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$\mathcal{J}_x - \mathcal{L}$ – open and some of their generalizations

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Abstract

The present work is concerned with the new set namely $\mathcal{J}_x - \mathcal{L}$ – open, which is devoted to the basic properties of $\mathcal{J}_x - \mathcal{L}$ – open and the relation between open set and $\mathcal{J}_x - \mathcal{L}$ – open set is shown. The main goals of this article are presenting a comprehensive study about of an $\mathcal{J}_x - \mathcal{L}$ – open set and giving many new results.

Subject Classification: *00A05, 00A06.*

Keywords: *Ttopology, Field, Ideal and open set.*

1. Introduction

Asaad, Ahmed, and Ebrahim [1], and Ahmed, and Ebrahim [2] have been studied some notions plays a significant turn in real analysis and algebra. Some types of classical sets have playing a substantial turn in the study of several properties in topological space as well as some authors introduced different generalized of an open set [3], [4] and where $\mathcal{H}$ in a space $(\mathcal{X}, \mathfrak{F})$ called

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$\mathcal{L}$ −open set when $\mathcal{H} \subseteq T \cup Int(\mathcal{H})$ $\forall T \in \mathfrak{F}$ & $T \neq \Phi$ [5]. The idea of semi-open set was introduced in [6] which is generalization for open. If we have topological space $(\mathcal{X}, \mathfrak{F})$, then an ordered pair $(\mathcal{X}, \mathcal{L}O_{\mathcal{X}})$ will be named $\mathcal{L}$ −space. The notion of α - open studied in [7] which is also generalization of open set. Ideal $\mathcal{L}$ −space is $\mathcal{L}$ −space $(\mathcal{X}, \mathcal{L}O_{\mathcal{X}})$ with ideal $\mathcal{I}_{\mathcal{X}}$ in $\mathcal{X}$ & its denoted as $(\mathcal{X}, \mathcal{L}O_{\mathcal{X}}, \mathcal{I}_{\mathcal{X}})$. The concept of regular open is studied in [8] which is stronger form open. The concept of h- open was studied in 2023 by [9]. There are many authors are studied some concepts related to open sets for example Abdalbaqi, Hadi, and Al-Swidi [10] and Khalaf et al. [11].

2. Main result in this work:

Definition 2.1: If $(\mathcal{X}, \mathcal{L}O_{\mathcal{X}})$ is $\mathcal{L}$ −space & $\mathcal{F} \subseteq \mathcal{X}$. A point $\varkappa \in \mathcal{F}$ is named $\mathcal{L}$ −interior point of $\mathcal{F}$ if there exist $\mathcal{H} \in \mathcal{L}O_{\mathcal{X}}$ such that $\in \mathcal{H} \subseteq \mathcal{F}$.

Definition 2.2: Assume $(\mathcal{X}, \mathcal{L}O_{\mathcal{X}})$ is $\mathcal{L}$ −space & $\mathcal{F} \subseteq \mathcal{X}$, then $lnt_{\mathcal{L}}(\mathcal{F})$ is known as a set of all $\mathcal{L}$ −interior points of $\mathcal{F}$.

Theorem 2.3: $lnt_{\mathcal{L}}(\mathcal{F}) = \bigcup_{i\in \mathrm{I}} \{\mathcal{H}_i : \mathcal{H}_i \text{ is } \mathcal{L} - open\ ,\ \mathcal{H}_i \subseteq \mathcal{F}\ \forall i \in \mathrm{I}\}$.

