

Convergence and best proximity point theorems via tripled G_{P_S} -mappings

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Abstract

In this article, we introduce a tripled G_{P_S} -cyclic mapping in P_S -dislocated metric space, and we prove some convergence and best proximity point theorems by using tripled G_{P_S} -cyclic mapping.

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1. Introduction

One of the most important theorems at the beginning of the 20th century happened around 1922; this theorem is known as Banach's thesis [1]. Since this theorem has many applications, many researchers have been interested in expanding and developing the fixed point (F.P) theory. [2-25] Keeler and Meir [4], they proved Banach's theorem via M-K contraction. Also, proved that a unique F.P of M-K contraction in metric space. After that, many authors worked on this concept, as Basha, Sadiq, and Jeyaraj [5] studied some theorems of best proximity points (B.P.P) for types of α -cyclic mappings. In 2013, Karapınar, Kumam, and Salimi [6] introduced the definition of α - ψ -M-K contraction maps. Many others offered various modifieds of the concept of M-K maps in multiple metric spaces [7-9].

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Definition 1.1: The non-empty set A , and the function $d_{\mathcal{P}\mathcal{S}}: A \times A \times A \rightarrow \mathbb{R}^+$ is defined as $\mathcal{P}\mathcal{S}$ -dislocated metric if there is $\Phi: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ a continuous function such that $\Phi(t) \leq t, \forall t \in \mathbb{R}^+$ and $\Phi(0) = 0$ exists $\forall a, z, e \in A$ the following conditions hold:

1. If $d_{\mathcal{P}\mathcal{S}}(a, z, e) = 0$ then $a = z = e$
2. $d_{\mathcal{P}\mathcal{S}}(a, a, z) = d_{\mathcal{P}\mathcal{S}}(z, z, a)$
3. $d_{\mathcal{P}\mathcal{S}}(a, z, e) \leq \Phi(d_{\mathcal{P}\mathcal{S}}(a, a, u) + d_{\mathcal{P}\mathcal{S}}(z, z, u) + d_{\mathcal{P}\mathcal{S}}(e, e, u)) \quad \forall u \in A$

The pair $(A, d_{\mathcal{P}\mathcal{S}})$ is defined as $\mathcal{P}\mathcal{S}$ -dislocated metric space. This is an example of this metric.

Example 1.2: Let $A = \mathbb{R}$, $d_{\mathcal{P}\mathcal{S}}(a, z, e) = \max\{|a - 5|, |z - 5|, |e - 5|\}$, for all $a, z, e \in A$.

Example 1.3: Let $A = \mathbb{R}$, $d_{\mathcal{P}\mathcal{S}}(a, z, e) = |a| + |z| + e^2$ is not $\mathcal{P}\mathcal{S}$ -dislocated metric space

Definition 1.4: [4] A pair of mappings $T: \mathcal{S} \rightarrow E$ and $f: E \rightarrow \mathcal{S}$ is said

1. $k_{\mathcal{P}\mathcal{S}}$ -cyclic map if there exists $k < \frac{1}{2}$ such that

$$d_{\mathcal{P}\mathcal{S}}(Ta, Ta, fb) \leq k [d_{\mathcal{P}\mathcal{S}}(a, a, Ta) + d_{\mathcal{P}\mathcal{S}}(b, b, fb)] + (1 - 2k)d_{\mathcal{P}\mathcal{S}}(\mathcal{S}, \mathcal{S}, E),$$
 for $a \in \mathcal{S}, b \in E$.
2. $G_{\mathcal{P}\mathcal{S}}$ -cyclic map if $\exists k < \frac{1}{2}$ such that

$$d_{\mathcal{P}\mathcal{S}}(Ta, Ta, fb) \leq k [d_{\mathcal{P}\mathcal{S}}(a, a, fb) + d_{\mathcal{P}\mathcal{S}}(b, b, Ta)] + (1 - 2k)d_{\mathcal{P}\mathcal{S}}(\mathcal{S}, \mathcal{S}, E),$$
 for $a \in \mathcal{S}, b \in E$.

We also recall M-K mapping. $\mathfrak{D}: [0, \infty) \rightarrow [0, \infty)$ is said to be M-K map if, $\forall g_s > 0, \delta \exists > 0$ such that, $\forall t \in [0, \infty)$ with $g_s \leq t < g_s + \delta \rightarrow \mathfrak{D}(t) < g_s$.

2. Main Results

In this section, we give a tripled $G_{\mathcal{P}\mathcal{S}}$ -cyclic mappings, and we prove convergence and best proximity theorems in $\mathcal{P}\mathcal{S}$ -dislocated metric space.

