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ʃ-focal nested set

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Abstract

In this work, we derived two new mathematical concepts in ʃ-topological spaces with significant effects on topological concepts, which are (ʃ-focal nested set, ʃ-focal ingrained set), and were constructed based on the grill and its properties. The most important properties of both were highlighted by providing various examples and an intensive study of their shared and interrelated properties.

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1. Introduction

The topological mathematical concepts of various types (interior, exterior, bounded, closure,) appear in multiple forms depending on the topological space or the shape of the space. Many scientists and researchers have studied these properties in various spaces such as metric spaces $(X, d)$and prximity spaces (X, δ), which is a bold generalization of metric spaces and is considered a qualitative leap in scientific research, especially in its applied field, as defined by Riesz in 1909 [1]. These concepts have also been studied concerning i −topological and ʃ −topological space particularly in relation to adherent points, which were termed by the local function defined by Kuratowski in 1933[2]. Then, the researcher Yazi studied it concerning the ideal proximity space, and specific mathematical concepts emerged, which he referred to (focal and nested) [3]. The researcher Ali studied the concepts occlusion set via

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$i - open$ set [4], while the researcher Raghad moved on to study local functions in spaces proximity, which introduced new concepts as well as follower set, takeoff set, sporadic family [5-8]. From here, the work became parallel and coordinated to extract and derive concepts, and this research specialized in the concept of focal sets.

2. Basic definitions and concepts

Definition 2.1: [6] Let ʆ is a family of a non-empty set in topological space is called a grill if satisfies two conditions: 1. Ş ∈ ʆ and Ş ⊆ Ŗ ⊆ K ⇒ Ŗ ∈ ʆ 2. if Ş ∪ Ŗ ∈ ʆ for Ş , Ŗ ⊆ K, then Ş ∈ ʆ or Ŗ ∈ ʆ.

Definition 2.2: Let ʆ be a grill define on a nonempty set K , the binary relation ≊ defined on K as follow: Ş ≊ Ŗ if and only if(Ş − Ŗ) ∪ (Ŗ − Ş) ∉ ʆ .

Example 2.3: Ꝁ = {ş, ŗ, ç} and ʆ = {Ꝁ, {ŗ}, {ş, ŗ}, {ŗ, ç}}, Ş = {ş}, Ŗ = {ş, ŗ}
Ş ≊ Ŗ because (Ş − Ŗ) ∪ (Ŗ − Ş) = ({ş} − {ş, ŗ}) ∪ ({ş, ŗ} − {ş}) = {ŗ} ∉ ʆ.

Definition 2.4: Let ʆ be a grill, a ʆ −topological space on K is a family $Ţ_ʆ$ of a subset of K that satisfies the conditions:

1. K, ∅ ∈ $Ţ_ʆ$.
2. Every Ų ⊆ $Ţ_ʆ$, there exist Ȩ ∈ $Ţ_ʆ$ such that ⋃ Ų ≊ Ȩ.
3. Every Ų, Ȩ ∈ $Ţ_ʆ$, there exist Ḩ ∈ $Ţ_ʆ$ $such\ that$ Ų ∩ Ȩ ≊ Ḩ.
4. $Ţ_ʆ$ ∩ (Ƥ(K)∖ ʆ) = {∅}.

Definitions 2.5: Let $K_ʆ^{Ţ_ʆ}$ be ʆ − topological space. Then, $K_ʆ^{Ţ_ʆ}$ is called advanced ʆ − topological space if it is closed under the intersection.

Definition 2.6: [7] A binary relation δ defined on the power set of K is called proximity on K if and only if it satisfies the following axioms:

1. ŞδŖ implies ŖδŞ
2. Ş ∪ ŖδȨ if and only if ŞδȨ or ŖδȨ
3. ŞδŖ implies $A \neq \emptyset$ and Ŗ $\neq \emptyset$
4. Ş ∩ Ŗ $\neq \emptyset$ implies ŞδŖ
5. Ş$\bar{\delta}$Ŗ implies there exists a subset Ų of K such that Ş$\bar{\delta}$Ų and K − Ų$\bar{\delta}$Ŗ .

   The pair (K , δ) is called a proximity space.

Proposition 2.7: [7] Let (K , δ) be a proximity space then:

1. If ŖδȨ , then ŞδȨ where Ŗ ⊆ Ş.
2. If ŖδȨ , then ŖδḨ where Ȩ ⊆ Ḩ.
3. If Ŗ$\bar{\delta}$Ȩ , then Ŗ$\bar{\delta}$Ḩ where Ḩ ⊆ Ȩ.

Definition 2.8: The quadruple $\left(K_{ʆ}^{Ţ_ʆ}, \delta\right)$ is called ʆ −topological proximity space where $K_{ʆ}^{Ţ_ʆ}$ is a ʆ −topological space and (K, δ) is a proximity space.

Definition 2.9: Let $K_{ʆ}^{Ţ_ʆ}$ be a ʆ $-$ *topological* space, then a subset Ş is named a ʆ −focal set of a point $k \in K$, if there exist $Ų \in Ţ_ʆ(k)$ such that $Ų \approxeq Ş$. The system of all ʆ −focal sets of point k is denoted by $ʆ_{\not{s}}(k) = \{Ş \subseteq K : \exists Ų \in Ţ_ʆ(k), Ų \approxeq Ş\}$.

Example 2.10: Let $K = \{ş, ŗ, ç\}$, $Ţ_ʆ = \{\emptyset, K, \{ş, ŗ\}, \{ş, ç\}\}$ and $ʆ = \{K, \{ş\}, \{ş, ŗ\}, \{ş, ç\}\}$, $Ţ_ʆ(ş) = \{K, \{ş, ŗ\}, \{ş, ç\}\}$, $Ţ_ʆ(ŗ) = \{K, \{ş, ŗ\}\}$, $Ţ_ʆ(ç) = \{K, \{ş, ç\}\}$, $Ƥ(K) = \{\emptyset, \{ş\}, \{ŗ\}, \{ç\}, \{ş, ŗ\}, \{ŗ, ç\}, \{ş, ç\}\}$, $ʆ_{\not{s}}(ş) = ʆ_{\not{s}}(ŗ) = ʆ_{\not{s}}(ç) = \{\{ş\}, \{ş, ŗ\}, \{ş, ç\}, K\}$.

Theorem 2.11: *Let* $X_{ʆ}^{Ţ_ʆ}$ *be a* ʆ $-$ *topological space and* $A, B \subseteq X$, *then each of the following properties are holds:*

1. *For every* $A \in Ţ_ʆ(x)$, *then* $A \in ʆ_{\not{s}}(x)$ *and* $\emptyset \notin ʆ_{\not{s}}(x)$.
2. *If* $A \in ʆ_{\not{s}}(x)$, *then* $X - A \notin ʆ_{\not{s}}(x)$.