Proof: Assume $\varkappa \in lnt_{\mathcal{L}}(\mathcal{F})$, then $\varkappa$ is an $\mathcal{L}$ −interior point for $\mathcal{F}$, hence $\exists \mathcal{H} \in \mathcal{L}O_{\mathcal{X}}$ s.t $\varkappa \in \mathcal{H} \subseteq \mathcal{F}$. Now, $\mathcal{H}$ is a $\mathcal{L}$ −open such that $\mathcal{H} \subseteq \mathcal{F}$. So, we have $\varkappa \in \bigcup_{i\in \mathrm{I}} \{\mathcal{H}_i : \mathcal{H}_i \text{ is } \mathcal{L}-\text{ open}, \mathcal{H}_i \subseteq \mathcal{F}\ \forall i \in \mathrm{I}\}$. implies, $lnt_{\mathcal{L}}(\mathcal{F}) \subseteq \bigcup_{i\in \mathrm{I}} \{\mathcal{H}_i : \mathcal{H}_i \text{ is } \mathcal{L} - open\ ,\ \mathcal{H}_i \subseteq \mathcal{F}\ \forall i \in \mathrm{I}\}$.

Now, let $\varkappa \in \bigcup_{i\in \mathrm{I}} \{\mathcal{H}_i : \mathcal{H}_i \text{ is } \mathcal{L} - open\ ,\ \mathcal{H}_i \subseteq \mathcal{F}\ \forall i \in \mathrm{I}\}$. Then there is $\mathcal{L}-$ open $\mathcal{H}^*$, $\mathcal{H}^* \subseteq \mathcal{F}$ such that $\varkappa \in \mathcal{H}^*$, hence $\varkappa \in \mathcal{F}$ is an $\mathcal{L}$ −interior point of $\mathcal{F}$, that is $\varkappa \in lnt_{\mathcal{L}}(\mathcal{F})$. Thus $lnt_{\mathcal{L}}(\mathcal{F}) \supseteq \bigcup_{i\in \mathrm{I}} \{\mathcal{H}_i : \mathcal{H}_i \text{ is } \mathcal{L} - open\ ,\ \mathcal{H}_i \subseteq \mathcal{F}\ \forall i \in \mathrm{I}\}$. This completes the proof.

Theorem 2.4: *If $(\mathcal{X}, \mathcal{L}O_{\mathcal{X}})$ is $\mathcal{L} - space$ & $\mathcal{F} \subseteq \mathcal{X}$, then $lnt_{\mathcal{L}}(\mathcal{F})$ is largest $\mathcal{L} -$ open which included in $\mathcal{F}$.*

Proof: By theorem (2.3), we get

$$lnt_{\mathcal{L}}(\mathcal{F}) = \bigcup_{i\in \mathrm{I}} \{\mathcal{H}_i : \mathcal{H}_i \text{ is } \mathcal{L} - open\ ,\ \mathcal{H}_i \subseteq \mathcal{F}\ \forall i \in \mathrm{I}\}.$$

The arbitrary union of $\mathcal{L}-$ open is also $\mathcal{L} - open$ hence $lnt_{\mathcal{L}}(\mathcal{F})$ is a $\mathcal{L}-$ open set. Let $\mathcal{H}_i$ is $\mathcal{L} - open\ ,\ \mathcal{H}_i \subseteq \mathcal{F}\ \forall i \in \mathrm{I}$. Then

$\bigcup_{i\in \mathrm{I}} \{\mathcal{H}_i : \mathcal{H}_i \text{ is } \mathcal{L} - open\ ,\ \mathcal{H}_i \subseteq \mathcal{F}\ \forall i \in \mathrm{I}\} \subseteq \mathcal{F}$. Hence, $lnt_{\mathcal{L}}(\mathcal{F}) \subseteq \mathcal{F}$.

Now, let $\mathcal{H}^*$ be a $\mathcal{L}-$ open such that $\mathcal{H}^* \subseteq \mathcal{F}$. Then

$\mathcal{H}^* \subseteq \bigcup_{i\in \mathrm{I}} \{\mathcal{H}_i : \mathcal{H}_i \text{ is } \mathcal{L} - open,\ \mathcal{H}_i \subseteq \mathcal{F}\ \forall i \in \mathrm{I}\}$, that is $\mathcal{H}^* \subseteq lnt_{\mathcal{L}}(\mathcal{F})$. Therefore, $lnt_{\mathcal{L}}(\mathcal{F})$ largest $\mathcal{L}-$ open included in $\mathcal{F}$.

