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Fixed point theorem over orthogonal vector metric space and application for solve integral equation

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Abstract

This research introduces the concept of an orthogonal set and explores the idea of orthogonality within the setting of a vector metric space. In addition, we provide fixed point theorems for the proposed concept and provide illustrated examples to demonstrate that our conclusions effectively generalize existing findings in the literature. In addition, the primary results are utilized to establish fixed point solutions for orthogonal in vector metric space. In addition, a unique and specific solution was created for the orthogonal shape in the vector space that we mentioned earlier. Finally, a solution to a problem was discovered through practical application.

Subject Classification: *31A10, 33E30.*

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1. Introduction

The Banach contraction principle, a well-known theory among mathematicians, is of great importance in analysis and in the field of fixed-point theory. This fixed-point theorem is widely applied in fields and has been extended in several ways [10]. One common way to improve Banach contraction principle involves replacing a space that specific public space. Civic and Alton introduced the concept of vector space, which researchers have thoroughly explored and has led to significant results. [1, 2, 3, 7, 15]. Gordgi discusses the concepts of orthogonal sets and orthogonal metric spaces in [5]. This study aims to introduce the concept of orthogonality in sets, specifically in the context of vector metric space. It explores the idea of the equivalent of a vector metric space within orthogonal sets. Let Ĕ be a Riesz distance and F be a non-empty set. A map ∝̃:F×F→Ĕ named vector or Ĕ-metric over F where its satisfies: [3, 13]:

a) ∝̃(I€̃_(1),I€̃_(2))=0 if and only if I€̃_(1)=I€̃_(2) ;
b) ∝̃(I€̃_(1),I€̃_(2),)≥∝̃(I€̃_(1),I€̃_3)+∝̃(I€̃_(2),I€̃_3).

For each I€̃_(1),I€̃_(2),I€̃_3∈F. The triple (F,∝̃,Ĕ) is referred to as a vector metric. A generalization for a metric is vector metric. The following statements apply to all arbitrary elements I€̃(1), I€̃_(2),I€̃_3,I€̃_4 in a vector metric space.

i) ∝̃(I€̃_(1),I€̃_(2)) ≥ 0;
ii) ∝̃(I€̃_(1),I€̃_(2))=∝̃(I€̃_(2),I€̃_(1));
iii) |∝̃(I€̃_(1),I€̃_(3))-∝̃(I€̃_(2),I€̃_(3))|≥∝̃(I€̃_(1),I€̃_(2));
iv) |∝ ̃(I€̃_(1),I€̃_(3))-∝ ̃(I€̃_(2),I€̃_(4))|≥∝ ̃(I€̃_(1),I€̃_(2)) +∝̃(I€̃_(3),I€̃_(4)).

Civic and Alton introduced the concept of vector space. Many researchers have extensively explored this topic yielding results [2, 5, 12].

Furthermore, the Banach contraction principle has been extended in ways, such as incorporating metric spaces. These spaces are characterized by the definition of an orthogonal relationship with a given set and are shown to be complete when any orthogonal Cauchy sequence converges. This property makes it valuable for studying fixed points and contractions. [11]

The latest research seeks to add to existing studies in this field by delving into the properties and uses of vector spaces and their corresponding counterparts in groups. By undertaking this exploration, our aim is to provide perspectives on the concepts of fixed points and contractions, within these extended spaces and thus advance our understanding of mathematical frameworks.

Many researchers have discovered ways to improve on the Banach contraction principle by replacing area with metric space. Civic and Alton introduced the concept of vector space, which gained the attention of many scientists and led to remarkable discoveries. [3, 7, 9, 14-17] and [18], [19].

If $\breve{\mathrm{E}}$ is Riesz space & $\mathfrak{F}$ non-empty set. A map $\tilde{\propto}: \mathfrak{F} \times \mathfrak{F} \to \breve{\mathrm{E}}$ is called vector metric (or $\breve{\mathrm{E}}$-metric) on $\mathfrak{F}$ if:

a) $\tilde{\propto}\left(\tilde{\text{Ѥ}}_1, \tilde{\text{Ѥ}}_2\right) = 0$ iff $\tilde{\text{Ѥ}}_1 = \tilde{\text{Ѥ}}_2$;

b) $\tilde{\propto}\left(\tilde{\text{Ѥ}}_1, \tilde{\text{Ѥ}}_2,\right) \leq \tilde{\propto}\left(\tilde{\text{Ѥ}}_1, \tilde{\text{Ѥ}}_3\right) + \tilde{\propto}(\tilde{\text{Ѥ}}_2, \tilde{\text{Ѥ}}_3)$.

For each $\tilde{\text{Ѥ}}_1, \tilde{\text{Ѥ}}_2, \tilde{\text{Ѥ}}_3 \in \mathfrak{F}$. The triple $(\mathfrak{F}, \tilde{\propto}, \breve{\mathrm{E}})$ is referred to as vector metric space.