Definition 2.1: Let $\mathcal{S}$ and E nonempty subsets of A . A mappings $T: \mathcal{S} \rightarrow E$ and $f: E \rightarrow \mathcal{S}$ are tripled- $G_{\mathcal{P}\mathcal{S}}$ -cyclic between $\mathcal{S}$ and E if there is a stronger M-K map $\Psi: [0, \infty) \rightarrow [0, \frac{1}{2})$ in A such that for $a \in \mathcal{S}, b \in E$,

$$d_{\mathcal{P}\mathcal{S}}(Ta, Ta, fb) - d_{\mathcal{P}\mathcal{S}}(\mathcal{S}, \mathcal{S}, E) \leq \Psi(d_{\mathcal{P}\mathcal{S}}(a, a, b)) \cdot [G_{\mathcal{P}\mathcal{S}}(a, a, b) - 2d_{\mathcal{P}\mathcal{S}}(\mathcal{S}, \mathcal{S}, E)] \quad (1)$$

When,

$$G_{\mathcal{P}\mathcal{S}}(a, a, b) = \frac{1}{2} \max\{d_{\mathcal{P}\mathcal{S}}(a, a, b), d_{\mathcal{P}\mathcal{S}}(a, a, Ta), d_{\mathcal{P}\mathcal{S}}(b, b, fb), d_{\mathcal{P}\mathcal{S}}(a, a, fb), d_{\mathcal{P}\mathcal{S}}(b, b, Ta)\}.$$

Lemma 2.2: *If T and F form a tripled- G_{P_S} -cyclic mapping. Then there exists $\{a_n\}$ in A s.t.*

$$\lim_{n \rightarrow \infty} d_{P_S}(a_n, a_n, a_{n+1}) = d_{P_S}(S, S, E).$$

Proof: Let $a_{2n+1} = Ta_{2n}$ and $a_{2n+2} = Fa_{2n+1}$ for each $n \in N$. So,

$$\begin{aligned} d_{P_S}(a_{2n+1}, a_{2n+1}, a_{2n+2}) - d_{P_S}(S, S, E) &= d_{P_S}(Ta_{2n}, Ta_{2n}, Fa_{2n+1}) - d_{P_S}(S, S, E) \\ &\leq \Psi \left(d_{P_S}(a_{2n}, a_{2n}, a_{2n+1}) \right) \cdot [G_{P_S}(a_{2n}, a_{2n}, a_{2n+1}) - 2d_{P_S}(S, S, E)] \end{aligned} \quad (2)$$

$$\begin{aligned} G_{P_S}(a_{2n}, a_{2n}, a_{2n+1}) &= \frac{1}{2} \max\{d_{P_S}(a_{2n}, a_{2n}, a_{2n+1}), d_{P_S}(a_{2n}, a_{2n}, Ta_{2n}), \\ &\quad d_{P_S}(a_{2n+1}, a_{2n+1}, Fa_{2n+1}), d_{P_S}(a_{2n}, a_{2n}, Fa_{2n+1}), d_{P_S}(a_{2n+1}, a_{2n+1}, Ta_{2n})\} \\ &= \frac{1}{2} \max\{d_{P_S}(a_{2n}, a_{2n}, a_{2n+1}), d_{P_S}(a_{2n}, a_{2n}, a_{2n+1}), d_{P_S}(a_{2n+1}, a_{2n+1}, a_{2n+2}), \\ &\quad d_{P_S}(a_{2n}, a_{2n}, a_{2n+2}), d_{P_S}(a_{2n+1}, a_{2n+1}, a_{2n+1})\} \end{aligned}$$

$$\begin{aligned} &\leq \max\{d_{P_S}(a_{2n}, a_{2n}, a_{2n+1}), d_{P_S}(a_{2n}, a_{2n}, a_{2n+1}), d_{P_S}(a_{2n+1}, a_{2n+1}, a_{2n+2}), \\ &\quad \Phi(d_{P_S}(a_{2n}, a_{2n}, a_{2n+1}) + d_{P_S}(a_{2n}, a_{2n}, a_{2n+1}) + d_{P_S}(a_{2n+2}, a_{2n+2}, a_{2n+1})), \\ &\quad d_{P_S}(a_{2n+1}, a_{2n+1}, a_{2n+1})\} \end{aligned}$$