Theorem 2.5: *Assume $(\mathcal{X}, \mathcal{L}O_{\mathcal{X}})$ is $\mathcal{L} - space$ & $\mathcal{F} \subseteq \mathcal{X}$, then $\mathcal{F}$ is $\mathcal{L} - open$ iff $lnt_{\mathcal{L}}(\mathcal{F}) = \mathcal{F}$.*

Proof: Assume $\mathcal{F}$ is a $\mathcal{L}-$ open. by theorem (2.4), we get

$lnt_{\mathcal{L}}(\mathcal{F}) \subseteq \mathcal{F}$. But $\mathcal{F}$ is a $\mathcal{L}-$ open & $\mathcal{F} \subseteq \mathcal{F}$ and $lnt_{\mathcal{L}}(\mathcal{F})$ is largest $\mathcal{L}-$ open in $\mathcal{F}$. Then $\mathcal{F} \subseteq lnt_{\mathcal{L}}(\mathcal{F})$. Hence $lnt_{\mathcal{L}}(\mathcal{F}) = \mathcal{F}$.

Conversely): suppose that $lnt_{\mathcal{L}}(\mathcal{F}) = \mathcal{F}$. By theorem (2.4), implies that $lnt_{\mathcal{L}}(\mathcal{F})$ is $\mathcal{L}-$ open set. This completes the proof.

Theorem 2.6: *Suppose* $(\mathcal{X}, \mathcal{LO}_{\mathcal{X}})$ *is* $\mathcal{L}$ *−space &* $\mathcal{F}, \mathcal{C} \subseteq \mathcal{X}$. *Then*

1. If $\mathcal{F} \subseteq \mathcal{C}$, then $lnt_{\mathcal{L}}(\mathcal{F}) \subseteq lnt_{\mathcal{L}}(\mathcal{C})$.
2. $lnt_{\mathcal{L}}(\mathcal{F} \cap \mathcal{C}) = lnt_{\mathcal{L}}(\mathcal{F}) \cap lnt_{\mathcal{L}}(\mathcal{C})$.
3. $lnt_{\mathcal{L}}\big(lnt_{\mathcal{L}}(\mathcal{F})\big) = lnt_{\mathcal{L}}(\mathcal{F})$.
4. $lnt_{\mathcal{L}}(\Phi) = \Phi$ and $lnt_{\mathcal{L}}(\mathcal{X}) = \mathcal{X}$.

Proof:

1. assume $\mathcal{F} \subseteq \mathcal{C}$. Since $lnt_{\mathcal{L}}(\mathcal{F})$ is a $\mathcal{L}-$ open in $\mathcal{F}$ and $\mathcal{F} \subseteq \mathcal{C}$, then $lnt_{\mathcal{L}}(\mathcal{F})$ is $\mathcal{L}-$ open in $\mathcal{C}$. But $lnt_{\mathcal{L}}(\mathcal{C})$ largest $\mathcal{L}-$ open in $\mathcal{C}$, so $lnt_{\mathcal{L}}(\mathcal{F}) \subseteq lnt_{\mathcal{L}}(\mathcal{C})$.
2. Since $lnt_{\mathcal{L}}(\mathcal{F})$ and $lnt_{\mathcal{L}}(\mathcal{C})$ are $\mathcal{L}-$ open in $\mathcal{F}$ and $\mathcal{C}$ respectively & since intersection for two $\mathcal{L}-$ open is also $\mathcal{L}-$ open, then $lnt_{\mathcal{L}}(\mathcal{F}) \cap lnt_{\mathcal{L}}(\mathcal{C})$ is $\mathcal{L}-$ open. Now, $lnt_{\mathcal{L}}(\mathcal{F}) \subseteq \mathcal{F}$ *and* $lnt_{\mathcal{L}}(\mathcal{C}) \subseteq \mathcal{C}$, then $lnt_{\mathcal{L}}(\mathcal{F}) \cap lnt_{\mathcal{L}}(\mathcal{C}) \subseteq \mathcal{F} \cap \mathcal{C}$, so $lnt_{\mathcal{L}}(\mathcal{F}) \cap lnt_{\mathcal{L}}(\mathcal{C})$ is a $\mathcal{L}-$ open in $\mathcal{F} \cap \mathcal{C}$. from theorem (2.4), implies $lnt_{\mathcal{L}}(\mathcal{F} \cap \mathcal{C})$ is largest $\mathcal{L}-$ open in $\mathcal{F} \cap \mathcal{C}$, thus $lnt_{\mathcal{L}}(\mathcal{F}) \cap lnt_{\mathcal{L}}(\mathcal{C}) \subseteq lnt_{\mathcal{L}}(\mathcal{F} \cap \mathcal{C})$. To prove $lnt_{\mathcal{L}}(\mathcal{F} \cap \mathcal{C}) \subseteq lnt_{\mathcal{L}}(\mathcal{F}) \cap lnt_{\mathcal{L}}(\mathcal{C})$. Let $\varkappa \in lnt_{\mathcal{L}}(\mathcal{F} \cap \mathcal{C})$, then $\varkappa$ is an $\mathcal{L}$ −interior point of $\mathcal{F} \cap \mathcal{C}$, hence there is $\mathcal{H} \in \mathcal{LO}_{\mathcal{X}}$ such that $\varkappa \in \mathcal{H} \subseteq \mathcal{F} \cap \mathcal{C}$.thus $\varkappa \in \mathcal{H} \subseteq \mathcal{F}$ and $\varkappa \in \mathcal{H} \subseteq \mathcal{C}$,therefore $\varkappa$ is an $\mathcal{L}$ −interior point of $\mathcal{F}$ and $\mathcal{C}$. Consequentially, $\varkappa \in lnt_{\mathcal{L}}(\mathcal{F}) \cap lnt_{\mathcal{L}}(\mathcal{C})$. This completes the proof.
3. Since $lnt_{\mathcal{L}}(\mathcal{F})$ is $\mathcal{L}-$ open & $lnt_{\mathcal{L}}(\mathcal{F}) \subseteq lnt_{\mathcal{L}}(\mathcal{F})$. But $lnt_{\mathcal{L}}\big(lnt_{\mathcal{L}}(\mathcal{F})\big)$ largest $\mathcal{L}-$ open in $lnt_{\mathcal{L}}(\mathcal{F})$, then $lnt_{\mathcal{L}}(\mathcal{F}) \subseteq lnt_{\mathcal{L}}\big(lnt_{\mathcal{L}}(\mathcal{F})\big)$. Now, applying theorem (2.4) with $lnt_{\mathcal{L}}(\mathcal{F})$ replaced by $lnt_{\mathcal{L}}\big(lnt_{\mathcal{L}}(\mathcal{F})\big)$, we get $lnt_{\mathcal{L}}\big(lnt_{\mathcal{L}}(\mathcal{F})\big)$ largest $\mathcal{L}-$ open in $lnt_{\mathcal{L}}(\mathcal{F})$, this completes proof.
4. Direct.

Proposition 2.7: $lnt\,(\mathcal{F}) \subset lnt_{\mathcal{L}}(\mathcal{F})$, when ever $\mathcal{F} \subseteq \mathcal{X}$.

Proof: It is clear that, $lnt\,(\mathcal{F})$ is open set & $lnt\,(\mathcal{F}) \subseteq \mathcal{F}$. Since each open implies to$\mathcal{L}$ −open, so $lnt\,(\mathcal{F})$ $\mathcal{L}$ −open. But $lnt_{\mathcal{L}}(\mathcal{F})$ largest $\mathcal{L}$ −open contained in $\mathcal{F}$. Then we get $lnt\,(\mathcal{F}) \subset lnt_{\mathcal{L}}(\mathcal{F})$.

Remark 2.8: If $\mathcal{F} \subseteq \mathcal{X}$, then it is not necessarily that $lnt\,(\mathcal{F}) = lnt_{\mathcal{L}}(\mathcal{F})$.