Proposition 2.12: Let $X_{ʆ_i}^{Ţ_{iʆ_i}}, i = 1,2$ be a ʆ −topological spaces such that $Ţ_{2_{ʆ_2}}$ is finer than $Ţ_{1_{ʆ_1}}$, $ʆ_2$ is finer than $ʆ_1$, then:

1. $ʆ_{\not{s}Ţ_{2_{ʆ_2}}}(x) \subseteq ʆ_{\not{s}Ţ_{1_{ʆ_1}}}(x)$.
2. $ʆ_{2\not{s}Ţ_{2_{ʆ_2}}}(x) \subseteq ʆ_{1\not{s}Ţ_{1_{ʆ_1}}}(x)$.

Definition 2.13: Let $(K_{ʆ}^{Ţ_ʆ}, \delta)$ be a ʆ −topological proximity space then a point k of K is called a ʆ −nested point of a subset Ş of K if there exist a proper subset Ų of K such that $Ų \in Ţ_ʆ(k)$ and $Ų\ \delta\ Ş$.

The set of all ʆ −nested points of K is called a ʆ −nested set denoted by $Ņ_{Ţ_ʆ}(Ş) = \{k \in K : \exists\, K \neq Ų \in Ţ_ʆ(k), Ų \delta\, Ş\}$.

By example 2.10, δ is proximity define by: $Ş\ \delta\ Ŗ\ \forall\, Ş \neq \emptyset, Ŗ \neq \emptyset$ except the relation : $\{ş\}\bar{\delta}\{ŗ, ç\}$, $\{ŗ, ç\}\bar{\delta}\{ş\}$, $\{ş\}\bar{\delta}\{ŗ\}$, $\{ŗ\}\bar{\delta}\{ş\}$, $\{ş\}\bar{\delta}\{ç\}$, $\{ç\}\bar{\delta}\{ş\}$, $Ş = \{ş\}$, $Ņ_{Ţ_ʆ}(Ş) = \{\{ş, ŗ\}, \{ş, ç\}\}$.

Proposition 2.14: Let $(K_{ʆ}^{Ţ_ʆ}, \delta)$ be a ʆ $-$ *topological* proximity space and let $Ş, Ŗ \subseteq K$, then each of the following properties are holds :

1. $Ş \subseteq Ŗ$, *then* $Ņ_{Ţ_ʆ}(Ş) \subseteq Ņ_{Ţ_ʆ}(Ŗ)$
2. $Ņ_{Ţ_ʆ}(Ş \cap Ŗ) \subseteq Ņ_{Ţ_ʆ}(Ş) \cap Ņ_{Ţ_ʆ}(Ŗ)$
3. $Ņ_{Ţ_ʆ}(Ş) \cup Ņ_{Ţ_ʆ}(Ŗ) = Ņ_{Ţ_ʆ}(Ş \cup Ŗ)$

4. If $Ş \in Ţ_{ʆ}$, then $Ş \subseteq Ņ_{Ţ_{ʆ}}(Ş)$.
5. $Ņ_{Ţ_{ʆ}}(Ş) \subseteq Ş$.

Definition 2.15: Let $(K_{ʆ}^{Ţ_{ʆ}}, \delta)$ be a $ʆ - topological$ proximity space, then $k\ in\ K$ is called an ingrained point of a subset $Ų\ of\ k$ such that $Ų \in Ţ_{ʆ}(k)$ and $Ų\ \bar{\delta}\ K \backslash Ş$.

The set of all ingrained points of K is called an ingrained set denoted by $Ç_{Ţ_{ʆ}}(Ş) = \{k \in K: \forall\, K \neq Ų \in Ţ_{ʆ}(k), Ų \bar{\delta}\, K \backslash Ş\}$.

By example 2.10, δ is a proximity define by: $Ş\ \delta\ Ŗ\ \forall\ Ş \neq \emptyset, Ŗ \neq \emptyset$ except the relation : $\{ş\}\bar{\delta}\{ŗ, ç\}$,$\{ŗ, ç\}\bar{\delta}\{ş\}$, $\{ş\}\bar{\delta}\{ŗ\}$,$\{ŗ\}\bar{\delta}\{ş\}$,$\{ş\}\bar{\delta}\{ç\}$,$\{ç\}\bar{\delta}\{ş\}$, $Ş = \{ş\}, K \backslash Ş = \{ŗ, ç\}$ $Ç_{Ţ_{ʆ}}(Ş) = \emptyset$.

3. The $ʆ$ −focal nested set

Definition 3.1: Let $(K_{ʆ}^{Ţ_{ʆ}}, \delta)$ be a $ʆ - topological$ proximity space then a point $k \in K$ is called a $ʆ$ −focal nested point of $Ş \subseteq K$ if and only if there exist $Ų \in ʆ_{\mathcal{G}}(k)$ is a proper subset of $K, k \in Ų$ such that $Ų\ \delta\ Ş$. The set of all $ʆ$ −focal nested points of a set $Ş$ is called a $ʆ$ −focal nested set and denoted by $Ņ_{ʆ_{\mathcal{G}}}(Ş)$, i.e $Ņ_{ʆ_{\mathcal{G}}}(Ş) = \{k \in K:\ \exists\ Ų \in ʆ_{\mathcal{G}}(k), Ų\ \delta\ Ş\}$ where $ʆ_{\mathcal{G}}(k) = \{Ų \in ʆ_{\mathcal{G}}(k), k \in Ų\}$.

- $k \notin Ņ_{ʆ_{\mathcal{G}}}(Ş)$ if and only if $\forall K \neq Ų \in ʆ_{\mathcal{G}}(k)$, $Ų\ \bar{\delta}\ Ş$.

By example 2.10, δ is proximity define by: $Ş\ \delta\ Ŗ\ \forall\ Ş \neq \emptyset, Ŗ \neq \emptyset$ except the relation:$\{ş\}\bar{\delta}\{ŗ, ç\}$,$\{ŗ, ç\}\bar{\delta}\{ş\}$, $\{ş\}\bar{\delta}\{ŗ\}$,$\{ŗ\}\bar{\delta}\{ş\}$,$\{ş\}\bar{\delta}\{ç\}$,$\{ç\}\bar{\delta}\{ş\}$, $Ş = \{ş\}$, $Ņ_{ʆ_{\mathcal{G}}}(Ş) = \{\{ş\}, \{ş, ŗ\}, \{ş, ç\}\}$.

The relation between the two types of $ʆ$ −nested set is discussed below

Proposition 3.2: Let $(K_{ʆ}^{Ţ_{ʆ}}, \delta)$ be a $ʆ - topological$ proximity space then $Ņ_{Ţ_{ʆ}}(Ş) \subseteq Ņ_{ʆ_{\mathcal{G}}}(Ş)$for each subset $Ş$ of K.

Proof: Let $k \in Ņ_{Ţ_{ʆ}}(Ş)$ by definition 2.14 $\exists\ K \neq Ų \in Ţ_{ʆ}(k)$, such that $Ų \delta\ Ş$ and by theorem (2.12 part (1)) we have $Ų \in ʆ_{\mathcal{G}}(k)$, hence $k \in Ņ_{ʆ_{\mathcal{G}}}(Ş)$.

