm{x})\ coth 0.5\ (\mathrm{y}+\mathrm{x})$

$u_{1x} = -csch^4\big(0.5\ (\mathrm{y}+\mathrm{x})\big) - 2csch^2\left(\frac{(\mathrm{y}+\mathrm{x})}{2}\right)coth^2\left(\frac{(\mathrm{y}+\mathrm{x})}{2}\right)$

$u_{1xx} = 4csch^4\big(0.5\ (\mathrm{y}+\mathrm{x})\big)coth 0.5\ (\mathrm{y}+\mathrm{x}) + 2csch^2 0.5\ (\mathrm{y}+\mathrm{x})coth^3\big(0.5\ (\mathrm{y}+$
$\mathrm{x})\big)$

$u_{1xxx} = -2csch^6 0.5(\mathrm{y}+\mathrm{x}) - 11csch^4 0.5\ (\mathrm{y}+\mathrm{x})coth^2 0.5\ (\mathrm{y}+\mathrm{x}) -$
$2csch^2 0.5\ (\mathrm{y}+\mathrm{x})coth^4 0.5\ (\mathrm{y}+\mathrm{x})$

$u_{1xxxx} = 17csch^6 0.5\ (\mathrm{y}+\mathrm{x})coth 0.5\ (\mathrm{y}+\mathrm{x}) + 26csch^4 0.5\ (\mathrm{y}+$
$\mathrm{x})coth^3 0.5\ (\mathrm{y}+\mathrm{x}) + 2csch^2 0.5\ (\mathrm{y}+\mathrm{x})coth^5 0.5\ (\mathrm{y}+\mathrm{x})$

Also, from (11), $N_1 = \frac{\partial}{\partial t}[-6(uu_{xx} + (u_x)^2)] = -6(u_0u_{1xx} + u_1u_{0xx} + 2u_{0x}u_{1x}) = 6\left[\frac{7}{2}\ csch^6\big(0.5\ (\mathrm{y}+\ \mathrm{x})\big)\ coth\ \big(0.5\ (\mathrm{y}+\mathrm{x})\big) + 4csch^4\ \big(0.5\ (\mathrm{y}+\mathrm{x})\big)coth^3\big(0.5\ (\mathrm{y}+\mathrm{x})\big)\right]$; From (19); $u_{2x}(\mathrm{y}+\mathrm{x}) = -\frac{1}{2!}\big(\mathcal{R}\big(u_1(\mathrm{y}+\mathrm{x})\big) + N_1\big)$

$u_{2x} = \frac{1}{2!}\big[-16csch^4\big(0.5\ (\mathrm{y}+\mathrm{x})\big)coth\big(0.5\ (\mathrm{y}+\mathrm{x})\big) - 8csch^2\big(0.5\ (\mathrm{y}+$
$\mathrm{x})\big)coth^3\big(0.5\ (\mathrm{y}+\mathrm{x})\big)\big] \rightarrow u_2 = \frac{1}{2!}\big[4csch^4\big(0.5\ (\mathrm{y}+\mathrm{x})\big) + 8csch^2\big(0.5\ (\mathrm{y}+$
$\mathrm{x})\big)coth^2\big(0.5\ (\mathrm{y}+\mathrm{x})\big)\big]$, and so on , from (3) : $u = \sum_{k=0}^{\infty} u_k(x,y)t^k$

$u =$
$\frac{1}{2}csch^2\left(\frac{(\mathrm{y}+\mathrm{x})}{2}\right) + 2tcsch^2\left(\frac{1}{2}(\mathrm{y}+\mathrm{x})\right)coth\left(\frac{(\mathrm{y}+\mathrm{x})}{2}\right) + \frac{t^2}{2!}\left(4csch^4\left(\frac{(\mathrm{y}+\mathrm{x})}{2}\right) + 8csch^2\left(\frac{(\mathrm{y}+\mathrm{x})}{2}\right)coth^2\left(\frac{(\mathrm{y}+\mathrm{x})}{2}\right)\right) + \cdots$. So $u(y,x,t) = \frac{1}{2}csch^2\left(\frac{(\mathrm{y}+\mathrm{x}-4\mathrm{t})}{2}\right)$ is the exact solution. Some researchers solved this example and get approximate solution see [15]. Others solved the example using more complicate method such Herioto bilinear method [18, 19].