Example 2.9: If $\mathcal{X} = \{u, z, w, e\}$ & $\mathfrak{F} = \{\Phi, \{u, z, w\}, \mathcal{X}\}$. Then

$$\mathcal{LO}_{\mathcal{X}} = \begin{Bmatrix} \Phi, \mathcal{X}, \{u\}, \{z\}, \{w\}, \{u, z\}, \\ \{u, w\}, \{z, w\}, \{u, z, w\} \end{Bmatrix}.$$

Now, $lnt_{\mathcal{L}}(\{z, w\}) = \{z, w\}$ and $lnt\,(\{z, w\}) = \Phi$. Then

$lnt\,(\{z, w\}) \subset lnt_{\mathcal{L}}(\{z, w\})$. So, $lnt_{\mathcal{L}}(\{z, w\}) \neq lnt\,(\{z, w\})$.

This example verifies an equality in Remark (2.8) is not hold.

Remark 2.10: In the following proposition we consider the necessary and sufficient condition for equality to be true in Remark (2.8).

Proposition 2.11: If a subset $\mathcal{F}$ of $\mathcal{X}$ is open set, then $lnt\ (\mathcal{F}) = lnt_{\mathcal{L}}(\mathcal{F})$.

Proof: Assume that $\mathcal{F}$ is open set, then $\mathcal{F} = lnt\ (\mathcal{F})$.

Since every open is $\mathcal{L}-$open set, then $\mathcal{F}$ is $\mathcal{L}-$open set, Consequently, $\mathcal{F} = lnt_{\mathcal{L}}(\mathcal{F})$. Hence proved.

Definition 2.12: If $(\mathcal{X}, \mathcal{LO}_{\mathcal{X}}, \mathcal{I}_{\mathcal{X}})$ is an ideal $\mathcal{L}-$space & $\mathcal{F} \subseteq \mathcal{X}$, then $\mathcal{F}$ is called $\mathcal{I}_{\mathcal{X}} - \mathcal{L}-$open set if $\mathcal{F} \subseteq lnt_{\mathcal{L}}(\mathcal{F}^*_{\mathcal{L}})$.

We shall denote the collection of $\mathcal{I}_{\mathcal{X}} - \mathcal{L}-$open set by $\mathcal{I}_{\mathcal{X}}\mathcal{LO}_{\mathcal{X}}$.

Remark 2.13: Assume that $(\mathcal{X}, \mathcal{LO}_{\mathcal{X}}, \mathcal{I}_{\mathcal{X}})$ is an ideal $\mathcal{L}-$space, then Φ is $\mathcal{I}_{\mathcal{X}} - \mathcal{L}-$open.

Proof: Since $\Phi^*_{\mathcal{L}} = \Phi$, then $lnt_{\mathcal{L}}(\Phi^*_{\mathcal{L}}) = lnt_{\mathcal{L}}(\Phi) = \Phi$. Hence proved.

Remark 2.14:

1. In general, $\mathcal{X}$ is not $\mathcal{I}_{\mathcal{X}} - \mathcal{L}-$open.
2. The relation between open set & $\mathcal{I}_{\mathcal{X}} - \mathcal{L}-$open is independent.
3. In general, $\mathcal{A}^*_{\mathcal{L}}(\mathcal{I}_{\mathcal{X}}, \mathcal{LO}_{\mathcal{X}}) \nsubseteq \mathcal{A}$ & $\mathcal{A} \nsubseteq \mathcal{A}^*_{\mathcal{L}}(\mathcal{I}_{\mathcal{X}}, \mathcal{LO}_{\mathcal{X}})$.