The converse of above proposition is not necessary true by example 2.10, δ is proximity define by: $Ş\ \delta\ Ŗ\ \forall\ Ş \neq \emptyset, Ŗ \neq \emptyset$ except the relation :$\{ş\}\bar{\delta}\{ŗ, ç\}$,$\{ŗ, ç\}\bar{\delta}\{ş\}$, $\{ş\}\bar{\delta}\{ŗ\}$,$\{ŗ\}\bar{\delta}\{ş\}$,$\{ş\}\bar{\delta}\{ç\}$,$\{ç\}\bar{\delta}\{ş\}$, $Ş = \{ş\}$, $Ņ_{Ţ_{ʆ}}(Ş) = \{\{ş, ŗ\}, \{ş, ç\}\}$,$Ņ_{ʆ_{\mathcal{G}}}(Ş) = \{\{ş\}, \{ş, ŗ\}, \{ş, ç\}\}$,$Ņ_{Ţ_{ʆ}}(Ş) \subseteq Ņ_{ʆ_{\mathcal{G}}}(Ş)$ but $Ņ_{ʆ_{\mathcal{G}}}(Ş) \not\subseteq Ņ_{Ţ_{ʆ}}(Ş)$.

Note 3.3: Let $\left(K_{ʆ}^{Ţ_{ʆ}}, \delta\right)$ be a $ʆ - topological$ proximity space and $Ş \subseteq K$, then for some $k \in Ş$, $Ņ_{ʆ_{\mathcal{G}}}(Ş) \subseteq \cup\{Ų \in ʆ_{\mathcal{G}}(k): Ų \subsetneq K, Ų\ \delta\ Ş\}$.

Proof: Let $k \in N_{\mathfrak{L}_{\mathfrak{s}}}(S)$ such that $k \in S$, so $\exists K \neq U \in \mathfrak{L}_{\mathfrak{s}}(k), U\ \delta\ S$ and by definition (2.6) $k \in U$, hence $k \in \cup\{U \in \mathfrak{L}_{\mathfrak{s}}(k) : U \subsetneq K, U\ \delta\ S\}$

The converse of above in bot necessary true for example: let $K = \{s, r, c\}$, $T_{\mathfrak{L}} = \{\emptyset, K, \{s\}, \{r\}\}$, $\mathfrak{L} = \{K, \{s\}, \{r\}, \{s, r\}, \{s, c\}, \{r, c\}\}$, $\mathfrak{L}_{\mathfrak{s}}(r) = \{\{r\}, \{s, r\}, \{r, c\}, K\}$, δ is proximity define by: $S\ \delta\ R\ \forall S \neq \emptyset, R \neq \emptyset$ except the relation :$\{r\}\bar{\delta}\{s, c\}$,$\{s, c\}\bar{\delta}\{r\}$, $\{s\}\bar{\delta}\{r\}$,$\{r\}\bar{\delta}\{s\}$,$\{r\}\bar{\delta}\{c\}$,$\{c\}\bar{\delta}\{r\}$, $S = \{r\}$, $N_{\mathfrak{L}_{\mathfrak{s}}}(S) = \{\{r\}, \{s, r\}, \{r, c\}\}$,$\{r\} \cup \{s, r\} \cup \{r, c\} = K \not\subseteq N_{\mathfrak{L}_{\mathfrak{s}}}(S)$.

Remark 3.4: Let $(K_{\mathfrak{L}}^{T_{\mathfrak{L}}}, \delta)$ be a $\mathfrak{L} - topological$ proximity space, then we have the following features of $\mathfrak{L}$ −focal nested set:

1. $N_{\mathfrak{L}_{\mathfrak{s}}}(S)$ is not necessary $\mathfrak{L}$ −clopen set in general, By above example, $\mathfrak{L}_{\mathfrak{s}}(s) = \{\{s\}, \{s, r\}, \{r, c\}, K\}$, $S = \{r\}$, $N_{\mathfrak{L}_{\mathfrak{s}}}(S) = \{s, r\} \notin T_{\mathfrak{L}}$ and not $\mathfrak{L}$ −closed set.
2. It is not necessary that $N_{\mathfrak{L}_{\mathfrak{s}}}(S) \subset S$ in general, for example: let $K = \{s, r, c\}$, $T_{\mathfrak{L}} = \{\emptyset, K, \{s\}, \{c\}, \{s, c\}\}$, $\mathfrak{L} = \{K, \{s\}, \{c\}, \{s, r\}, \{s, c\}, \{r, c\}\}$, $\mathfrak{L}_{\mathfrak{s}}(r) = \{\{s, c\}, K\}$, δ is proximity define by: $S\ \delta\ R\ \forall S \neq \emptyset, R \neq \emptyset$ except the relation :$\{s\}\bar{\delta}\{r, c\}$,$\{r, c\}\bar{\delta}\{s\}$, $\{s\}\bar{\delta}\{r\}$,$\{r\}\bar{\delta}\{s\}$,$\{s\}\bar{\delta}\{c\}$,$\{c\}\bar{\delta}\{s\}$, $S = \{s\}$, $N_{\mathfrak{L}_{\mathfrak{s}}}(S) = \{s, c\} \not\subseteq S$
3. If $S = N_{\mathfrak{L}_{\mathfrak{s}}}(S)$ is not necessary $S \in T_{\mathfrak{L}}$, By example in note 3.3 $\mathfrak{L}_{\mathfrak{s}}(c) = \{\{s, r\}, K\}$, $S = \{s, r\}$, $N_{\mathfrak{L}_{\mathfrak{s}}}(S) = \{s, r\} = S$ but $S \notin T_{\mathfrak{L}}$

Proposition 3.5: Let $(K_{\mathfrak{L}}^{T_{\mathfrak{L}}}, \delta)$ be a $\mathfrak{L} - topological$ proximity space such that $S, R \subseteq K$,then each of the following are holds:

1. If $S \subseteq R$ $then$ $N_{\mathfrak{L}_{\mathfrak{s}}}(S) \subseteq N_{\mathfrak{L}_{\mathfrak{s}}}(R)$.
2. $N_{\mathfrak{L}_{\mathfrak{s}}}(S \cap R) \subseteq N_{\mathfrak{L}_{\mathfrak{s}}}(S) \cap N_{\mathfrak{L}_{\mathfrak{s}}}(R)$.
3. $N_{\mathfrak{L}_{\mathfrak{s}}}(S \cup R) = N_{\mathfrak{L}_{\mathfrak{s}}}(S) \cup N_{\mathfrak{L}_{\mathfrak{s}}}(R)$.
4. $N_{\mathfrak{L}_{\mathfrak{s}}}(S) \subseteq N_{\mathfrak{L}_{\mathfrak{s}}}\left(N_{\mathfrak{L}_{\mathfrak{s}}}(S)\right)$.
5. $N_{\mathfrak{L}_{\mathfrak{s}}}(\emptyset) = \emptyset$
6. If $\emptyset \neq S \in T_{\mathfrak{L}}(k)$, then $S \subseteq N_{\mathfrak{L}_{\mathfrak{s}}}(S)$.

