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# Certain properties of analytic fractional functions on the open unit disk

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**Abstract**

This research paper introduces a new generalization of the normalized fractional differential operator, which opens a new direction in our research area. We utilize this new operator to establish a subclass of univalent functions. Also, various characteristics of this subclass, including coefficient inequality and some topological properties in the Hardy space of analytic functions are studied.

***Subject Classification:*** *Primary 30C45, Secondary 30C33.*

***Keywords:*** *Analytic functions, Fractional differential operator, Coefficient inequality, Hardy space.*

## 1. Introduction

For $0 \leq \alpha \leq 1,\ \mu(\alpha) \geq 1$ such that $\mu(0) = 1$, let $\mathbb{A}_\mu$ denote the class of fractional functions of the form [1]:

$$f(z) = z + \sum_{\nu=2}^{\infty} a_\nu z^{\mu\nu+\alpha}, \tag{1}$$

are analytic in $\mathcal{D} = \{z : z \in \mathbb{C} \text{ such that } |z| < 1\}$. If, we set $\mathbb{A}_1 := \mathbb{A}$ then it is easily observed that the analytic function $f \in \mathbb{A}_\mu$ is normalized in the case when

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$\alpha = 0$ and $\mu(0) = 1$. For all $z \in \mathcal{D}$, the *convolution* $(\cdot * \cdot)$ of two analytic functions $f$ given by (1) and $\tau$ presented by:

$$\tau(z) = z + \sum_{\nu=2}^{\infty} b_\nu z^{\mu\nu+\alpha},$$

is defined by

$$(f * \tau)(z) = z + \sum_{\nu=2}^{\infty} a_\nu b_\nu z^{\mu\nu+\alpha} = (\tau * f)(z).$$

For $f \in \mathbb{A}_\mu$, let $\mathfrak{K}(\mu, \alpha, \mathcal{t})$ denote the class of fractional functions of the form $\Re[f'(z)] > \mathcal{t}$ for $0 \le \mathcal{t} < 1$. In terms of integral [2-5] the authors studied the fractional differential and integral operators of analytic functions. For $0 < \sigma \le 1, 0 < \kappa \le 1$ such that $1 > \kappa - \sigma \ge 0$, $f(z) \in \mathbb{A}, z \in \mathcal{D}$, the fractional differential Tremblay operator $\mathbb{T}_z^{\kappa,\sigma}$ considered in [2, 3] and defined by:

$$\mathbb{T}_z^{\kappa,\sigma} f(z) := \frac{\Gamma(\sigma)}{\Gamma(\kappa)\Gamma(1-\kappa+\sigma)} \left(z^{1-\sigma} \frac{d}{dz}\right) \int_0^z \frac{t^{\kappa-1} f(\tau)}{(z-\tau)^{\kappa-\sigma}} d\tau. \tag{2}$$

In analytic presentation, if we apply the operator (2) to functions belonging to the class $\mathbb{A}_{\mu,}$ defined in (1), then we have the following definition:

**Definition 1**: The fractional differential Tremblay operator $\mathrm{T}_{z,\mu}^{\kappa,\sigma}$ of function $f \in \mathbb{A}_\mu$ given in (1) is defined by:

$$\mathrm{T}_{z,\mu}^{\kappa,\sigma} f(z) := \frac{\Gamma(\sigma)}{\Gamma(\kappa)\Gamma(1-\kappa+\sigma)} \left(z^{1-\sigma} \frac{d}{dz}\right) \int_0^z \frac{t^{\kappa-1}\left(\tau + \sum_{\nu=2}^{\infty} a_\nu \tau^{\mu\nu+\alpha}\right)}{(z-\tau)^{\kappa-\sigma}} d\tau, \tag{3}$$

$$= \frac{\Gamma(\sigma)}{\Gamma(\kappa)\Gamma(1-\kappa+\sigma)} \left(z^{1-\sigma} \frac{d}{dz}\right) \int_0^z \frac{t^{\kappa}}{(z-\tau)^{\kappa-\sigma}} d\tau + \frac{\Gamma(\sigma)}{\Gamma(\kappa)\Gamma(1-\kappa+\sigma)} \left(z^{1-\sigma} \frac{d}{dz}\right) \sum_{\nu=2}^{\infty} a_\nu \int_0^z \frac{t^{\kappa-1+\mu\nu+\alpha}}{(z-\tau)^{\kappa-\sigma}} d\tau$$

and after a simple calculation, the operator $\mathrm{T}_{z,\mu}^{\kappa,\sigma}$ given by (3) is presented as:

$$\mathrm{T}_{z,\mu}^{\kappa,\sigma} f(z) = \frac{\Gamma(\sigma)\Gamma(\kappa+1)}{\Gamma(\sigma+1)\Gamma(\kappa)} z + \sum_{\nu=2}^{\infty} \frac{\Gamma(\sigma)\Gamma(\kappa+\mu\nu+\alpha)}{\Gamma(\kappa)\Gamma(\sigma+\mu\nu+\alpha)} a_\nu z^{\mu\nu+\alpha}. \tag{4}$$

In the normalization series, we considered to apply (4) to define the following new fractional linear operator $\chi_{z,\mu}^{\kappa,\sigma} f(z): \mathbb{A}_\mu \to \mathbb{A}_\mu$ by:

$$\chi_{z,\mu}^{\kappa,\sigma} f(z) = \left(\frac{\sigma}{\kappa}\right) T_{z,\mu}^{\kappa,\sigma} f(z), \quad z \in \mathcal{D}$$

$$= z + \sum_{\nu=2}^{\infty} \binom{\kappa+\mu\nu+\alpha-1}{\sigma+\mu\nu+\alpha-1} \binom{\kappa}{\kappa-\sigma}^{-1} a_\nu z^{\mu\nu+\alpha}, \tag{5}$$

$(0 < \sigma \le 1, 0 < \kappa \le 1$ and $0 \le \alpha \le 1$, $\mu(\alpha) \ge 1$ such that $\mu(0) = 1)$

where $\binom{a}{b} = \frac{\Gamma(a)}{\Gamma(b)\Gamma(a-b+1)}$ is the binomial coefficients.