Example 2.15: If $\mathcal{X} = \{u, z, w\}$ & $\mathfrak{F} = \{\Phi, \{u, z\}, \mathcal{X}\}$ with $\mathcal{I}_{\mathcal{X}} = \{\Phi, \{u\}\}$. Then $\mathcal{LO}_{\mathcal{X}} = \{\Phi, \{u\}, \{z\}, \{u, z\}, \mathcal{X}\}$. So, we have:

For, $\Phi \Rightarrow \Phi^*_{\mathcal{L}} = \Phi$, for $\{u\} \Rightarrow \{u\}^*_{\mathcal{L}} = \Phi$, for $\{z\} \Rightarrow \{z\}^*_{\mathcal{L}} = \{z, w\}$, for $\{w\} \Rightarrow \{w\}^*_{\mathcal{L}} = \{w\}$, for $\{u, z\} \Rightarrow \{u, z\}^*_{\mathcal{L}} = \{z, w\}$, for $\{u, w\} \Rightarrow \{u, w\}^*_{\mathcal{L}} = \{w\}$, for $\{z, w\} \Rightarrow \{z, w\}^*_{\mathcal{L}} = \{z, w\}$ and for $\mathcal{X} \Rightarrow \mathcal{X}^*_{\mathcal{L}} = \{z, w\}$.

We note that $\{u, z\}^*_{\mathcal{L}} \nsubseteq \{u, z\}$ & $\{u, z\} \nsubseteq \{u, z\}^*_{\mathcal{L}}$.

Now, since $lnt_{\mathcal{L}}(\Phi^*_{\mathcal{L}}) = lnt_{\mathcal{L}}(\Phi) = \Phi$, then Φ is $\mathcal{I}_{\mathcal{X}} - \mathcal{L}-$open.

Since $lnt_{\mathcal{L}}(\{u\}^*_{\mathcal{L}}) = lnt_{\mathcal{L}}(\Phi) = \Phi$, then $\{u\} \nsubseteq lnt_{\mathcal{L}}(\{u\}^*_{\mathcal{L}})$, so $\{u\}$ is not $\mathcal{I}_{\mathcal{X}} - \mathcal{L}-$open.

Since $lnt_{\mathcal{L}}(\{z\}^*_{\mathcal{L}}) = lnt_{\mathcal{L}}(\{z, w\}) = \{z\}$, then $\{z\}$ is $\mathcal{I}_{\mathcal{X}} - \mathcal{L}-$open.

Since $lnt_{\mathcal{L}}(\{w\}^*_{\mathcal{L}}) = lnt_{\mathcal{L}}(\{w\}) = \Phi$, then $\{w\} \nsubseteq lnt_{\mathcal{L}}(\{u\}^*_{\mathcal{L}})$, so $\{w\}$ is not $\mathcal{I}_{\mathcal{X}} - \mathcal{L}-$open.

Since $lnt_{\mathcal{L}}(\{u, z\}^*_{\mathcal{L}}) = lnt_{\mathcal{L}}(\{z, w\}) = \{z\}$, then $\{u, z\} \nsubseteq lnt_{\mathcal{L}}(\{u, z\}^*_{\mathcal{L}})$, so $\{u, z\}$ is not $\mathcal{I}_{\mathcal{X}} - \mathcal{L}-$open.

Since $lnt_{\mathcal{L}}(\{u, w\}^*_{\mathcal{L}}) = lnt_{\mathcal{L}}(\{w\}) = \Phi$, then $\{u, w\} \nsubseteq lnt_{\mathcal{L}}(\{u, w\}^*_{\mathcal{L}})$, so $\{u, w\}$ is not $\mathcal{I}_{\mathcal{X}} - \mathcal{L}-$open.

Since $lnt_{\mathcal{L}}(\{z, w\}^*_{\mathcal{L}}) = lnt_{\mathcal{L}}(\{z, w\}) = \{z\}$, then $\{z, w\} \nsubseteq lnt_{\mathcal{L}}(\{z, w\}^*_{\mathcal{L}})$, so $\{z, w\}$ is not $\mathcal{I}_{\mathcal{X}} - \mathcal{L}-$open.

Since $lnt_{\mathcal{L}}(\mathcal{X}^*_{\mathcal{L}}) = lnt_{\mathcal{L}}(\{z, w\}) = \{z\}$, then $\mathcal{X} \nsubseteq lnt_{\mathcal{L}}(\mathcal{X}^*_{\mathcal{L}})$, so $\mathcal{X}$ is not $\mathcal{I}_{\mathcal{X}} - \mathcal{L}-$open.