The following statements hold for all arbitrary elements $\tilde{\text{Ѥ}}_1, \tilde{\text{Ѥ}}_2, \tilde{\text{Ѥ}}_3, \tilde{\text{Ѥ}}_4$ in a vector metric space.

i) $\tilde{\propto}(\tilde{\text{Ѥ}}_1, \tilde{\text{Ѥ}}_2) \geq 0$;

ii) $\tilde{\propto}(\tilde{\text{Ѥ}}_1, \tilde{\text{Ѥ}}_2) = \tilde{\propto}(\tilde{\text{Ѥ}}_2, \tilde{\text{Ѥ}}_1)$;

iii) $\left|\tilde{\propto}\left(\tilde{\text{Ѥ}}_1, \tilde{\text{Ѥ}}_3\right) - \tilde{\propto}(\tilde{\text{Ѥ}}_2, \tilde{\text{Ѥ}}_3)\right| \leq \tilde{\propto}(\tilde{\text{Ѥ}}_1, \tilde{\text{Ѥ}}_2)$;

iv) $\left|\tilde{\propto}\left(\tilde{\text{Ѥ}}_1, \tilde{\text{Ѥ}}_3\right) - \tilde{\propto}\left(\tilde{\text{Ѥ}}_2, \tilde{\text{Ѥ}}_4\right)\right| \leq \tilde{\propto}\left(\tilde{\text{Ѥ}}_1, \tilde{\text{Ѥ}}_2\right) + \tilde{\propto}(\tilde{\text{Ѥ}}_3, \tilde{\text{Ѥ}}_4)$.

Definition 1.1: [7] A sequence $\{\tilde{\text{Ѥ}}_n\}$ in $(\mathfrak{F}, \tilde{\propto}, \breve{\mathrm{E}})$ is called vectorial converges ($\breve{\mathrm{E}}$-converges) to some $\tilde{\text{Ѥ}} \in \breve{\mathrm{E}}$, if there is a sequence $\{\tilde{\wp}_n\}$ in $\breve{\mathrm{E}}$ such that $\tilde{\wp}_n \downarrow 0$ and $\tilde{\propto}(\tilde{\text{Ѥ}}_n, \tilde{\text{Ѥ}}) \leq \tilde{\wp}_n$ for each n.

Definition 1.2: [7] A sequence $\{\tilde{\text{Ѥ}}_n\}$ in $(\mathfrak{F}, \tilde{\propto}, \breve{\mathrm{E}})$ is $\breve{\mathrm{E}}$-Cauchy where there is sequence $\{\tilde{\text{Ѥ}}_n\}$ in $\breve{\mathrm{E}}$ s.t $\{\tilde{\text{Ѥ}}_n\}\downarrow 0$ & $\tilde{\propto}(\tilde{\text{Ѥ}}_n, \tilde{\text{Ѥ}}_{n+p}) \leq \tilde{\wp}_n$ for every n & p.

Definition 1.3: [7] Each $\breve{\mathrm{E}}$-Cauchy sequence in $\mathfrak{F}$ is $\breve{\mathrm{E}}$-complete where its $\breve{\mathrm{E}}$-converge to limit in $\mathfrak{F}$.

Definition 1.4: [6] If $\mathfrak{F} \neq \emptyset$ & $\perp \subseteq \mathfrak{F} \times \mathfrak{F}$ is binary relation. If $\perp$ satisfies

If there is $\tilde{\text{Ѥ}}_1 \in \mathfrak{F}$ so that $(\forall \tilde{\text{Ѥ}}_2 \in \mathfrak{F}, \tilde{\text{Ѥ}}_1 \perp \tilde{\text{Ѥ}}_2)$ or $(\forall \tilde{\text{Ѥ}}_2 \in \mathfrak{F}, \tilde{\text{Ѥ}}_2 \perp \tilde{\text{Ѥ}}_1)$, it is known as an orthogonal set, or simply $\tilde{O}$-set. This is called $\tilde{O}$-set by $(\mathfrak{F}, \perp)$.

Definition 1.5: [8] Suppose that $(\mathfrak{F}, \perp)$ is an $\tilde{O}$-set. An sequence $\{\tilde{\text{Ѥ}}_n\}$ is orthogonal if $(\forall n \in N, \tilde{\text{Ѥ}}_n \perp \tilde{\text{Ѥ}}_{n+1})$ or ($(\forall n \in N, \tilde{\text{Ѥ}}_{n+1} \perp \tilde{\text{Ѥ}}_n)$.

Definition 1.6: [4] A map $\tilde{\propto}: \mathfrak{F} \to \mathfrak{F}$ orthogonal continuous in $\tilde{\text{Ѥ}} \in \mathfrak{F}$ if for $\tilde{O}$-sequence $\{\tilde{\text{Ѥ}}_n\}_{n\in\tilde{N}}$ in $\mathfrak{F}$ s.t $\tilde{\text{Ѥ}}_n \to \tilde{\text{Ѥ}}$ then $P(\tilde{\text{Ѥ}}_n) \to P(\tilde{\text{Ѥ}})$. Also, P is $\tilde{O}$-continuous on $\mathfrak{F}$ if its $\tilde{O}$-continuous at all $\tilde{\text{Ѥ}} \in \mathfrak{F}$.

Definition 1.7: [4] Let $(\mathfrak{F}, \perp)$ is an $\tilde{O}$-set. A mapping $\tilde{\propto}: \mathfrak{F} \to \mathfrak{F}$ is $\tilde{O}$-preserving if $P\tilde{\text{Ѥ}}_1 \perp P\tilde{\text{Ѥ}}_2$ where $\tilde{\text{Ѥ}}_1 \perp \tilde{\text{Ѥ}}_2$.

