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Modified adomian decomposition method for solving PDEs of fractional order

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Abstract

A modify approach for approximating an analytical solution of some PDEs with fractional order is the Variational Adomian decomposition methodology (VIADM). We provide several examples to support our conclusions. the results of the solutions process showed that the suggested approach is highly simple, dependable, and effective. The outcomes demonstrate the effectiveness and accuracy of current technology in solving a variety of applied scientific nonlinear challenges. All of the calculations and visualizations were done using the MATLAB R2024B software. In Caputo Sense, fractional derivatives are mentioned. Additionally, a graphical representation of the solutions to a few cases was created. Solution graphs are displayed for problems of both integer and fractional order.

Subject Classification: *35R11, 35R60.*

Keywords: *Differential equations, Fractional order, Decomposition method, Semi analytical method.*

1. Introduction

Due to its focus on the study and use of differential equations and derivatives of any order, the fractional calculus holds an essential place in numerous domains. Fractional calculus may be seen as a topic that is both ancient and new. It begins with a conjecture by G.W. Leibniz and L. Euler, who proposed applying this connection to non-integer values of n. Historically,

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Locroix (1819) is the first derivative of arbitrary order to be discussed; J.B. Fourier developed it. Furthermore, Abel used fractional calculus to solve an integral equation, which was the first use of fractional derivatives. Liouville tried to define fractional derivatives logically. Fractional differential equations (FDEs) are a recent development that generalize PDEs. A variety of numerical and analytical methods, including the ADM, HPM, VIM, SVIM, and others, have been proposed to the results of fractional non-integer differential equations [1–14].

2. Solutions of Differential Equations by VIADM

Consider the fractional differential equation as follows:

$$cD_t^{\alpha} y(r,ț) + L\big(y(r,ț)\big) + N\big(y(r,ț)\big) = f(r,ț) \tag{1}$$

Where, $cD_ț^{\alpha}$ is the *Caputo* fractional order derivative, L, N are linear differential operator and nonlinear terms respectively, then Eq. (1) is given by:

$$y_{n+1}(r,ț) = y_n(r,ț) + \int_0^t \lambda(r,ș)\ [{}^cD_t^{\alpha} y_n(r,ș) + L(\tilde{y}_n(r,ț)) + N(\tilde{y}_n(r,ț)) - f(r,ș)]\, ds \tag{2}$$

Then

$$y_{n+1}(r,ț) = y_n(r,ț) + J_t^{\alpha}\ [\lambda(r,ș)\, \{D_t^{\alpha}(y_n(r,ș)) + L(\tilde{y}_n(r,ț)) + N(\tilde{y}_n(r,ț)) - g(r,ș)\}] \tag{3}$$

$$y_{n+1}(r,ț) = y_n(r,ț) - (\beta - 1)\, J_t^{\alpha} \left[\frac{\partial^{\beta}}{\partial ț^{\beta}} y_n(r,ș) + L\big(y_n(r,ș)\big) + N\big(y_n(r,ș)\big) - g(r,ș)\right] \tag{4}$$

The unknown linear function $y = \sum_{n=0}^{\infty} y_n$ and $N(y) = Ɣ(y)$ can be expressed by $N\big(y(r\,,ț)\big) = Ɣ(y) = \sum_{n=0}^{\infty} A_n$, s.t $A_0 = Ɣ(y_0)$, $A_1 = Ɣ'(y_0) y_1$, $A_2 = Ɣ'(y_0) y_2 + \frac{Ɣ''(y_0)}{2!} y_1^2$, $A_3 = Ɣ'(y_0) y_3 + Ɣ''(y_0)\, y_1\, y_2 + \frac{Ɣ''(y_0)}{3} y_1^3$, $A_4 = Ɣ'(y_0) y_4 + \left(\frac{1}{2!} y_2^2 + y_1\, y_2\right) Ɣ''(y_0) + \frac{1}{2!} y_1^2\, y_2\ Ɣ'''(y_0) + \frac{1}{4!} y_1^4 Ɣ''''(y_0)$.