Proof:

1. Let $k \in N_{\mathfrak{L}_{\mathfrak{s}}}(S)$, so there exist $U \in \mathfrak{L}_{\mathfrak{s}}(k), U\ \delta\ S$ but $S \subseteq R$, then by proposition (2.7 part 2) $U\ \delta\ R$,hence $k \in N_{\mathfrak{L}_{\mathfrak{s}}}(R)$.

2. Since $Ș \cap Ŗ \subseteq Ș$, by (1) $Ņ_{ʆ_{ɕ}}(Ș \cap Ŗ) \subseteq Ņ_{ʆ_{ɕ}}(Ș)$ and $Ș \cap Ŗ \subseteq Ŗ$, by (1) $Ņ_{ʆ_{ɕ}}(Ș \cap Ŗ) \subseteq N_{ʆ_{ɕ}}(Ŗ)$then, $Ņ_{ʆ_{ɕ}}(Ș \cap Ŗ) \subseteq Ņ_{ʆ_{ɕ}}(Ș) \cap Ņ_{ʆ_{ɕ}}(Ŗ)$.
3. Since $Ș \subseteq Ș \cup Ŗ$, by (1) $Ņ_{ʆ_{ɕ}}(Ș) \subseteq Ņ_{ʆ_{ɕ}}(Ș \cup Ŗ)$ and $Ŗ \subseteq Ș \cup Ŗ$,by (1) $Ņ_{ʆ_{ɕ}}(Ŗ) \subseteq Ņ_{ʆ_{ɕ}}(Ș \cup Ŗ)$then, $Ņ_{ʆ_{ɕ}}(Ș) \cup Ņ_{ʆ_{ɕ}}(Ŗ) \subseteq Ņ_{ʆ_{ɕ}}(Ș \cup Ŗ)$. Conversely, let $ꝃ \in Ņ_{ʆ_{ɕ}}(Ș \cup Ŗ)$,then there exist $Ų \in ʆ_{ɕ}(ꝃ), ꝃ \in Ų$ such that $Ų\ \delta\ Ș \cup Ŗ$, hence by definition (2.6 part (2)) we have $Ų\ \delta\ Ș\ or\ Ų\ \delta\ Ŗ$ and from that we get $ꝃ \in Ņ_{ʆ_{ɕ}}(Ș) \cup Ņ_{ʆ_{ɕ}}(Ŗ)$.
4. Let $ꝃ \in Ņ_{ʆ_{ɕ}}(Ș)$, if possible that $ꝃ \notin Ņ_{ʆ_{ɕ}}\left(Ņ_{ʆ_{ɕ}}(Ș)\right)$, then for each $Ų \in ʆ_{ɕ}(Ș)$, $Ų\,\bar{\delta}\,Ņ_{ʆ_{ɕ}}(Ș)$ but $ꝃ \in Ų$, $ꝃ\,\bar{\delta}\,Ņ_{ʆ_{ɕ}}(Ș)$, $ꝃ \notin Ņ_{ʆ_{ɕ}}(Ș)$ which is a contradiction , thus $ꝃ \in Ņ(Ņ_{ʆ_{ɕ}}(Ș))$.
5. If not, $\exists\, ꝃ \in Ņ_{ʆ_{ɕ}}(\emptyset)$, then there exist $Ų \in ʆ_{ɕ}(ꝃ)$ such that $Ų\ \delta\ \emptyset$ this contradiction $Ņ_{ʆ_{ɕ}}(\emptyset) = \emptyset$
6. If $Ș = \emptyset$ & $Ņ_{ʆ_{ɕ}}(\emptyset) = \emptyset$ so $\emptyset = Ș \subseteq Ņ_{ʆ_{ɕ}}(\emptyset) = \emptyset$. If $Ș \neq \emptyset$, so $\exists\, ꝃ \in Ș$, but $Ș\ \delta\ Ș\ and\ Ș \in Ţ_{ʆ}(ꝃ)$, so $ꝃ \in Ņ_{ʆ_{ɕ}}(Ș)$.

The converse of (1) is not necessary true for example: let $Ꝃ = \{ș, ŗ, ç\}$, $Ţ_{ʆ} = \{\emptyset, Ꝃ, \{ș\}, \{ŗ\}\}$, $ʆ = \{Ꝃ, \{ș\}, \{ŗ\}, \{ș, ŗ\}, \{ș, ç\}, \{ŗ, ç\}\}$, $ʆ_{ɕ}(ŗ) = \{\{ŗ\}, \{ș, ŗ\}, \{ŗ, ç\}, Ꝃ\}$, δ is proximity define by: $Ș\ \delta\ Ŗ\ \forall\ Ș \neq \emptyset, Ŗ \neq \emptyset$ except the relation :$\{ç\}\bar{\delta}\,\{ș, ŗ\}$,$\{ș, ŗ\}\bar{\delta}\,\{ç\}$, $\{ç\}\bar{\delta}\,\{ŗ\}$,$\{ŗ\}\bar{\delta}\,\{ç\}$,$\{ș\}\bar{\delta}\,\{ç\}$,$\{ç\}\bar{\delta}\,\{ș\}$$Ș = \{ŗ\}$, $Ŗ = \{ç\}$, $Ņ_{ʆ_{ɕ}}(Ș) = \{\{ŗ\}, \{ș, ŗ\}, \{ș, ç\}$, $Ņ_{ʆ_{ɕ}}(Ŗ) = \{ș, ç\}$, $Ņ_{ʆ_{ɕ}}(Ŗ) \subseteq Ņ_{ʆ_{ɕ}}(Ș)\,but\ Ŗ \nsubseteq Ș$.

Also the converse of (2) is not necessary true because By example 2.10, δ is proximity define by: $Ș\ \delta\ Ŗ\ \forall\ Ș \neq \emptyset, Ŗ \neq \emptyset$ except the relation :$\{ŗ\}\bar{\delta}\,\{ș, ç\}$,$\{ș, ç\}\bar{\delta}\,\{ŗ\}$, $\{ș\}\bar{\delta}\,\{ŗ\}$,$\{ŗ\}\bar{\delta}\,\{ș\}$,$\{ŗ\}\bar{\delta}\,\{ç\}$,$\{ç\}\bar{\delta}\,\{ŗ\}$, $Ș = \{ș, ŗ\}$, $Ŗ = \{ŗ, ç\}$, $Ș \cap Ŗ = \{ŗ\}$, $Ņ_{ʆ_{ɕ}}(Ș) = Ņ_{ʆ_{ɕ}}(Ŗ) = \{\{ș\}, \{ș, ŗ\}, \{ș, ç\}\} = Ņ_{ʆ_{ɕ}}(Ș) \cap Ņ_{Ţ_{ʆ}}(Ŗ)$,$Ņ_{ʆ_{ɕ}}(Ș \cap Ŗ) = \{ș, ŗ\} \subseteq Ņ_{ʆ_{ɕ}}(Ș) \cap Ņ_{ʆ_{ɕ}}(Ŗ)$but $Ņ_{ʆ_{ɕ}}(Ș) \cap Ņ_{ʆ_{ɕ}}(Ŗ) \nsubseteq Ņ_{ʆ_{ɕ}}(Ș \cap Ŗ)$.