2. Main Results

This section contains the key outcomes as well as more definitions and properties

Definition 2.1: A map $\widetilde{\propto}: \mathfrak{F} \times \mathfrak{F} \to \breve{E}$ is called vector metric on an orthogonal set $(\mathfrak{F}, \perp)$, if:

i) $\widetilde{\propto}\left(\widetilde{E}_1, \widetilde{E}_2\right) = 0$ iff $\widetilde{E}_1 = \widetilde{E}_2$ for any points $\widetilde{E}_1, \widetilde{E}_2 \in \mathfrak{F}$ such that $\widetilde{E}_1 \perp \widetilde{E}_2$ and $\widetilde{E}_2 \perp \widetilde{E}_1$;

ii) $\widetilde{\propto}\left(\widetilde{E}_1, \widetilde{E}_2\right) \leq \widetilde{\propto}\left(\widetilde{E}_1, \widetilde{E}_2\right) + \widetilde{\propto}\left(\widetilde{E}_2, \widetilde{E}_3\right)$ for any point $\widetilde{E}_1, \widetilde{E}_2, \widetilde{E}_3 \in \mathfrak{F}$ such that $\widetilde{E}_1 \perp \widetilde{E}_2, \widetilde{E}_2 \perp \widetilde{E}_3$ and $\widetilde{E}_1 \perp \widetilde{E}_3$.

here orthogonal set $\mathfrak{F}$ denoted by $(\mathfrak{F}, \widetilde{\propto}, \perp)$ and named orthogonal vector metric space.

Definition 2.2: A mapping $P: \mathfrak{F} \to \mathfrak{F}$ in $(\mathfrak{F}, \widetilde{\propto}, \breve{E})$ is $\tilde{O}$-preserving if there is sequence $\{\widetilde{E}_n\}$ in $\breve{E}$ s.t $P\widetilde{E}_1 \neq P\widetilde{E}_2$ whenever $\widetilde{E}_1 \perp \widetilde{E}_2$ and $\widetilde{\propto}(\widetilde{E}_n, \widetilde{E}_{n+p}) \leq \widetilde{\wp}_n$ for every n and p.

Definition 2.2: A mapping $P: \mathfrak{F} \to \mathfrak{F}$ in orthogonal vector metric space $(\mathfrak{F}, \widetilde{\propto}, \breve{E})$ is called orthogonality continuous ($\tilde{O}$-continuous) in $\widetilde{E} \in \mathfrak{F}$ if for each $\tilde{O}$-sequence $\{\widetilde{E}_n\}_{n \in \tilde{N}}$ such that $\widetilde{E}_n \to \widetilde{E}$, then $P(\widetilde{E}_n) \to P(\widetilde{E})$ Also, P is $\tilde{O}$-continuous on $\mathfrak{F}$ if P is $\tilde{O}$-continuous in each $\widetilde{E} \in \mathfrak{F}$.

Theorem 2.3: *If $\mathfrak{F}$ is $\tilde{O}$-complete orthogonal vector metric space with $\breve{E}$ is Archimedean and $P: \mathfrak{F} \to \mathfrak{F}$ is an $\tilde{O}$-preserving and $\tilde{O}$-continuous mapping so that*

$$\widetilde{\propto}\left(P\widetilde{E}_1, P\widetilde{E}_2\right) \leq \tilde{z}\, \widetilde{\propto}\left(\widetilde{E}_1, \widetilde{E}_2\right) \tag{1}$$

for all $\widetilde{E}_1, \widetilde{E}_2 \in \mathfrak{F}$ with $\widetilde{E}_1 \perp \widetilde{E}_2$ wherever $\tilde{z} \in [0,1)$.

So $\widetilde{\propto}$ have a unique fixed point $\widetilde{E}^* \in \mathfrak{F}$ & $\widetilde{\propto}\left(\widetilde{E}_1^*, \widetilde{E}_2^*\right) = 0$.

Proof: According to the definition of orthogonally, there is $\widetilde{E}_1 \in \widetilde{\mathfrak{F}}$ such that for every $\widetilde{E}_2 \in \mathfrak{F}, \widetilde{E}_1 \perp \widetilde{E}_2$ or for every $\widetilde{E}_2 \in \mathfrak{F}, \widetilde{E}_2 \perp \widetilde{E}_1$.

It follows that $\widetilde{E}_1 \perp P\widetilde{E}_1$ or $P\widetilde{E}_1 \perp \widetilde{E}_1$. Let $\widetilde{E}_1 = P\widetilde{E}_0$, $\widetilde{E}_2 = P\widetilde{E}_1, \widetilde{E}_3 = P\widetilde{E}_2, \ldots, \widetilde{E}_{n+1} = P\widetilde{E}_n$, whenever $n \in \tilde{N}$, since P be a $\tilde{O}$-preserving and $\breve{E}$ Archimedean then $\{\widetilde{E}_n\}$ is an $\tilde{O}$-sequence .

Then, by (1), we get

$$\widetilde{\propto}(P\widetilde{E}_n, \widetilde{E}_{n+1}) = \widetilde{\propto}\left(P\widetilde{E}_{n-1}, P\widetilde{E}_n\right) \leq \tilde{z}^n\, \widetilde{\propto}\left(\widetilde{E}_0, \widetilde{E}_1\right).$$

For all n $\in \tilde{N}$. For each $m \geq 1$ and $\tilde{p} \geq 1$ it follows that