Therefore, $\mathcal{I}_{\mathcal{X}}\mathcal{LO}_{\mathcal{X}} = \{\Phi, \{z\}\}$.

We note that, $\{u, z\}$ is an open but it is not $\mathcal{I}_{\mathcal{X}} - \mathcal{L}$ −open and $\{z\}$ is $\mathcal{I}_{\mathcal{X}} - \mathcal{L}$ −open but it is not open.

Proposition 2.16: Assume that $(\mathcal{X}, \mathcal{L}O_{\mathcal{X}}, \mathcal{I}_{\mathcal{X}})$ is an ideal $\mathcal{L}$ −space & $\mathcal{I}_{\mathcal{X}}$ is $\mathcal{L}$ −codense ideal, then $\mathcal{X}$ is $\mathcal{I}_{\mathcal{X}} - \mathcal{L}$ −open.

Proof: Since $\mathcal{I}_{\mathcal{X}}$ is $\mathcal{L}$ −codense ideal, then $\mathcal{X}_{\mathcal{L}}^* = \mathcal{X}$, hence $lnt_{\mathcal{L}}(\mathcal{X}_{\mathcal{L}}^*) = lnt_{\mathcal{L}}(\mathcal{X}) = \mathcal{X}$, thus $\mathcal{X}$ is $\mathcal{I}_{\mathcal{X}} - \mathcal{L}$ −open.

Proposition 2.17: If $(\mathcal{X}, \mathcal{L}O_{\mathcal{X}})$ is $\mathcal{L}$ −space & $\mathcal{I}_{\mathcal{X}}$ is $\mathcal{L}$ −codense ideal, then $\mathcal{X}$ is $\mathcal{I}_{\mathcal{X}} - \mathcal{L}$ −open.

Proof: Since $\mathcal{I}_{\mathcal{X}}$ is $\mathcal{L}$ −codense ideal, then $\mathcal{X}_{\mathcal{L}}^* = \mathcal{X}$, hence $lnt_{\mathcal{L}}(\mathcal{X}_{\mathcal{L}}^*) = lnt_{\mathcal{L}}(\mathcal{X}) = \mathcal{X}$, thus $\mathcal{X}$ is $\mathcal{I}_{\mathcal{X}} - \mathcal{L}$ −open.

Proposition 2.18: If $\mathcal{F}$ $\mathcal{I}_{\mathcal{X}} - \mathcal{L}$ −open & $\mathcal{F}_{\mathcal{L}}^*$ $\mathcal{L}$ −open, so $\mathcal{F} \subseteq \mathcal{F}_{\mathcal{L}}^*$.

Proof: Suppose $\mathcal{F}_{\mathcal{L}}^*$ is $\mathcal{L}$ −open, then $lnt_{\mathcal{L}}(\mathcal{F}_{\mathcal{L}}^*) = \mathcal{F}_{\mathcal{L}}^*$. Since $\mathcal{F}$ is $\mathcal{I}_{\mathcal{X}} - \mathcal{L}$ −open set, then $\mathcal{F} \subseteq lnt_{\mathcal{L}}(\mathcal{F}_{\mathcal{L}}^*)$. So, $\mathcal{F} \subseteq \mathcal{F}_{\mathcal{L}}^*$.

Proposition 2.19: If $\mathcal{F}$ is $\mathcal{I}_{\mathcal{X}} - \mathcal{L}$ −open and it is $\mathcal{L}$ −closed such that $\mathcal{F}_{\mathcal{L}}^*$ is $\mathcal{L}$ −open, then $\mathcal{F} = \mathcal{F}_{\mathcal{L}}^*$.

Proof: The proof is hold by Proposition (2.18).