Since $\Phi(t) \leq t$ and $d_{P_S}(a, a, e) = d_{P_S}(e, e, a)$. Then

$$\begin{aligned} &\leq \frac{1}{2} \max\{d_{P_S}(a_{2n}, a_{2n}, a_{2n+1}), d_{P_S}(a_{2n}, a_{2n}, a_{2n+1}), d_{P_S}(a_{2n+1}, a_{2n+1}, a_{2n+2}), \\ &\quad d_{P_S}(a_{2n}, a_{2n}, a_{2n+1}) + d_{P_S}(a_{2n}, a_{2n}, a_{2n+1}) + d_{P_S}(a_{2n+1}, a_{2n+1}, a_{2n+2}), \\ &\quad d_{P_S}(a_{2n+1}, a_{2n+1}, a_{2n+1})\} \\ &= \frac{3}{2} \max\{d_{P_S}(a_{2n}, a_{2n}, a_{2n+1}), d_{P_S}(a_{2n+1}, a_{2n+1}, a_{2n+2})\} \\ G_{P_S}(a_{2n}, a_{2n}, a_{2n+1}) &\leq \frac{3}{2} \max\{d_{P_S}(a_{2n}, a_{2n}, a_{2n+1}), d_{P_S}(a_{2n+1}, a_{2n+1}, a_{2n+2})\} \\ G_{P_S}(a_{2n}, a_{2n}, a_{2n+1}) &< \frac{4}{2} \max\{d_{P_S}(a_{2n}, a_{2n}, a_{2n+1}), d_{P_S}(a_{2n+1}, a_{2n+1}, a_{2n+2})\} \\ G_{P_S}(a_{2n}, a_{2n}, a_{2n+1}) &< 2 \max\{d_{P_S}(a_{2n}, a_{2n}, a_{2n+1}), d_{P_S}(a_{2n+1}, a_{2n+1}, a_{2n+2})\} \end{aligned} \quad (3)$$

Taking into account (2) and (3), we have that for each $n \in N$.

$$\begin{aligned} d_{P_S}(a_{2n+1}, a_{2n+1}, a_{2n+2}) - d_{P_S}(S, S, E) &\leq \Psi \left(d_{P_S}(a_{2n}, a_{2n}, a_{2n+1}) \right) \cdot [G_{P_S}(a_{2n}, a_{2n}, a_{2n+1}) \\ &\quad - 2d_{P_S}(S, S, E)] \\ &< \max\{d_{P_S}(a_{2n}, a_{2n}, a_{2n+1}), d_{P_S}(a_{2n+1}, a_{2n+1}, a_{2n+2})\} \\ &\quad - d_{P_S}(S, S, E) \end{aligned}$$

And

$$d_{P_S}(a_{2n+1}, a_{2n+1}, a_{2n+2}) - d_{P_S}(S, S, E) < d_{P_S}(a_{2n}, a_{2n}, a_{2n+1}) - d_{P_S}(S, S, E) \quad (4)$$

And for each $n \in N$

$$\begin{aligned} d_{p_s}(a_{2n}, a_{2n}, a_{2n+1}) - d_{p_s}(\$, \$, E) &= d_{p_s}(Fa_{2n-1}, Fa_{2n-1}, Ta_{2n}) - d_{p_s}(\$, \$, E) \\ &\leq \Psi \left(d_{p_s}(a_{2n}, a_{2n}, a_{2n-1}) \right) \cdot [G_{p_s}(a_{2n}, a_{2n}, a_{2n-1}) - 2d_{p_s}(\$, \$, E)] \end{aligned} \quad (5)$$

$$\begin{aligned} G_{p_s}(a_{2n}, a_{2n}, a_{2n-1}) &\leq \frac{1}{2} \max\{d_{p_s}(a_{2n}, a_{2n}, a_{2n-1}), d_{p_s}(a_{2n}, a_{2n}, a_{2n+1}), \\ d_{p_s}(a_{2n}, a_{2n}, a_{2n-1}), d_{p_s}(a_{2n}, a_{2n}, a_{2n}), d_{p_s}(a_{2n-1}, a_{2n-1}, a_{2n}) + \\ d_{p_s}(a_{2n-1}, a_{2n-1}, a_{2n}) + d_{p_s}(a_{2n}, a_{2n}, a_{2n+1})\} \\ &\leq \frac{1}{2} [\max\{d_{p_s}(a_{2n}, a_{2n}, a_{2n-1}), d_{p_s}(a_{2n}, a_{2n}, a_{2n+1})\} + \\ \max\{d_{p_s}(a_{2n}, a_{2n}, a_{2n-1}), d_{p_s}(a_{2n}, a_{2n}, a_{2n+1}) + \\ \max\{d_{p_s}(a_{2n}, a_{2n}, a_{2n-1}), d_{p_s}(a_{2n}, a_{2n}, a_{2n+1})\}] \\ &< \frac{4}{2} [\max\{d_{p_s}(a_{2n}, a_{2n}, a_{2n-1}), d_{p_s}(a_{2n}, a_{2n}, a_{2n+1})\} \end{aligned} \quad (6)$$