$$\widetilde{\propto}\left(\widetilde{₠}_{m+\tilde{p}},\widetilde{₠}_m\right)\leq\widetilde{\propto}\left(\widetilde{₠}_{m+\tilde{p}},\widetilde{₠}_{m+\tilde{p}-1}\right)+\widetilde{\propto}\left(\widetilde{₠}_{m+\tilde{p}-1},\widetilde{₠}_m\right)$$
$$=\widetilde{\propto}\left(\widetilde{₠}_{m+\tilde{p}},\widetilde{₠}_{m+\tilde{p}-1}\right)+\widetilde{\propto}\left(\widetilde{₠}_{m+\tilde{p}-1},\widetilde{₠}_m\right)$$
$$\leq\left(\widetilde{₠}_{m+\tilde{p}},\widetilde{₠}_{m+\tilde{p}-1}\right)+\widetilde{\propto}\left(\widetilde{₠}_{m+\tilde{p}-1},\widetilde{₠}_{m+\tilde{p}-2}\right)+\widetilde{\propto}\left(\widetilde{₠}_{m+\tilde{p}-2},\widetilde{₠}_{m+\tilde{p}-3}\right)+\cdots+$$
$$+\widetilde{\propto}\left(\widetilde{₠}_{m+2},\widetilde{₠}_{m+1}\right)+\widetilde{\propto}\left(\widetilde{₠}_{m+1},\widetilde{₠}_m\right)$$
$$\leq\ \tilde{z}^{m+\tilde{p}-1}\ \widetilde{\propto}\left(\widetilde{₠}_1,\widetilde{₠}_0\right)+\tilde{z}^{m+\tilde{p}-2}\ \widetilde{\propto}\left(\widetilde{₠}_1,\widetilde{₠}_0\right)$$
$$+\ \tilde{z}^{m+\tilde{p}-3}\ \widetilde{\propto}\left(\widetilde{₠}_1,\widetilde{₠}_0\right)+\cdots+$$
$$+\tilde{z}^{m+1}\ \widetilde{\propto}\left(\widetilde{₠}_1,\widetilde{₠}_0\right)+\tilde{z}^{m}\ \widetilde{\propto}\left(\widetilde{₠}_1,\widetilde{₠}_0\right)$$
$$\leq\left(\tilde{z}^{m+\tilde{p}-1}+\tilde{z}^{m+\tilde{p}-2}+\tilde{z}^{m+\tilde{p}-3}+\cdots+\tilde{z}^{m+1}+\tilde{z}^{m}\right)\widetilde{\propto}\left(\widetilde{₠}_1,\widetilde{₠}_0\right)$$
$$\leq\ \frac{\tilde{z}^m}{1-\tilde{z}}\ \widetilde{\propto}\left(\widetilde{₠}_1,\widetilde{₠}_0\right).$$

Using the limit as $m\to\infty$, we get $\lim_{m\to\tilde{\infty}}\widetilde{\propto}\left(\widetilde{₠}_{m+\tilde{p}},\widetilde{₠}_m\right)=0$

Now, since Ĕ Archimedean, so $\{\widetilde{₠}_n\}$ be a $\breve{E}$-Cauchy $\tilde{O}$-seq. and $\tilde{O}$-complete, there is $\widetilde{₠_1^*}\in\mathfrak{F}$. So that $\widetilde{₠}_n\to\widetilde{₠^*}\ as\ n\to\tilde{\infty}$. Since P is $\tilde{O}$-continuous we obtain

$$P\widetilde{₠^*}=\ P(lim_{n\to\tilde{\infty}}\ \widetilde{₠}_n)=lim_{n\to\tilde{\infty}}\ P\ \widetilde{₠}_n=lim_{n\to\tilde{\infty}}\ \widetilde{₠}_{n+1}=\widetilde{₠^*}$$

Hence $\widetilde{₠^*}$ fixed point for P.

Assume $\widetilde{₠_2^*}\in\mathfrak{F}$ fixed point for P. So we obtain $P^{\mathrm{n}}\widetilde{₠^*}=\widetilde{₠^*}$ and $P^{\mathrm{n}}\widetilde{₠_2^*}=\widetilde{₠_2^*}$ forl n $\in\tilde{N}$. According to the definition of orthogonally, there is $\widetilde{₠}_1\in\mathfrak{F}$ so that $\left[\widetilde{₠}_1\perp\widetilde{₠^*}\right.$ and $\left.\widetilde{₠}_1\perp\ \widetilde{₠_2^*}\right]$ or $[\widetilde{₠^*}\perp\widetilde{₠}_1$and $\widetilde{₠_2^*}\perp\widetilde{₠}_1]$

Since P is $\tilde{O}$-preserving, hence $[P^n\widetilde{₠}_1\perp P^n\widetilde{₠^*}\ and\ P^n\widetilde{₠}_1\perp P^n\widetilde{₠_2^*}]$ or $[P^n\widetilde{₠^*}\perp P^n\widetilde{₠}_1\ and\ P^n\widetilde{₠_2^*}\ \perp P^n\widetilde{₠}_1]$ for all n $\in\widetilde{N}$. therefore,

$$\widetilde{\propto}\left(\widetilde{₠_1^*},\widetilde{₠_2^*}\right)=\widetilde{\propto}\left(P^n\widetilde{₠_1^*},P^n\widetilde{₠_2^*}\right)=\widetilde{\propto}\left(P^n\widetilde{₠_1^*},P^n\widetilde{₠}_1\right)+\widetilde{\propto}\left(P^n\widetilde{₠}_1,P^n\widetilde{₠_2^*}\right)$$
$$\leq\ \tilde{z}^n\ \widetilde{\propto}\left(\widetilde{₠_1^*},\widetilde{₠}_1\right)+z^n\ \widetilde{\propto}\ (\widetilde{₠}_1,\widetilde{₠_2^*}).$$

Taking limit as n $\to\tilde{\infty}$, we obtain $\widetilde{\propto}\left(\widetilde{₠_1^*},\widetilde{₠_2^*}\right)=0$,so $\widetilde{₠_1^*}=\widetilde{₠_2^*}$.