So, the iteration formula of Eq. (4) reduces to:

$$y_{n+1}(r,ț) = y_n(r,ț) - (\beta - 1)\, J_t^{\alpha} \left[\frac{\partial^{\beta}}{\partial t^{\beta}} y_n(r,s) + L\big(y_n(r,s)\big) + \sum_{n=0}^{\infty\infty} A_n - g(r,s)\right].$$

3. Applications and Results

3.1 Example: (1) Consider advection partial differential equation of the time fractional as

$$D_ț^{\alpha}\, y(\varrho,ț) + y(\varrho,ț)\, y_{\varrho}(\varrho,ț) - \varrho - \varrho\, ț^2 = 0, >\ 0\ , 0\ \leq \alpha\ \leq\ 1 \tag{5}$$

With

$$y(\varrho,0) = 0 \tag{6}$$

Solution: By using FVIADM to find the solution of Eq. (5), the iteration formula is given by:

$$y_{n+1}(\varrho, ţ) = y_n(\varrho, ţ) - \int_0^{ţ} \left(\frac{\partial^\alpha}{\partial t^\alpha} y_n(\varrho, ş) + y_n(\varrho, ş) \frac{\partial y_n(\varrho, ş)}{\partial \varrho} - (\varrho + \varrho ş^2) \right) dş \tag{7}$$

Applying the *Adomian polynomials* for nonlinear term of above equation, then Eq. (7) becomes:

$$y_{n+1}(\varrho, ţ) = y_n(\varrho, ţ) - \int_0^{ţ} \left(\frac{\partial^\alpha}{\partial t^\alpha} y_n(\varrho, ş) + \sum_{i=0}^{\infty} A_i - (\varrho + \varrho ş^2) \right) dş$$

By using FVIADM, starting with $y_0(\varrho, t) = 0$, then

$$y_{n+1}(\varrho, ţ) = y_n(\varrho, ţ) + J_ţ^\alpha(\varrho + \varrho\, ţ^2) - J^\alpha\left(y_n \frac{\partial\, y_n}{\partial\, \varrho}\right) - J_ţ^\alpha\left(\frac{\partial\, y_n}{\partial\, ţ^\alpha}\right)$$

$$y_1(\varrho, ţ) = \varrho\left(\frac{ţ^\alpha}{\Gamma(\alpha+1)} + \frac{tţ^{\alpha+2}}{\Gamma(\alpha+3)}\right) = \varrho \frac{ţ^\alpha}{\Gamma(\alpha+1)} + \varrho \frac{ţ^{\alpha+2}}{\Gamma(\alpha+3)}$$

$$y_2(\varrho, t) = \varrho\left(\frac{ţ^\alpha}{\Gamma(\alpha+1)} + \frac{ţ^{\alpha+2}}{\Gamma(\alpha+3)} - \frac{ţ^{3\alpha}\Gamma(1+2\alpha)}{\Gamma(1+\alpha)^2\Gamma(1+3\alpha)} - \frac{4\Gamma(3+2\alpha)\, ţ^{3\alpha+2}}{\Gamma(1+\alpha)\,\Gamma(3+\alpha)\,\Gamma(3+3\alpha)} - \frac{4\Gamma(5+2\alpha)\, ţ^{3\alpha+4}}{\Gamma(3+\alpha)^2\,\Gamma(5+3\alpha)}\right).$$

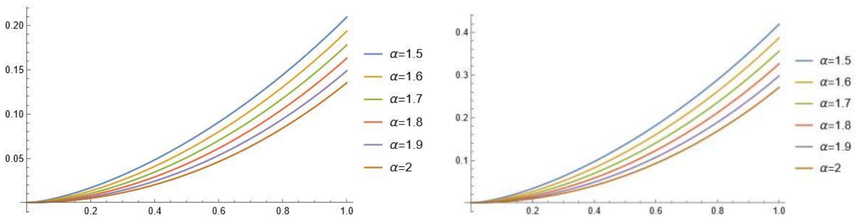

Figure 1
The approximate analytical solutions of y_1 (where $x = 0.25$ ($left$), $x = 0.5$($right$))

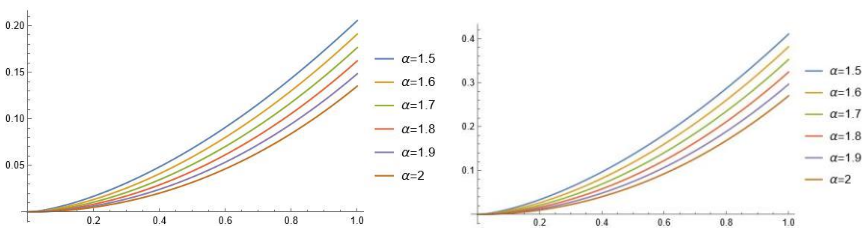

Figure 2
The approximate analytical solutions of y_2 ($x = 0.25$ ($left$), $x = 0.5$ ($right$))

3.2 *Results and Discussion*

The following Figures present the approximate analytical solutions and absolute error at α =0.5, 0.6, 0.8 and $\alpha = 1$. Here, we use some terms to approximate the exact solution, the proposed method FVIADM has a high convergence order and higher accuracy we get. All the approximate results on the Graphs have shown are much closed and explain the reliability of the present technique as shown in the following figures 1, 2, and 3.