The converse of (4) is not necessary true because by above example, $Ș = \{ŗ\}$, $Ņ_{ʆ_{ɕ}}(Ș) = \{ș, ŗ\}$, $Ņ_{ʆ_{ɕ}}\left(Ņ_{ʆ_{ɕ}}(Ș)\right) = \{\{ș\}, \{ș, ŗ\}, \{ș, ç\}\}$ and $Ņ_{ʆ_{ɕ}}(Ņ_{ʆ_{ɕ}}(Ș)) \nsubseteq Ņ_{ʆ_{ɕ}}(Ș)$.

Corollary 3.6: *Let* $(Ꝃ_{ʆ}^{Ţ_{ʆ}}, \delta)$ *be advance*$ʆ - topological$ *proximity space, then* $Ņ_{ʆ_{ɕ}}(Ș \cap Ŗ) = Ņ_{ʆ_{ɕ}}(Ș) \cap Ņ_{ʆ_{ɕ}}(Ŗ)$.

Proposition 3.7: Let $\left(Ꝃ_{ʆ_i}^{Ţ_{ʆ_i}}, \delta\right)$ $i = 1,2$ be a $ʆ - topological$ proximity space, if $ʆ_2$ is finer than $ʆ_1$ then $Ņ_{ʆ_{1ɕ}}(Ș) \subseteq Ņ_{ʆ_{2ɕ}}(Ș)$ for each $Ș$ of $Ꝃ$.

Proof: Let $\Bbbk \in Ņ_{ʆ_{1∮}}(\text{Ş})$ so there exist $\text{Ų} \in ʆ_{1∮}(\Bbbk)$ such that $\text{Ų}\ \delta\ \text{Ş}$, by proposition (2.13) $\text{Ų} \in ʆ_{2∮}(\Bbbk)$ and $\text{Ų}\ \delta\ \text{Ş}$ so $\Bbbk \in Ņ_{ʆ_{2∮}}(\text{Ş})$.

Proposition 3.8: In $ʆ-$topological proximity space $(K_{ʆ}^{Ţ_ʆ}, \delta)$, $Ņ_{ʆ_∮}(\text{Ş}) \neq \emptyset$ for each $ʆ-$focal proper subset $\text{Ş}\ of\ K$.

Proof: Let Ş is a $ʆ-$focal proper subset $of\ K$. Then $\text{Ş} \in ʆ_∮(k)$, that is, $\text{Ş}\ \delta\ \text{Ş}$, thus $\text{Ş} \in Ņ_{ʆ_∮}(\text{Ş})$, hence $Ņ_{ʆ_∮}(\text{Ş}) \neq \emptyset$

Proposition 3.9: Let $(K_{ʆ}^{Ţ_ʆ}, \delta)$ be a $ʆ-topological$ proximity space and $\text{Ş} \in Ţ_ʆ(k)\ \forall\ k \in K$ then:

1. $\text{Ų} \cap Ņ_{Ţ_ʆ}(\text{Ş}) \subseteq Ņ_{Ţ_ʆ}(\text{Ş} \cap \text{Ų})$.
2. $\text{Ų} \cap Ņ_{Ţ_ʆ}(\text{Ş}) \subseteq Ņ_{ʆ_∮}(\text{Ş} \cap \text{Ų})$.

Proof:

1. Let $ŗ \in \text{Ų} \cap Ņ_{Ţ_ʆ}(\text{Ş}) \Rightarrow ŗ \in \text{Ų}\ and\ ŗ \in Ņ_{Ţ_ʆ}(\text{Ş}), \exists\ Ḩ \in Ţ_ʆ(ŗ),\ Ḩ\ \delta\ \text{Ş}$. By proposition 2.14 part 5, $Ņ_{Ţ_ʆ}(\text{Ş}) \subseteq \text{Ş}$, then $ŗ \in \text{Ş}$, that is, $ŗ \in (Ḩ \cap \text{Ų} \cap \text{Ş})$. By Definition 2.6 part 4, $Ḩ\delta\text{Ų} \cap \text{Ş}$ thus $ŗ \in Ņ_{Ţ_ʆ}(\text{Ş} \cap \text{Ų})$.
2. by (1) $\text{Ų} \cap Ņ_{Ţ_ʆ}(\text{Ş}) \subseteq Ņ_{Ţ_ʆ}(\text{Ş} \cap \text{Ų})$ and by proposition 3.3$Ņ_{Ţ_ʆ}(\text{Ş} \cap \text{Ų}) \subseteq Ņ_{ʆ_∮}(\text{Ş} \cap \text{Ų})$, $\text{Ų} \cap Ņ_{Ţ_ʆ}(\text{Ş}) \subseteq Ņ_{ʆ_∮}(\text{Ş} \cap \text{Ų})$

4. $ʆ-$Focal ingrained set

Definition 4.1: Let $(K_{ʆ}^{Ţ_ʆ}, \delta)$ be a $ʆ-topological$ proximity space then a point $k \in K$ is called a $ʆ-$focal ingrained point of $\text{Ş} \subseteq K$ if and only if for each $\text{Ų} \in ʆ_∮(k)$ is a proper subset of $K, k \in \text{Ų}$ such that $\text{Ų}\ \bar{\delta}\ K - \text{Ş}$. The set of all $ʆ-$focal ingrained points of a set Ş is called a $ʆ-$focal ingrained set and denoted by $Ç_{ʆ_∮}(\text{Ş})$, i.e $Ç_{ʆ_∮}(\text{Ş}) = \{k \in K :\ \exists\ \text{Ų} \in ʆ_∮(k),\ \text{Ų}\ \bar{\delta}\ K\backslash \text{Ş}\}$ where$ʆ_∮(k) = \{\text{Ų} \in ʆ_∮(k), k \in \text{Ų}\}$.

By example 2.11, δ is a proximity define by: $\text{Ş}\ \delta\ Ŗ\ \forall\ \text{Ş} \neq \emptyset, Ŗ \neq \emptyset$ except the relation: $\{ş\}\bar{\delta}\{ŗ, ç\}$,$\{ŗ, ç\}\bar{\delta}\{ş\}$, $\{ş\}\bar{\delta}\{ŗ\}$,$\{ŗ\}\bar{\delta}\{ş\}$,$\{ş\}\bar{\delta}\{ç\}$,$\{ç\}\bar{\delta}\{ş\}$, $\text{Ş} = \{ş\}$, $K\backslash\text{Ş} = \{ŗ, ç\}$, $Ç_{ʆ_∮}(\text{Ş}) = \{ş\}$.