Theorem 2.4: *Assume that $(\mathfrak{F},\widetilde{\propto})$ is an $\tilde{O}$-complete orthogonal vector metric space and $P:\mathfrak{F}\to\mathfrak{F}$ is an $\tilde{O}$-preserving and $\tilde{O}$-continuous mapping such that*

$$\widetilde{\propto}\left(P\widetilde{₠}_1,P\widetilde{₠}_2\right)\leq\tilde{z}\,max\{\ \widetilde{\propto}\left(\widetilde{₠}_1,\widetilde{₠}_2\right),\widetilde{\propto}\left(\widetilde{₠}_1,P\widetilde{₠}_1\right),\widetilde{\propto}\left(\widetilde{₠}_2,P\widetilde{₠}_2\right)\qquad(2)$$

for all $\widetilde{₠}_1,\widetilde{₠}_2\in\mathfrak{F}$ wherever $0\leq\tilde{z}<1$.so P have unique fixed point $\widetilde{₠^*}\in\mathfrak{F}$ & $\widetilde{\propto}\left(\widetilde{₠^*},\widetilde{₠^*}\right)=0$

Proof: According to definition for orthogonally, thus there is $\widetilde{₠}_1\in\mathfrak{F}$ such that for every $\widetilde{₠}_2\in\ \mathfrak{F}$, $\widetilde{₠}_1\perp\widetilde{₠}_2$ or for every $\widetilde{₠}_2\in\mathfrak{F}$, $\widetilde{₠}_2\perp\widetilde{₠}_1$.

It follows that $\widetilde{₠}_1\perp P\widetilde{₠}_1$ or $\mathrm{P}\widetilde{₠}_1\perp\widetilde{₠}_1$. Let $\widetilde{₠}_1=P\widetilde{₠}_0$, $\widetilde{₠}_2=P\widetilde{₠}_1$, $\widetilde{₠}_3=P\widetilde{₠}_2,\ldots,\widetilde{₠}_{n+1}=P\widetilde{₠}_n$, for $n\in\tilde{N}$, since P is $\tilde{O}$-preserving & Ě is Archimedean then $\{\widetilde{₠}_n\}$ is an $\tilde{O}$-sequence. Then, by (2), we have

$$\widetilde{\propto}\left(\widetilde{₠}_{n+1},\widetilde{₠}_n\right)=\widetilde{\propto}\ (P\widetilde{₠}_n,P\widetilde{₠}_{n-1})$$
$$\leq\tilde{z}\max\{\widetilde{\propto}\left(\widetilde{₠}_n,\widetilde{₠}_{n-1}\right),\widetilde{\propto}\left(\widetilde{₠}_n,P\widetilde{₠}_n\right),\widetilde{\propto}\ (\widetilde{₠}_{n-1},P\widetilde{₠}_{n-1})\}$$

$$\leq \tilde{z}\max\{\tilde{\propto}\left(\tilde{\text{Ɇ}}_n,\tilde{\text{Ɇ}}_{n-1}\right),\tilde{\propto}\left(\tilde{\text{Ɇ}}_n,\tilde{\text{Ɇ}}_{n+1}\right),\tilde{\propto}\left(\tilde{\text{Ɇ}}_{n-1},\tilde{\text{Ɇ}}_n\right)\}$$
$$\leq \tilde{z}\max\{\tilde{\propto}\left(\tilde{\text{Ɇ}}_n,\tilde{\text{Ɇ}}_{n-1}\right),\tilde{\propto}\left(\tilde{\text{Ɇ}}_n,\tilde{\text{Ɇ}}_{n+1}\right)\}.$$

If for some $n, max\left\{\tilde{\propto}\left(\tilde{\text{Ɇ}}_n,\tilde{\text{Ɇ}}_{n-1}\right),\tilde{\propto}\left(\tilde{\text{Ɇ}}_n,\tilde{\text{Ɇ}}_{n+1}\right)\right\}=\tilde{\propto}\left(\tilde{\text{Ɇ}}_{n+1},\tilde{\text{Ɇ}}_n\right)$, we have

$$\tilde{\propto}\left(\tilde{\text{Ɇ}}_{n+1},\tilde{\text{Ɇ}}_n\right)\leq \tilde{z}\,\tilde{\propto}\left(\tilde{\text{Ɇ}}_{n+1},\tilde{\text{Ɇ}}_n\right)<\tilde{\propto}\left(\tilde{\text{Ɇ}}_{n+1},\tilde{\text{Ɇ}}_n\right)$$

which is contradiction.

Thus, $max\left\{\tilde{\propto}\left(\tilde{\text{Ɇ}}_n,\tilde{\text{Ɇ}}_{n-1}\right),\tilde{\propto}\left(\tilde{\text{Ɇ}}_n,\tilde{\text{Ɇ}}_{n+1}\right)\right\}=\tilde{\propto}\left(\tilde{\text{Ɇ}}_n,\tilde{\text{Ɇ}}_{n-1}\right)$ for all $n\geq 1$. Again, we have

$$\tilde{\propto}\left(\tilde{\text{Ɇ}}_{n+1},\tilde{\text{Ɇ}}_n\right)\leq \tilde{z}\,\tilde{\propto}\left(\tilde{\text{Ɇ}}_n,\tilde{\text{Ɇ}}_{n-1}\right),\text{ we obtain}$$
$$\tilde{\propto}\left(\tilde{\text{Ɇ}}_{n+1},\tilde{\text{Ɇ}}_n\right)\leq \tilde{z}^{\mathrm{n}}\,\tilde{\propto}\left(\tilde{\text{Ɇ}}_1,\tilde{\text{Ɇ}}_0\right) \tag{3}$$