Proposition 2.20: If we have $(\mathcal{X}, \mathfrak{F})$ is topological space & $(\mathcal{X}, \mathcal{L}O_{\mathcal{X}}, \mathcal{I}_{\mathcal{X}})$ is an ideal $\mathcal{L}$ −space with $\mathcal{I}_{\mathcal{X}}$ is $\mathcal{L}$ −codense, then every open set is $\mathcal{I}_{\mathcal{X}} - \mathcal{L}$ −open.

Proof: Suppose $\mathcal{I}_{\mathcal{X}}$ is $\mathcal{L}$ −codense and $\mathcal{F}$ is open, then we get $\mathcal{F}$ is $\mathcal{L}$ −open, hence we have $\mathcal{F}_{\mathcal{L}}^* = cl_{\mathcal{L}}(\mathcal{F})$, but $\mathcal{F} \subseteq cl_{\mathcal{L}}(\mathcal{F})$, then we get $\mathcal{F} \subseteq \mathcal{F}_{\mathcal{L}}^*$.

We have $lnt\,(\mathcal{F}) = lnt_{\mathcal{L}}(\mathcal{F})$. But $\mathcal{F}$ is open, then $lnt\,(\mathcal{F}) = \mathcal{F}$, hence $\mathcal{F} = lnt_{\mathcal{L}}(\mathcal{F})$. Now, we have $\mathcal{F} \subseteq \mathcal{F}_{\mathcal{L}}^*$, then $lnt_{\mathcal{L}}(\mathcal{F}) \subseteq lnt_{\mathcal{L}}(\mathcal{F}_{\mathcal{L}}^*)$, hence $\mathcal{F} \subseteq lnt_{\mathcal{L}}(\mathcal{F}_{\mathcal{L}}^*)$ since $lnt_{\mathcal{L}}(\mathcal{F}) = \mathcal{F}$. Therefore, $\mathcal{F}$ is $\mathcal{I}_{\mathcal{X}} - \mathcal{L}$ −open set.

Conclusion

The results are studied in this work:

1. $lnt_{\mathcal{L}}(\mathcal{F}) = \cup_{i\in \mathrm{I}} \{\mathcal{H}_i : \mathcal{H}_i \text{ is } \mathcal{L} - open\,,\ \mathcal{H}_i \subseteq \mathcal{F}\ \forall i \in \mathrm{I}\}$.
2. Assume $(\mathcal{X}, \mathcal{L}O_{\mathcal{X}})$ is $\mathcal{L} - space$ & $\mathcal{F} \subseteq \mathcal{X}$,then $\mathcal{F}$ is $\mathcal{L} -$ open iff $lnt_{\mathcal{L}}(\mathcal{F}) = \mathcal{F}$.
3. $lnt\,(\mathcal{F}) \subset lnt_{\mathcal{L}}(\mathcal{F})$, for any subset $\mathcal{F}$ of $\mathcal{X}$.
4. $\mathcal{X}$ is $\mathcal{I}_{\mathcal{X}} - \mathcal{L}$ −open.
5. Assume that $(\mathcal{X}, \mathcal{L}O_{\mathcal{X}}, \mathcal{I}_{\mathcal{X}})$ is an ideal $\mathcal{L}$ −space & $\mathcal{I}_{\mathcal{X}}$ is $\mathcal{L}$ −codense ideal, then $\mathcal{X}$ is $\mathcal{I}_{\mathcal{X}} - \mathcal{L}$ −open.
6. If $(\mathcal{X}, \mathcal{L}O_{\mathcal{X}})$ is $\mathcal{L}$ −space & $\mathcal{I}_{\mathcal{X}}$ is $\mathcal{L}$ −codense ideal, then $\mathcal{X}$ is $\mathcal{I}_{\mathcal{X}} -$ $\mathcal{L}$ −open.
7. If $\mathcal{F}$ is $\mathcal{I}_{\mathcal{X}} - \mathcal{L}$ −open and it is $\mathcal{L}$ −closed such that $\mathcal{F}_{\mathcal{L}}^*$ is $\mathcal{L}$ −open, then $\mathcal{F} = \mathcal{F}_{\mathcal{L}}^*$.

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