Taking into account (5) and (6), we get that for each $n \in N$

$$\begin{aligned} d_{p_s}(a_{2n}, a_{2n}, a_{2n+1}) - d_{p_s}(\$, \$, E) &< \Psi \left(d_{p_s}(a_{2n-1}, a_{2n-1}, a_{2n}) \right) \cdot \\ 2[\max\{d_{p_s}(a_{2n-1}, a_{2n-1}, a_{2n}), d_{p_s}(a_{2n}, a_{2n}, a_{2n+1})\} - d_{p_s}(\$, \$, E)] \\ &< \max\{d_{p_s}(a_{2n-1}, a_{2n-1}, a_{2n}), d_{p_s}(a_{2n}, a_{2n}, a_{2n+1})\} - d_{p_s}(\$, \$, E) \end{aligned} \quad (7)$$

And so we conclude that

$$d_{p_s}(a_{2n}, a_{2n}, a_{2n+1}) - d_{p_s}(\$, \$, E) < d_{p_s}(a_{2n-1}, a_{2n-1}, a_{2n}) - d_{p_s}(\$, \$, E) \quad (8)$$

Generally, by (4) and (8), we get that for each $n \in N$

$$\begin{aligned} d_{p_s}(a_{n+1}, a_{n+1}, a_{n+2}) &< d_{p_s}(a_n, a_n, a_{n+1}) \\ d_{p_s}(a_{n+1}, a_{n+1}, a_{n+2}) - d_{p_s}(\$, \$, E) &\leq \Psi \left(d_{p_s}(a_n, a_n, a_{n+1}) \right) \cdot \\ 2 \cdot [d_{p_s}(a_n, a_n, a_{n+1}) - d_{p_s}(\$, \$, E)] \end{aligned}$$

Thus, $\langle a_n \rangle = \langle d_{p_s}(a_n, a_n, a_{n+1}) \rangle$ is decreasing and bounded, and hence, it is convergent. Let $\lim_{n \rightarrow \infty} d_{p_s}(a_n, a_n, a_{n+1}) = g \geq 0$. i.e.,

$$g \leq d_{p_s}(a_n, a_n, a_{n+1}) < g + \delta \quad (9)$$

$$\text{And hence, } \Psi(d_{p_s}(a_n, a_n, a_{n+1})) \leq \gamma_g, \forall n \geq n_0 \quad (10)$$

Now,

$$\begin{aligned} d_{p_s}(a_n, a_n, a_{n+1}) - d_{p_s}(\$, \$, E) \\ &\leq \Psi \left(d_{p_s}(a_{n-1}, a_{n-1}, a_n) \right) \cdot 2 \cdot [d_{p_s}(a_{n-1}, a_{n-1}, a_n) \\ &\quad - d_{p_s}(\$, \$, E)] \\ d_{p_s}(a_n, a_n, a_{n+1}) - d_{p_s}(\$, \$, E) &\leq 2\gamma_g \cdot [d_{p_s}(a_{n-1}, a_{n-1}, a_n) - d_{p_s}(\$, \$, E)] \\ &\leq (2\gamma_g)^2 \cdot [d_{p_s}(a_{n-2}, a_{n-2}, a_{n-1}) - d_{p_s}(\$, \$, E)] \end{aligned}$$

continue, we have

$$\leq (2\gamma_g)^{n-n^0} \cdot [d_{P_S}(a_{n^0}, a_{n^0}, a_{n^0+1}) - d_{P_S}(\mathcal{S}, \mathcal{S}, E)]$$

Since $\gamma_g \in [0, 1)$, then we get $\lim_{n \rightarrow \infty} d_{P_S}(a_n, a_n, a_{n+1}) - d_{P_S}(\mathcal{S}, \mathcal{S}, E) = 0$.

$$\lim_{n \rightarrow \infty} d_{P_S}(a_n, a_n, a_{n+1}) = d_{P_S}(\mathcal{S}, \mathcal{S}, E).$$

Theorem 2.3: Under conditions of above lemma. If $a_{2n+1} = Ta_{2n}$ and $a_{2n+2} = Fa_{2n+1}$. Suppose that $\langle a_{2n} \rangle$ has a subsequence converging to $a \in \mathcal{S}$ such that $d_{P_S}(a, a, a) \leq d_{P_S}(\mathcal{S}, \mathcal{S}, E)$. Then a is best proximity point of T .