for all $\mathrm{n}\geq 0$, $m,n\in\tilde{N}$ with $m>n$, we have

$$\tilde{\propto}\left(\tilde{\text{Ɇ}}_n,\tilde{\text{Ɇ}}_m\right)\leq\tilde{\propto}\left(\tilde{\text{Ɇ}}_n,\widetilde{\text{Ɇ}_{n+1}}\right)+\tilde{\propto}\left(\widetilde{\text{Ɇ}_{n+1}},\tilde{\text{Ɇ}}_n\right)$$
$$\leq\tilde{\propto}\left(\tilde{\text{Ɇ}}_n,\tilde{\text{Ɇ}}_{n+1}\right)+\tilde{\propto}\left(\tilde{\text{Ɇ}}_{n+1},\tilde{\text{Ɇ}}_{n+2}\right)+\tilde{\propto}\left(\widetilde{\mathfrak{Z}_{n+2}},\tilde{\text{Ɇ}}_m\right)$$
$$\leq\tilde{\propto}\left(\tilde{\text{Ɇ}}_n,\tilde{\text{Ɇ}}_{n+1}\right)+\tilde{\propto}\left(\tilde{\text{Ɇ}}_{n+1},\tilde{\text{Ɇ}}_{n+2}\right)+\tilde{\propto}\left(\tilde{\text{Ɇ}}_{n+2},\tilde{\text{Ɇ}}_{n+3}\right)+\cdots+\left(\tilde{\text{Ɇ}}_{m-1},\tilde{\text{Ɇ}}_m\right)$$

From inequality (3), we get

$$\tilde{\propto}\left(\tilde{\text{Ɇ}}_n,\tilde{\text{Ɇ}}_m\right)\leq\tilde{z}^n\,\tilde{\propto}\left(\tilde{\text{Ɇ}}_1,\tilde{\text{Ɇ}}_0\right)+\tilde{z}^{n+1}\,\tilde{\propto}\left(\tilde{\text{Ɇ}}_1,\tilde{\text{Ɇ}}_0\right)+\tilde{z}^{n+2}\,\tilde{\propto}\left(\tilde{\text{Ɇ}}_1,\tilde{\text{Ɇ}}_0\right)+\cdots$$
$$+\tilde{z}^{m-1}\,\tilde{\propto}\left(\tilde{\text{Ɇ}}_1,\tilde{\text{Ɇ}}_0\right)\leq\tilde{z}^n(1+\tilde{z}^1+\tilde{z}^2+\cdots)\,\tilde{\propto}\left(\tilde{\text{Ɇ}}_1,\tilde{\text{Ɇ}}_0\right)$$
$$=\frac{\tilde{\mathrm{z}}^n}{1-\tilde{z}}\,\tilde{\propto}\left(\tilde{\text{Ɇ}}_1,\tilde{\text{Ɇ}}_0\right)$$

Since $\tilde{\mathrm{z}}\in[0,1), m,n\to\tilde{\infty}$ and according to the above inequality that $\lim_{\tilde{n},\tilde{m}\to\tilde{\infty}}\tilde{\propto}\left(\tilde{\text{Ɇ}}_n,\tilde{\text{Ɇ}}_m\right)=0$.

Therefore, $\{\tilde{\text{Ɇ}}_n\}$ is an $\tilde{E}$-Cauchy sequence.

Since $\mathfrak{F}$ $\tilde{O}$-complete, there is $\widetilde{\text{Ɇ}^*}\in\mathfrak{F}$ so $\tilde{\text{Ɇ}}_n\to\widetilde{\text{Ɇ}^*}$ as $n\to\tilde{\infty}$. As P is $\tilde{O}$-continuous, one write

$$P\widetilde{\text{Ɇ}^*}=P\left(\lim_{n,\to\tilde{\infty}}\tilde{\text{Ɇ}}_n\right)=\lim_{n\to\tilde{\infty}}P\tilde{\text{Ɇ}}_n=\lim_{n\to\tilde{\infty}}\tilde{\text{Ɇ}}_{n+1}=\widetilde{\text{Ɇ}^*}.$$

Therefore, $\widetilde{\text{Ɇ}^*}$ is fixed point. We show that $\widetilde{\text{Ɇ}_2^*}\neq\widetilde{\text{Ɇ}_1^*}\in\mathfrak{F}$ as fixed point of P. So, we obtain $P^n\widetilde{\text{Ɇ}_1^*}=\widetilde{\text{Ɇ}_1^*}$ and $P^n\widetilde{\text{Ɇ}_2^*}=\widetilde{\text{Ɇ}_2^*}$. By the definition of orthogonally, there is $\tilde{\text{Ɇ}}_1\in\mathfrak{F}$ so that $[\tilde{\text{Ɇ}}_1\perp\widetilde{\text{Ɇ}_1^*}$ and $\tilde{\text{Ɇ}}_1\perp\widetilde{\text{Ɇ}_2^*}]$ or $\left[\widetilde{\text{Ɇ}_1^*}\perp\tilde{\text{Ɇ}}_1\right.$ and $\left.\widetilde{\text{Ɇ}_2^*}\perp\tilde{\text{Ɇ}}_1\right]$.

As P is $\tilde{O}$-preserving, we have $[P^n\tilde{\text{Ɇ}}_1\perp P^n\widetilde{\text{Ɇ}_1^*}$ and $P^n\tilde{\text{Ɇ}}_1\perp P^n\widetilde{\text{Ɇ}_2^*}]$ or $\left[P^n\widetilde{\text{Ɇ}_1^*}\perp P^n\tilde{\text{Ɇ}}_1\right.$ and $\left.P^n\widetilde{\text{Ɇ}_2^*}\perp P^n\tilde{\text{Ɇ}}_1\right]$.