3.2 *Example 2*

Consider the nonlinear 4th order non-linear, in-homogeneous PDE [20]

$u_{tt} - u_{xx} - \frac{1}{2}(u^2)_{xx} - u_{yy} - u_{xxxx} = 0$, with ICs:

$$u(x,y,0) = 6sech^2\left(\frac{(\mathrm{x}+\mathrm{y})}{\sqrt{2}}\right) \text{ and}$$

$$u_t(x,y,0) = 24\left(\frac{1}{\sqrt{2}}\right)sech^2\left(\frac{(\mathrm{x}+\mathrm{y})}{\sqrt{2}}\right)tanh\left(\frac{(\mathrm{x}+\mathrm{y})}{\sqrt{2}}\right)$$

We used suggested method to solve the model equation. Firstly determine

$\mathfrak{l}(u) = \frac{\partial^2}{\partial t^2}, \mathcal{R}(u) = -u_{xx} - u_{yy} - u_{xxxx}, \aleph(u) = -\frac{1}{2}(u^2)_{xx}, g(x,y,t) = 0$

$$\Longrightarrow u_{tt} = u_{xx} + \frac{1}{2}(u^2)_{xx} + u_{yy} + u_{xxxx}$$

From ICs:

$u_0 = 6sech^2\left(\frac{(x+y)}{\sqrt{2}}\right)$, and $u_1 = 24\left(\frac{1}{\sqrt{2}}\right)sech^2\left(\frac{(x+y)}{\sqrt{2}}\right)tanh\left(\frac{(x+y)}{\sqrt{2}}\right)$

$u_{0x} = u_{0y} = \frac{-12}{\sqrt{2}}\text{sech}^2\left(\frac{(x+y)}{\sqrt{2}}\right)\tanh\left(\frac{(x+y)}{\sqrt{2}}\right)$; $u_{0xx} = -6sech^4\left(\frac{(x+y)}{\sqrt{2}}\right) + 12sech^2\left(\frac{(x+y)}{\sqrt{2}}\right)tanh^2\left(\frac{(x+y)}{\sqrt{2}}\right)$ and so on

$u_2(x,y) = -\frac{1}{2!}\left(\mathcal{R}(u_0(x,y)) + N_0\right) =$

$\frac{1}{2!}\left[-24sech^4\left(\frac{(x+y)}{\sqrt{2}}\right) + 48sech^2\left(\frac{(x+y)}{\sqrt{2}}\right)tanh^2\left(\frac{(x+y)}{\sqrt{2}}\right)\right]$

$u_{1x} = u_{1y} = 12\text{sech}^4\left(\frac{(x+y)}{\sqrt{2}}\right) - 24\text{sech}^2\left(\frac{(x+y)}{\sqrt{2}}\right)\tanh^2\left(\frac{(x+y)}{\sqrt{2}}\right)$

$u_{1xx} = \left(\frac{-96}{\sqrt{2}}\right)\text{sech}^4\left(\frac{(x+y)}{\sqrt{2}}\right)\tanh\left(\frac{(x+y)}{\sqrt{2}}\right) + \left(\frac{48}{\sqrt{2}}\right)\text{sech}^2\left(\frac{(x+y)}{\sqrt{2}}\right)\tanh^3\left(\frac{(x+y)}{\sqrt{2}}\right)$

$u_{1xxx} =$

$-48sech^6\left(\frac{(x+y)}{\sqrt{2}}\right) + 264sech^4\left(\frac{(x+y)}{\sqrt{2}}\right)tanh^2\left(\frac{(x+y)}{\sqrt{2}}\right) - 48sech^2\left(\frac{(x+y)}{\sqrt{2}}\right)$

$tanh^4\left(\frac{(x+y)}{\sqrt{2}}\right)$. Now, should be calculate $u_3(x,y)$

$u_3(x,y) = \frac{1}{3!}\left(\frac{-384}{\sqrt{2}}\tanh\left(\frac{(x+y)}{\sqrt{2}}\right)\text{sech}^4\left(\frac{(x+y)}{\sqrt{2}}\right) + \frac{192}{\sqrt{2}}\tanh^3\left(\frac{(x+y)}{\sqrt{2}}\right)\text{sech}^2\left(\frac{(x+y)}{\sqrt{2}}\right)\right)$

and so on, depending on (3), we have: $u = \sum_{k=0}^{\infty} u_k(x,y)t^k$.

$$u = 6sech^2\left(\frac{(x+y-2t)}{\sqrt{2}}\right).$$

This is exact solution. Some researchers solved this example by more complicate method such Herioto bilinear method [11, 21].

4. Conclusion

The precise solutions for 4th order non-linear PDEs are of significant importance for the explanation of some practical physical problems. The main aim of this article is to obtain the exact solution of this type of PDEs. By comparing the results in this article with the other research works, we find that the most accurate solutions as well as the function solution in this article is more general than the solution in the existing references. Because of not need any restriction conditions unlike VIM, which required additional time to calculate Lagrange multiplier and did not require further calculations like Adomian polynomials to handle the nonlinear parts in the ADM. So suggested approach can be used to solve any differential equations.

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