Remark 4.2:

1. $\Bbbk\backslash Ņ_{ʆ_∮}(\Bbbk\backslash \text{Ş}) = Ç_{ʆ_∮}(\text{Ş})$
2. $Ņ_{ʆ_∮}(\Bbbk\backslash \text{Ş}) = \Bbbk\backslash Ç_{ʆ_∮}(\text{Ş})$.
3. $\Bbbk\backslash Ç_{ʆ_∮}(\Bbbk\backslash \text{Ş}) = Ņ_{ʆ_∮}(\text{Ş})$

Proof:

1. $ꝁ \in Ꝁ\backslash Ņ_{ʆ_{\wp}}(Ꝁ\backslash Ş)\ Ꝁ$ if and only if $ꝁ \notin Ņ_{ʆ_{\wp}}(Ꝁ\backslash Ş)$ if and only if $\forall\, Ꝁ \neq Ų \in ʆ_{\wp}(ꝁ)$ such that $Ų\bar{\delta}\, Ꝁ\backslash Ş$ if and only if $ꝁ \in Ç_{ʆ_{\wp}}(Ş)$.
2. $ꝁ \in Ņ_{Ţ_{ʆ}}(Ꝁ\backslash Ş)$ if and only if $ꝁ \notin Ꝁ\backslash Ņ_{ʆ_{\wp}}(Ꝁ\backslash Ş) = Ç_{ʆ_{\wp}}(Ş)$ by above 1 $ꝁ \notin Ç_{ʆ_{\wp}}(Ş)$ if and only if $ꝁ \in Ꝁ\backslash Ç_{ʆ_{\wp}}(Ş)$.
3. $ꝁ \in Ꝁ\backslash Ç_{ʆ_{\wp}}(Ꝁ\backslash Ş)$ if and only if $ꝁ \notin Ç_{ʆ_{\wp}}(Ꝁ\backslash Ş)$ if and only if $\exists\, Ꝁ \neq Ų \in ʆ_{\wp}(ꝁ)$ such that $Ų\ \delta\ Ꝁ(Ꝁ\backslash Ş) = Ş$ if and only if $ꝁ \in Ņ_{ʆ_{\wp}}(Ş)$.

Proposition 4.3: Let $(Ꝁ_{ʆ}^{Ţ_{ʆ}}, \delta)$ be a $ʆ - topological$ proximity space, then $Ç_{ʆ_{\wp}}(Ş) \subseteq Ç_{Ţ_{ʆ}}(Ş)$ for each subset $Ş\ of\ X$.

Proof: $ꝁ\ \in Ç_{ʆ_{\wp}}(Ş)$ if and only if $ꝁ\ \in Ꝁ\backslash Ņ_{ʆ_{\wp}}(Ꝁ\backslash Ş)$ if and only if $ꝁ \notin Ņ_{ʆ_{\wp}}(Ꝁ\backslash Ş)$ by proposition (3.3) $ꝁ\ \notin Ņ_{Ţ_{ʆ}}(Ꝁ\backslash Ş)$ then $ꝁ\ \in Ꝁ\backslash Ņ_{Ţ_{ʆ}}(Ꝁ\backslash Ş)$ then $ꝁ \in Ç_{Ţ_{ʆ}}(Ş)$.

The converse of above proposition is not necessary true by example, let $Ꝁ = \{ş, ɽ, ç\}$, $Ţ_{ʆ} = \{\emptyset, Ꝁ, \{ş\}, \{ç\}, \{ş, ç\}\}$, $ʆ = \{Ꝁ, \{ş\}, \{ç\}, \{ş, ɽ\}, \{ş, ç\}, \{ɽ, ç\}\}$,$ʆ_{\wp}(ş) = \{\{ş\}, \{ş, ɽ\}, \{ş, ç\}, Ꝁ\}$, δ is proximity define by: $Ş\ \delta\ Ŗ\ \forall\, Ş \neq \emptyset, Ŗ \neq \emptyset$ except the relation : $\{ɽ\}\bar{\delta}\{ş, ç\}, \{ş, ç\}\bar{\delta}\{ɽ\}, \{ş\}\bar{\delta}\{ɽ\}, \{ɽ\}\bar{\delta}\{ş\}, \{ɽ\}\bar{\delta}\{ç\}, \{ç\}\bar{\delta}\{ɽ\}$, $Ş = \{ş, ç\}$, $Ꝁ\backslash Ş = \{ɽ\}$, $Ç_{ʆ_{\wp}}(Ş) = \{\{ş\}, \{ş, ç\}\}$, $Ç_{Ţ_{ʆ}}(Ş) = \{\{ş\}, \{ç\}, \{ş, ç\}\}$, $Ç_{ʆ_{\wp}}(Ş) \subseteq Ç_{Ţ_{ʆ}}(Ş)$ but $Ç_{Ţ_{ʆ}}(Ş) \nsubseteq Ç_{ʆ_{\wp}}(Ş)$.

Note 4.4: Let $\left(Ꝁ_{ʆ}^{Ţ_{ʆ}}, \delta\right)$ be a $ʆ - topological$ proximity space, then $Ç_{ʆ_{\wp}}(Ş) \subseteq \cup \left\{Ų\ \in ʆ_{\wp}(ꝁ), Ų\ \bar{\delta}\, Ꝁ\backslash Ş, ꝁ \in\ Ş\right\}$.

Proof: Let $ꝁ\ \in Ç_{ʆ_{\wp}}(Ş)$,$\forall\, Ꝁ \neq Ų\ \in ʆ_{\wp}(ꝁ)$ such that $Ų\ \bar{\delta}\, Ꝁ\backslash Ş$, $Ų\ \cap Ꝁ\backslash Ş = \emptyset \Longrightarrow Ų\ \subseteq\ Ş$ but $ꝁ \in Ų$ $hence, ꝁ \in \cup \left\{Ų\ \in ʆ_{\wp}(ꝁ), Ų\ \bar{\delta}\, Ꝁ\backslash Ş, ꝁ \in\ Ş\right\}$.

Remark 4.5:

1. $Ç_{ʆ_{\wp}}(Ş)$ is not necessary $ʆ$ −clopen set in general. By example in definition 4.1 $Ç_{ʆ_{\wp}}(Ş) = \{ş\} \notin Ţ_{ʆ}$ and not $ʆ$ −closed set.
2. It is not necessary that $Ç_{ʆ_{\wp}}(Ş) \subseteq Ş$ in general. By example in proposition 4.3 $Ç_{ʆ_{\wp}}(Ş) \nsubseteq Ş$.
3. If $Ş = Ç_{ʆ_{\wp}}(Ş)$ is not necessary $Ş\ \in Ţ_{ʆ}$. For example, let $Ꝁ = \{ş, ɽ, ç\}$, $Ţ_{ʆ} = \{\emptyset, Ꝁ, \{ş\}, \{ɽ\}\}$, $ʆ = \{Ꝁ, \{ş\}, \{ɽ\}, \{ş, ɽ\}, \{ş, ç\}, \{ɽ, ç\}\}$,$ʆ_{\wp}(ç) = \{\{ş, ɽ\}, Ꝁ\}$, δ is proximity define by: $Ş\ \delta\ Ŗ\ \forall\, Ş \neq \emptyset, Ŗ \neq \emptyset$ except the relation :$\{ç\}\bar{\delta}\{ş, ɽ\}$,$\{ş, ɽ\}\bar{\delta}\{ç\}$, $\{ç\}\bar{\delta}\{ɽ\}$,$\{ɽ\}\bar{\delta}\{ç\}$,$\{ş\}\bar{\delta}\{ç\}$,$\{ç\}\bar{\delta}\{ş\}$, $Ş = \{ş, ɽ\}$, $Ꝁ\backslash Ş = \{ç\}$, $Ç_{ʆ_{\wp}}(Ş) = \{ş, ɽ\} = Ş$ but $Ş \notin Ţ_{ʆ}$.