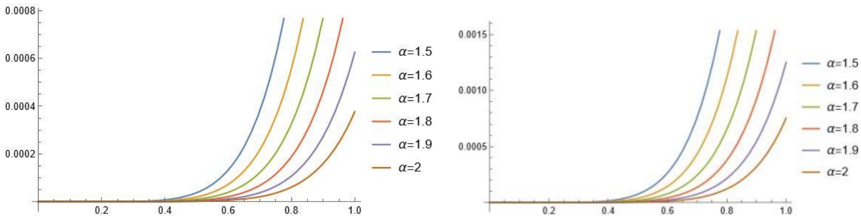

Figure 3
The absolute Error of $y_1 - y_2$

3.2 Example: (2) Consider fourth-order fractional singular PDEs:

$$\frac{\partial^\alpha y}{\partial t^\alpha} + \left(\frac{\varrho}{sinsin\, \varrho}\right)\frac{\partial^4 y}{\partial \varrho^4} - \frac{\partial^4 y}{\partial \varrho^4} = 0 \quad 0 < \varrho < 1 \ , \ t > 0 \quad 1 < \alpha \le 2 \tag{8}$$

With $y(\varrho, 0) = \varrho - sin\, sin\, \varrho, \quad 0 < \varrho < 1$

$$\frac{\partial y}{\partial t}(\varrho, 0) = -(\varrho - sin\, sin\, \varrho\,), \quad 0 < \varrho < 1$$

And $y(\varrho, 0) = 0, \quad y(1, t) = e^{-t}\,(1 - sin\, sin\, 1\,), \quad t > 0$

$$\frac{\partial^2 y}{\partial \varrho^2}(0, t) = 0\,, \quad \frac{\partial^2 y}{\partial \varrho^2}(1, t) = e^{-t}\, sin\, sin\, 1, \quad t > 1$$

Solution: By applying VIM, Eq. (8) can be expressed as:

$$y_{n+1}(\varrho, t) = y_n(\varrho, t) - (\alpha - 1)J_t^\alpha\left[\frac{\partial^\alpha y_n(\varrho,t)}{\partial t^\alpha} + \frac{\partial^4 y_n}{\partial \varrho^4}\left(-1 + \frac{\varrho}{sinsin\, \varrho}\right)\right] \tag{9}$$

Then

$$y_{n+1}(\varrho, t) = y_n(\varrho, t) - (\alpha - 1)J_t^\alpha\left[\frac{\partial^\alpha y_n(\varrho,t)}{\partial t^\alpha} + \frac{\partial^4 y_n}{\partial \varrho^4}\left(-1 + \frac{\varrho}{sinsin\, \varrho}\right)\right] \tag{10}$$

$\sum_{i=0}^{\infty} \quad Ai = \frac{\varrho}{sinsin\, \varrho} = y.y^* = \sum_{i=0}^{\infty} \quad y_i y_i^*, A_0 = \frac{\varrho}{sinsin\, \varrho},$

$A_1 = 2\frac{\varrho}{sinsin\, \varrho}, A_2 = 2$

Where, $y = \varrho, \quad y^* = \frac{1}{sinsin\, \varrho}\ , \ y_i = (y_0 + y_1 + y_2 + \cdots y_n)(y_0^*, y_1^*, y_2^* + \cdots)$

$A_0 = y_0 y_0^* = \frac{\varrho}{sinsin\, \varrho}$, $\quad A_1 = y_1 y_0^* + y_0 y_1^* = 2\frac{\varrho}{sinsin\, \varrho}$, $A_2 = 2\ y_2 y_0^* + y_1 y_1^* = \frac{2\varrho}{sinsin\, \varrho} + \left(\frac{1}{sinsin\, \varrho}\right)^2$