For all $\tilde{n}\in\tilde{N}$, then we have

$$\widetilde{\propto}\left(\widetilde{\text{Ҝ}_1^*}, \widetilde{\text{Ҝ}_2^*}\right) = \widetilde{\propto}\left(P^n\widetilde{\text{Ҝ}_1^*}, P^n\widetilde{\text{Ҝ}_2^*}\right)$$
$$\leq \tilde{z}\max\{\widetilde{\propto}\left(\widetilde{\text{Ҝ}_1^*}, \widetilde{\text{Ҝ}_2^*}\right), \widetilde{\propto}\left(\widetilde{\text{Ҝ}_1^*}, P\widetilde{\text{Ҝ}_1^*}\right), \widetilde{\propto}\left(\widetilde{\text{Ҝ}_2^*}, P\widetilde{\text{Ҝ}_2^*}\right)\}$$
$$= \tilde{z}\max\{\widetilde{\propto}\left(\widetilde{\text{Ҝ}_1^*}, \widetilde{\text{Ҝ}_2^*}\right), \widetilde{\propto}\left(\widetilde{\text{Ҝ}_1^*}, \widetilde{\text{Ҝ}_1^*}\right), \widetilde{\propto}\left(\widetilde{\text{Ҝ}_2^*}, \widetilde{\text{Ҝ}_2^*}\right)\}$$
$$= \tilde{z}\,\widetilde{\propto}\left(\widetilde{\text{Ҝ}_1^*}, \widetilde{\text{Ҝ}_2^*}\right)$$
$$< \widetilde{\propto}\left(\widetilde{\text{Ҝ}_1^*}, \widetilde{\text{Ҝ}_2^*}\right)$$

It is a contradiction, we obtain $\widetilde{\propto}\left(\widetilde{\text{Ҝ}_1^*}, \widetilde{\text{Ҝ}_2^*}\right) = 0$.

Corollary 2.5: *If* $(\mathfrak{F}, \widetilde{\propto})$ *is an* $\tilde{O}$*-complete orthogonal vector metric space and* $P: \mathfrak{F} \to \mathfrak{F}$ *is an* $\tilde{O}$*–preserving and* $\tilde{O}$*–continuous mapping so that*

$$\widetilde{\propto}\left(\mathrm{P}\widetilde{\text{Ҝ}}_1, \mathrm{P}\widetilde{\text{Ҝ}}_2\right) \leq \tilde{z}\max\{\widetilde{\propto}\left(\widetilde{\text{Ҝ}}_1, \mathrm{P}\widetilde{\text{Ҝ}}_1\right), \widetilde{\propto}\left(\widetilde{\text{Ҝ}}_2, \mathrm{P}\widetilde{\text{Ҝ}}_2\right)\}$$

for all $\widetilde{\text{Ҝ}}_1, \widetilde{\text{Ҝ}}_2 \in \mathfrak{F}$ *wherever* $0 \leq \tilde{z} < 1$. *hence* P *have unique fixed point* $\widetilde{\text{Ҝ}^*} \in \mathfrak{F}$ *&* $\widetilde{\propto}\left(\widetilde{\text{Ҝ}^*}, \widetilde{\text{Ҝ}^*}\right) = 0$

Example 2.6: Consider $\mathfrak{F} = L_2$. Given $\widetilde{\propto} : \mathfrak{F} \times \mathfrak{F} \to \tilde{R}^+$ *as* $\widetilde{\propto}\left(\widetilde{\text{Ҝ}}_1, \widetilde{\text{Ҝ}}_2\right) = |\sum_{n=1}^{\infty}\left(\widetilde{\text{Ҝ}}_1, \widetilde{\text{Ҝ}}_2\right)^2|^{\frac{1}{2}}$.

Define (1) on $\mathfrak{F}$ by $\widetilde{\text{Ҝ}}_1 \perp \widetilde{\text{Ҝ}}_2$ iff $\widetilde{\text{Ҝ}}_1 = \widetilde{\text{Ҝ}}_2$ or $\widetilde{\text{Ҝ}}_1, \widetilde{\text{Ҝ}}_2 \geq 0$. Then $(\mathfrak{F}, \widetilde{\propto})$ is an $\tilde{O}$-complete orthogonal vector metric space. Define the mapping $P: \mathfrak{F} \to \mathfrak{F}$ by $\mathrm{P}(\widetilde{\text{Ҝ}}) = \left\{\frac{\widetilde{\text{Ҝ}}_n}{2}\right\}$.

Let $\widetilde{\text{Ҝ}}_1, \widetilde{\text{Ҝ}}_2 \in \mathfrak{F}$ with $\widetilde{\text{Ҝ}}_1 \perp \widetilde{\text{Ҝ}}_2$.

1. if $\widetilde{\text{Ҝ}}_1 = \widetilde{\text{Ҝ}}_2$, then

$$\widetilde{\propto}\left(P\widetilde{\text{Ҝ}}_1, P\widetilde{\text{Ҝ}}_2\right) = 0 = \frac{1}{2}\,\widetilde{\propto}\,(P\widetilde{\text{Ҝ}}_1, P\widetilde{\text{Ҝ}}_2)$$

is satisfied.