Proposition 4.6: Let $(Ꝁ_{ʆ}^{Ţ_{ʆ}}, \delta)$ be a $ʆ-topological$ proximity space such that $Ş, Ŗ \subseteq Ꝁ$, then the following are holds:

1. If $Ş \subseteq Ŗ$ $then$ $Ç_{ʆ_∮}(Ş) \subseteq Ç_{ʆ_∮}(Ŗ)$.
2. $Ç_{ʆ_∮}(Ş \cap Ŗ) = Ç_{ʆ_∮}(Ş) \cap Ç_{ʆ_∮}(Ŗ)$.
3. If $(Ꝁ_{ʆ}^{Ţ_{ʆ}}, \delta)$ be advance, the$n$ $Ç_{ʆ_∮}(Ş \cup Ŗ) = Ç_{ʆ_∮}(Ş) \cup Ç_{ʆ_∮}(Ŗ)$.
4. $Ç_{ʆ_∮}\left(Ç_{ʆ_∮}(Ş)\right) \subseteq Ç_{ʆ_∮}(Ş)$.
5. $Ç_{ʆ_∮}(\emptyset) = \emptyset$
6. If $\emptyset \neq Ş \in Ţ_{ʆ}(ꝁ)$, then $Ç_{ʆ_∮}(Ꝁ \backslash Ş) \subseteq Ꝁ \backslash Ş$.

Proof:

1. $Ş \subseteq Ŗ$ if and only if $Ꝁ \backslash Ŗ \subseteq Ꝁ \backslash Ş$ by proposition(3.5 part 1) $Ņ_{ʆ_∮}(Ꝁ \backslash Ŗ) \subseteq Ņ_{ʆ_∮}(Ꝁ \backslash Ş)$ if and only if $Ꝁ \backslash Ņ_{ʆ_∮}(Ꝁ \backslash Ş) \subseteq Ꝁ \backslash Ņ_{ʆ_∮}(Ꝁ \backslash Ŗ)$ if and only if $Ç_{ʆ_∮}(Ş) \subseteq Ç_{ʆ_∮}(Ŗ)$.
2. $ꝁ \in Ç_{ʆ_∮}(Ş \cap Ŗ)$ if and only if $ꝁ \in Ꝁ \backslash Ņ_{ʆ_∮}(Ꝁ \backslash Ş \cap Ŗ)$ if and only if $ꝁ \notin Ņ_{ʆ_∮}(Ꝁ \backslash Ş \cap Ŗ)$ if and only if $ꝁ \notin Ņ_{ʆ_∮}(Ꝁ \backslash Ş \cup Ꝁ \backslash Ŗ)$, by proposition(3.5 part 3) $ꝁ \notin ((Ņ_{ʆ_∮}(Ꝁ \backslash Ş) \cup Ņ_{ʆ_∮}(Ꝁ \backslash Ŗ))$if and only if $ꝁ \notin Ņ_{ʆ_∮}(Ꝁ \backslash Ş)$ and $ꝁ \notin Ņ_{ʆ_∮}(Ꝁ \backslash Ŗ)$ if and only if $ꝁ \in Ꝁ \backslash Ņ_{ʆ_∮}(Ꝁ \backslash Ş)$ and $ꝁ \in Ꝁ \backslash Ņ_{ʆ_∮}(Ꝁ \backslash Ŗ)$ if and only if $ꝁ \in Ç_{ʆ_∮}(Ş)$ and $ꝁ \in Ç_{ʆ_∮}(Ŗ)$ if and only if $ꝁ \in (Ç_{ʆ_∮}(Ş) \cap Ç_{ʆ_∮}(Ŗ))$.
3. Since $Ş \subseteq Ş \cup Ŗ$ so by 1 $Ç_{ʆ_∮}(Ş) \subseteq Ç_{ʆ_∮}(Ş \cup Ŗ)$ and $Ŗ \subseteq Ş \cup Ŗ$ so by 1 $Ç_{ʆ_∮}(Ŗ) \subseteq Ç_{ʆ_∮}(Ş \cup Ŗ)$ we get that $Ç_{ʆ_∮}(Ş) \cup Ç_{ʆ_∮}(Ŗ) \subseteq Ç_{ʆ_∮}(Ş \cup Ŗ)$. Conversely, let $ꝁ \in Ç_{ʆ_∮}(Ş \cup Ŗ)$ if and only if $ꝁ \in Ꝁ \backslash Ņ_{ʆ_∮}(Ꝁ \backslash Ş \cup Ŗ)$ if and only if $ꝁ \notin Ņ_{ʆ_∮}(Ꝁ \backslash Ş \cup Ŗ)$ if and only if $ꝁ \notin Ņ_{ʆ_∮}(Ꝁ \backslash Ş \cap Ꝁ \backslash Ŗ)$, by corollary (3.6) $ꝁ \notin ((Ņ_{ʆ_∮}(Ꝁ \backslash Ş) \cap Ņ_{ʆ_∮}(Ꝁ \backslash Ŗ))$if and only if $ꝁ \notin Ņ_{ʆ_∮}(Ꝁ \backslash Ş)$ or $ꝁ \notin Ņ_{ʆ_∮}(Ꝁ \backslash Ŗ)$ if and only if $ꝁ \in Ꝁ \backslash Ņ_{ʆ_∮}(Ꝁ \backslash Ş)$ or $ꝁ \in Ꝁ \backslash Ņ_{ʆ_∮}(Ꝁ \backslash Ŗ)$ if and only if $ꝁ \in Ç_{ʆ_∮}(Ş)$ or $ꝁ \in Ç_{ʆ_∮}(Ŗ)$ if and only if $ꝁ \in (Ç_{ʆ_∮}(Ş) \cup Ç_{ʆ_∮}(Ŗ))$.
4. $ꝁ \in Ç_{ʆ_∮}(Ç_{ʆ_∮}(Ş))$ if and only if $ꝁ \in Ꝁ \backslash Ņ_{ʆ_∮}(Ꝁ \backslash Ç_{ʆ_∮}(Ş))$ if and only if $ꝁ \in Ꝁ \backslash Ņ_{ʆ_∮}(Ꝁ \backslash Ꝁ \backslash Ņ_{ʆ_∮}(Ꝁ \backslash Ş))$,if and only if $ꝁ \in Ꝁ \backslash Ņ_{ʆ_∮}(Ņ_{ʆ_∮}(Ꝁ \backslash Ş))$ $ꝁ \notin Ņ_{ʆ_∮}(Ņ_{ʆ_∮}(Ꝁ \backslash Ş))$, by proposition (3.5 part 4) then $ꝁ \notin Ņ_{ʆ_∮}(Ꝁ \backslash Ş), ꝁ \in Ꝁ \backslash Ņ_{ʆ_∮}(Ꝁ \backslash Ş) = Ç_{ʆ_∮}(Ş)$
5. If not, $\exists\, ꝁ \in Ç_{ʆ_∮}(\emptyset)$, then there exist $Ų \in ʆ_∮(ꝁ)$ such that $Ų\ \bar{\delta}\ Ꝁ \backslash \emptyset = Ꝁ$ this contradiction $Ç_{ʆ_∮}(\emptyset) = \emptyset$
6. $Ş \in Ţ_{ʆ}(ꝁ)$,by proposition (3.5 part 6), $Ş \subseteq Ņ_{ʆ_∮}(Ş)$, $Ş \subseteq Ꝁ \backslash Ç_{ʆ_∮}(Ꝁ \backslash Ş)$, $Ç_{ʆ_∮}(Ꝁ \backslash Ş) \subseteq Ꝁ \backslash Ş$.