And $y_0(\varrho, t) = (\varrho - sin\ sin\ \varrho\ ) - (\varrho - sin\ sin\ \varrho\ )t$

$$y_1(\varrho, t) = y_0(\varrho, t) - (\alpha - 1) J_t^\alpha \left(\frac{\partial^\alpha y_0(\varrho,t)}{\partial t^\alpha} + \left(-1 + \frac{\varrho}{sinsin\, \varrho}\right)\frac{\partial^4 y_0}{\partial \varrho^4}\right)$$

$$= (\varrho - sin\ sin\ \varrho\ ) - (\varrho - sin\ sin\ \varrho\ )t + (\alpha - 1)(\varrho - sin\ sin\ \varrho\ )\frac{t^\alpha}{\Gamma(\alpha+1)} - (\alpha - 1)(\varrho - sin\ sin\ \varrho\ ))\ \frac{t^{\alpha+1}}{\Gamma(\alpha+2)}$$

$$y_2(\varrho, t) = y_1(\varrho, t) - (\alpha - 1)\ J_t^\alpha \left[\frac{\partial^\alpha y_1(\varrho,t)}{\partial t^\alpha} + \left(-1 + \frac{\varrho}{sinsin\, \varrho}\right)\frac{\partial^4 y_1}{\partial \varrho^4}\right]$$

$$= (\varrho - sin\ sin\ \varrho\ ) - (\varrho - sin\ sin\ \varrho\ )t - (\alpha - 3)(\alpha - 1)(\varrho - sin\ sin\ \varrho\ )\ \frac{t^\alpha}{\Gamma(\alpha+1)} + (\alpha - 3)(\alpha - 1)(\varrho - sin\ sin\ \varrho\ )\frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + (\alpha - 1)^2(\varrho - sin\ sin\ \varrho\ )\ \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - (\alpha - 1)^2(\varrho - sin\ sin\ \varrho\ )\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}$$

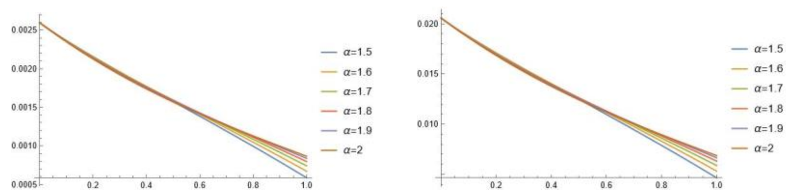

Figure 4
Show the approximate analytical solutions of y_1

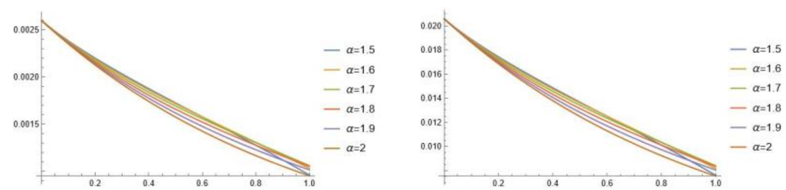

Figure 5
Show the approximate analytical solutions of y_2

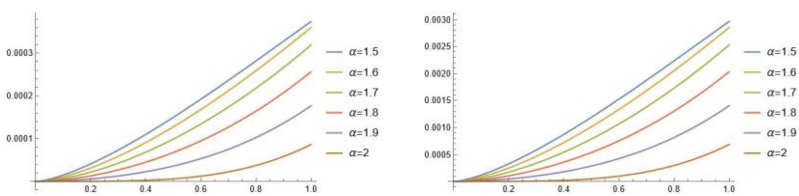

Figure 6
Show the absolute Error of $y_1 - y_2$

3.3 *Results and Discussion*

Figures 4, 5, and 6 present the absolute error at various values of $\alpha = 1.5, 1.6 \ldots$ and 2. Here, we use several terms to approximate the exact solution. The proposed method FVIADM has a high convergence order and higher accuracy. The approximate results on the graphs shown are very close and explain the reliability of the present technique.

4. Conclusion

When dealing with specific processes, differential equations are important. Nonlinear partial differential equations may not necessarily have analytical solutions. In this example, we use a semi-analytical approach to provide series solutions. The solutions to these difficulties are sought as a series. Identifying the series from initial conditions serves as the foundation for semi-analytical approaches. As a result, I must perform a convergence study of these strategies. Some semi-analytic approaches can achieve extremely good convergence with only a few arithmetic operations of the series, but further arithmetic operations may be required to improve convergence for larger problems and analytical solution.

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