2. if $\widetilde{\text{Ҝ}}_1, \widetilde{\text{Ҝ}}_2 \geq 0$, then

$$\widetilde{\propto}\,(P\widetilde{\text{Ҝ}}_1, P\widetilde{\mathfrak{Z}_2}) = \left[\sum_{n=1}^{\infty}\left(\frac{\widetilde{\text{Ҝ}}_{1n}}{2} - \frac{\widetilde{\text{Ҝ}}_{2n}}{2}\right)\right]^{\frac{1}{2}}$$
$$= \left[\sum_{n=1}^{\infty}\left(\frac{\widetilde{\text{Ҝ}}_{1n} - \widetilde{\text{Ҝ}}_{2n}}{2}\right)^2\right]^{\frac{1}{2}}$$
$$= \frac{1}{2}\left[\sum_{n=1}^{\infty}\left(\widetilde{\text{Ҝ}}_{1n} - \widetilde{\text{Ҝ}}_{2n}\right)^2\right]^{\frac{1}{2}}$$
$$= \frac{1}{2}\,\widetilde{\propto}\,(\widetilde{\text{Ҝ}}_1, \widetilde{\text{Ҝ}}_2)$$

3. Application

The aim for this part is to utilize theorem (2.4) to explore existence & uniqueness for solutions of a Fredholm integral equation. We put a space C(I) for continuous map. defined over I=[0,1] , equipped with a orthogonal vector metric space given as

$$\widetilde{\propto}\left(\widetilde{\text{Ꞓ}}_1, \widetilde{\text{Ꞓ}}_2\right) = \; |\widetilde{\text{Ꞓ}}_1(\tilde{t}) - \widetilde{\text{Ꞓ}}_2(\tilde{t})|\text{, for all } \widetilde{\text{Ꞓ}}_1, \widetilde{\text{Ꞓ}}_2 \in \mathrm{C(I)}$$

This space may be alternatively endowed with orthogonal given as

$$\widetilde{\text{Ꞓ}}_1, \widetilde{\text{Ꞓ}}_2 \in \mathrm{C(I)}, \widetilde{\text{Ꞓ}}_1 \perp \widetilde{\text{Ꞓ}}_2 \;\leftrightarrow\; \widetilde{\text{Ꞓ}}_1 \leq \widetilde{\text{Ꞓ}}_2 \,.$$

Now, let's examine the following integral equation of Fredholm type:

$$\widetilde{\text{Ꞓ}}_1(\tilde{t}) = \int_0^1 k\left(t, s, \widetilde{\text{Ꞓ}}_1(\tilde{t})\right) ds \text{ for all } t, s \in [0,1] \tag{4}$$

Where $k \in ([0,1], R)$.

So $(\mathfrak{F}, \widetilde{\propto}, \breve{\mathrm{E}})$ orthogonal complete vector metric space.

Theorem 3.1: *For* $\widetilde{\text{Ꞓ}}_1, \widetilde{\text{Ꞓ}}_2 \in ([0,1], R)$. *Assume*

$$\left|k\left(t, s, \widetilde{\text{Ꞓ}}_1(\tilde{t})\right) + k\left(t, s, \widetilde{\text{Ꞓ}}_2(\tilde{t})\right)\right| \leq \ddot{z}\left|\widetilde{\text{Ꞓ}}_1(\tilde{t}) + \widetilde{\text{Ꞓ}}_2(\tilde{t})\right|$$

$for\ all\ t, s \in [0,1], when\ \ddot{z} \in [0,1)$.

Then (4) have unique solution on the interval $[0,1]$

Proof: Define $\mathrm{P}: \mathfrak{F} \to \mathfrak{F}$ is given by $\mathrm{P}(\widetilde{\text{Ꞓ}}_1(\tilde{t})) = \int_0^1 k\left(t, s, \widetilde{\text{Ꞓ}}_1(\tilde{t})\right) ds$ for all $t, s \in [0,1]$.

So existence for fixed point over operator *P* same as existence for solution of (4) .

Thus, $\widetilde{\text{Ꞓ}}_1 \leq \widetilde{\mathfrak{Z}_2}, \widetilde{\text{Ꞓ}}_1 \perp \widetilde{\text{Ꞓ}}_2$ and $t, s \in [0,1]$. We obtain

$$\tilde{\alpha}\left(P\widetilde{\text{Ꞓ}}_1, P\widetilde{\text{Ꞓ}}_2\right) = \quad |k\widetilde{\text{Ꞓ}}_1(\tilde{t}), k\widetilde{\text{Ꞓ}}_2(\tilde{t})|$$

$$= \left|\int_0^1 k\left(t, s, \widetilde{\text{Ꞓ}}_1(\tilde{t})\right) - k\left(t, s, \widetilde{\text{Ꞓ}}_2(\tilde{t})\right)\, ds\right|$$

$$\leq \int_0^1 \left|k\left(t, s, \widetilde{\text{Ꞓ}}_1(\tilde{t})\right) - k\left(t, s, \widetilde{\text{Ꞓ}}_2(\tilde{t})\right)\right| ds$$

$$\leq \; \tilde{z} \int_0^1 \left|\widetilde{\text{Ꞓ}}_1(\tilde{t}) - \widetilde{\text{Ꞓ}}_2(\tilde{t})\right|\, ds$$

$$\leq \; \hat{z}(\widetilde{\text{Ꞓ}}_1(\tilde{t}) - \widetilde{\text{Ꞓ}}_2(\tilde{t})) \int_0^1 ds$$

$$\leq \; \hat{z} \; \widehat{\propto} \; (\widetilde{\text{Ꞓ}}_1, \widetilde{\text{Ꞓ}}_2)$$

Hence *P* has a unique solution on $\left[0, \tilde{1}\right]$.

4. Conclusion

The idea of an orthogonal vector metric space, which is based on a fixed point of an orthogonal set, has been described in this article, and various fixed theorems have been proven in this space. Ultimately, our findings show that has unique solution for Fredholm integral.

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