Corollary 4.7: *If* $Ş$ *is* $\int$ − *closed then* $Ç_{\int_{\mathcal{G}}}(Ş) \subseteq Ş$.

Proof: $k \in Ş, k \notin K\backslash Ş$, $K\backslash Ş \in Ţ_{\int}(k)$,$K\backslash Ş \subseteq Ņ_{\int_{\mathcal{G}}}(K\backslash Ş)$by proposition (3.6 part6), $K\backslash Ņ_{\int_{\mathcal{G}}}(K\backslash Ş) \subseteq K \backslash (K\backslash Ş) = Ş$, $Ç_{\int_{\mathcal{G}}}(Ş) \subseteq Ş$.

Proposition 4.8: Let $\left(K_{\int_i}^{Ţ_{\int_i}}, \delta\right) i = 1,2$ be a $\int - topological$ proximity space, if $\int_2$ is finer than $\int_1$ then $Ç_{\int_{1\mathcal{G}}}(Ş) \subseteq Ç_{\int_{2\mathcal{G}}}(Ş)$ for each $Ş$ of K.

Proof: Let $k \in Ç_{\int_{1\mathcal{G}}}(Ş)$ if and only if $k \in K\backslash Ņ_{\int_{2\mathcal{G}}}(K\backslash Ş)$ if and only if $k \notin Ņ_{\int_{2\mathcal{G}}}(K\backslash Ş)$ by proposition (3.9) $k \notin Ņ_{\int_{1\mathcal{G}}}(K\backslash Ş)$, $k \in K\backslash Ņ_{\int_{1\mathcal{G}}}(K\backslash Ş) = Ç_{\int_{1\mathcal{G}}}(Ŗ)$.

Proposition 4.9: In $\int$ −topological proximity space $(K_{\int}^{Ţ_{\int}}, \delta)$, $Ç_{\int_{\mathcal{G}}}(Ş) = \emptyset$ for each $\int$ −focal proper subset $Ş$ of K.

Proof: If possible $Ç_{\int_{\mathcal{G}}}(Ş) \neq \emptyset$, then $\exists k \in Ç_{\int_{\mathcal{G}}}(Ş)$ and $\forall K \neq Ų \in \int_{\mathcal{G}}(k)$ such that $Ų\, \bar{\delta}\, K\backslash Ş$ and this means $Ų \cap K\backslash Ş = \emptyset$ by proposition (2.12 part 2) $K\backslash Ų \notin \int_{\mathcal{G}}(k)$ and $Ş \subset K\backslash Ų$, hence $Ş \notin \int_{\mathcal{G}}(k)$ which contradiction because $Ş$ is a $\int$ −focal set.

Proposition 4.10: Let $(K_{\int}^{Ţ_{\int}}, \delta)$ be a $\int - topological$ proximity space and $Ş \in Ţ_{\int}(k)$ then:

1. $Ų \cap K\backslash Ç_{Ţ_{\int}}(K\backslash Ş) \subseteq K\backslash Ç_{Ţ_{\int}}(K\backslash Ş \cap Ų)$.
2. $Ų \cap K\backslash Ç_{Ţ_{\int}}(K\backslash Ş) \subseteq K\backslash Ç_{\int_{\mathcal{G}}}(K\backslash Ş \cap Ų)$. $Ç_{Ţ_{\int}}(Ş) = \{k \in K: \forall K \neq Ų \in Ţ_{\int}(k), Ų\bar{\delta}\, K\backslash Ş\}$.

Proof:

1. Let $ŗ \in Ų \cap K\backslash Ç_{Ţ_{\int}}(K\backslash Ş) \Rightarrow ŗ \in Ų$ and $ŗ \in K\backslash Ç_{Ţ_{\int}}(K\backslash Ş)$, thus $ŗ \notin Ç_{Ţ_{\int}}(K\backslash Ş)$, $\exists K \neq Ų \in Ţ_{\int}(k)$ such that $Ų\, \delta K\backslash Ş$. By proposition 2.7 part 1, $Ų\, \delta(K\backslash Ş \cap Ų)$, that is, $ŗ \notin Ç_{Ţ_{\int}}(K\backslash Ş \cap Ų)$, thus $ŗ \in Ų\, K\backslash Ç_{Ţ_{\int}}(K\backslash Ş \cap Ų)$.

2. By part (1) $Ų \cap K\backslash Ç_{Ţ_{\int}}(K\backslash Ş) \subseteq K\backslash Ç_{Ţ_{\int}}(K\backslash Ş \cap Ų)$ and by proposition (3.3) $K\backslash Ç_{Ţ_{\int}}(K\backslash Ş \cap Ų) \subseteq K\backslash Ç_{\int_{\mathcal{G}}}(K\backslash Ş \cap Ų)$, $Ų \cap K\backslash Ç_{Ţ_{\int}}(K\backslash Ş) \subseteq K\backslash Ç_{\int_{\mathcal{G}}}(K\backslash Ş \cap Ų)$.

5. Conclusion

The researchers concluded in this study that it is possible to connect the $\int$ −Focal set concept with the $\int$ −Nested set concept based on proximity space, thus finding new concepts similar to the previous ones. The $\int$ −Focal Nested set contains the $\int$ −Nested set, and the $\int$ −Focal nested set does not necessarily have to be an ʆ −open or ʆ −closed set, as explained in Note 3.5 (1). Additionally, a complement can be found to define the ʆ −Focal Nestd set, which can be observed in Note 4.2.

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