Proof: If $\langle a_{2n_k} \rangle$ converge to $a \in \mathcal{S}$. So, $d_{P_S}(a_{2n_k-1}, a_{2n_k-1}, a_{n_k})$ converges to $d_{P_S}(\mathcal{S}, \mathcal{S}, E)$. Now,

$$\begin{aligned} d_{P_S}(a_{2n_k}, a_{2n_k}, Ta) &= d_{P_S}(Ta, Ta, a_{2n_k}) = d_{P_S}(Ta, Ta, Fa_{2n_k-1}) \\ &\leq \psi \left(d_{P_S}(a, a, a_{2n_k-1}) \right) [G_{P_S}(a, a, a_{2n_k-1}) - 2d_{P_S}(\mathcal{S}, \mathcal{S}, E)] + d_{P_S}(\mathcal{S}, \mathcal{S}, E) \\ &< \gamma_g \cdot [G_{P_S}(a, a, a_{2n_k-1}) - 2d_{P_S}(\mathcal{S}, \mathcal{S}, E)] + d_{P_S}(\mathcal{S}, \mathcal{S}, E) \end{aligned} \quad (11)$$

$$\begin{aligned} G_{P_S}(a, a, a_{2n_k-1}) &= \frac{1}{2} \max\{d_{P_S}(a, a, a_{2n_k-1}), d_{P_S}(a, a, Ta), \\ &\quad d_{P_S}(a_{2n_k-1}, a_{2n_k-1}, a_{2n_k}), d_{P_S}(a, a, a_{2n_k}), d_{P_S}(a_{2n_k-1}, a_{2n_k-1}, Ta)\} \\ &\leq \frac{1}{2} \max\{2d_{P_S}(a, a, a_{2n_k}), d_{P_S}(a_{2n_k-1}, a_{2n_k-1}, a_{2n_k}), d_{P_S}(a, a, Ta), \\ &\quad d_{P_S}(a_{2n_k-1}, a_{2n_k-1}, a_{2n_k}), d_{P_S}(a, a, a_{2n_k}), 2d_{P_S}(a_{2n_k-1}, a_{2n_k-1}, a_{2n_k}) + \\ &\quad 2d_{P_S}(Ta, Ta, a_{2n_k}) + 2d_{P_S}(a, a, a_{2n_k})\} \\ &\leq d_{P_S}(a, a, a_{2n_k}) + d_{P_S}(a_{2n_k}, a_{2n_k}, a_{2n_k-1}) + d_{P_S}(a_{2n_k}, a_{2n_k}, Ta) \end{aligned} \quad (12)$$

By (11) and (12) we get

$$\begin{aligned} d_{P_S}(a_{2n_k}, a_{2n_k}, Ta) &\leq \gamma_g \cdot [G_{P_S}(a, a, a_{2n_k-1}) - 2d_{P_S}(\mathcal{S}, \mathcal{S}, E)] + d_{P_S}(\mathcal{S}, \mathcal{S}, E) \\ d_{P_S}(\mathcal{S}, \mathcal{S}, E) &\leq d_{P_S}(a_{2n_k}, a_{2n_k}, Ta) \leq \frac{\gamma_g}{1-\gamma_g} [d_{P_S}(a, a, a_{2n_k}) \\ &\quad + d_{P_S}(a_{2n_k}, a_{2n_k}, a_{2n_k-1})] + \frac{1-3\gamma_g}{1-\gamma_g} d_{P_S}(\mathcal{S}, \mathcal{S}, E) \end{aligned}$$

$$\begin{aligned} \text{When } k \rightarrow \infty, d_{P_S}(\mathcal{S}, \mathcal{S}, E) &\leq d_{P_S}(a, a, Ta) \leq \frac{\gamma_g}{1-\gamma_g} [d_{P_S}(a, a, a) + \\ &\quad d_{P_S}(\mathcal{S}, \mathcal{S}, E)] + \frac{1-3\gamma_g}{1-\gamma_g} d_{P_S}(\mathcal{S}, \mathcal{S}, E) \\ &\leq \frac{1-\gamma_g}{1-\gamma_g} d_{P_S}(\mathcal{S}, \mathcal{S}, E) \end{aligned}$$

Then, $d_{P_S}(a, a, Ta) = d_{P_S}(\mathcal{S}, \mathcal{S}, E)$. Then a is a best proximity point of T .

3. Conclusions

In this article, a generalize of tripled G_{P_S} -cyclic mappings introduced and convergence and best proximity theorems in P_S -dislocated metric space are showed

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