Noted that, when $\mu(0) = 1$ in (5), then $\chi_{z,1}^{\kappa,\sigma} = T_z^{\kappa,\sigma}$ see [3].

Let $\mathbb{A}_\mu^- \subset \mathbb{A}_\mu$ be the class of all analytic and univalent functions with negative coefficients, defined by [1]:

$$f(z) = z - \sum_{\nu=2}^{\infty} |a_\nu| z^{\mu\nu+\alpha}, \quad z \in \mathcal{D}. \tag{6}$$

This paper aims to systematically investigate a new class $\mathbb{P}_\sigma^\kappa(\mu, \alpha, \varpi, \sigma)$ of fractional functions $f(z) \in \mathbb{A}_\mu$ which satisfies the following condition:

$$\left|\frac{z^{-1}\chi_{z,\mu}^{\kappa,\sigma}f(z)-1}{z^{-1}\chi_{z,\mu}^{\kappa,\sigma}f(z)+(1-2\varpi)}\right|<\sigma,\ z\in\mathcal{D} \tag{7}$$

$(0<\sigma\leq 1, 0<\kappa\leq 1,\ \ 0\leq\alpha\leq 1,\ \mu(\alpha)\geq 1,\ \ 0\leq\varpi<1$and $0\leq\sigma\leq 1)$.

For the class of functions belonging to $\mathbb{P}_{\sigma}^{\kappa}(\mu,\alpha,\varpi,\sigma)$, we demonstrate several sharp results, including coefficient of fractional function theorem. Inclusion theorem concerning the Hardy space of analytic fractional function is also established.

## 2. The class $\mathbb{P}_{\sigma}^{\kappa}(\mu,\alpha,\varpi,\sigma)$

In this section, we demonstrate the following result, which provides the coefficient bounds for functions $f(z)\in\mathbb{P}_{\sigma}^{\kappa}(\mu,\alpha,\varpi,\sigma)$.

**Theorem 1:** *Let $f(z)$ defined by (6), be in the class $\mathbb{P}_{\sigma}^{\kappa}(\mu,\alpha,\varpi,\sigma)$ if and only if*

$$\sum_{\nu=2}^{\infty}\binom{\kappa+\mu\nu+\alpha-1}{\sigma+\mu\nu+\alpha-1}\binom{\kappa}{\kappa-\sigma}^{-1}(1+\sigma)a_{\nu}\leq 2(1-\varpi). \tag{8}$$

***Proof:*** Let assuming that the inequality (8) holds true and $|z|=1$, then we have

$$\left|\frac{\chi_{z,\mu}^{\kappa,\sigma}f(z)}{z}-1\right|-\sigma\left|\frac{\chi_{z,\mu}^{\kappa,\sigma}f(z)}{z}+(1-2\varpi)\right|=$$

$$\left|-\sum_{\nu=2}^{\infty}\binom{\kappa+\mu\nu+\alpha-1}{\sigma+\mu\nu+\alpha-1}\binom{\kappa}{\kappa-\sigma}^{-1}a_{\nu}z^{\mu\nu+\alpha-1}\right|-$$

$$\sigma\left|2(1-\varpi)-\sum_{\nu=2}^{\infty}\binom{\kappa+\mu\nu+\alpha-1}{\sigma+\mu\nu+\alpha-1}\binom{\kappa}{\kappa-\sigma}^{-1}a_{\nu}z^{\mu\nu+\alpha-1}\right|\leq$$

$$\sum_{\nu=2}^{\infty}\binom{\kappa+\mu\nu+\alpha-1}{\sigma+\mu\nu+\alpha-1}\binom{\kappa}{\kappa-\sigma}^{-1}(1+\sigma)|a_{\nu}|-2\sigma(1-\varpi)\leq 0,$$

by hypothesis and applying the maximum modulus theorem, we obtain $f(z)\in\mathbb{P}_{\sigma}^{\kappa}(\mu,\alpha,\varpi,\sigma)$. Now, to prove the converse, let $f(z)\in\mathbb{P}_{\sigma}^{\kappa}(\mu,\alpha,\varpi,\sigma)$, so that the condition (7) readily yields

$$\left|\frac{z^{-1}\chi_{z,\mu}^{\kappa,\sigma}f(z)-1}{z^{-1}\chi_{z,\mu}^{\kappa,\sigma}f(z)+(1-2\varpi)}\right|=\frac{\left|\sum_{\nu=2}^{\infty}\binom{\kappa+\mu\nu+\alpha-1}{\sigma+\mu\nu+\alpha-1}\binom{\kappa}{\kappa-\sigma}^{-1}a_{\nu}z^{\mu\nu+\alpha-1}\right|}{\left|2(1-\varpi)-\sum_{\nu=2}^{\infty}\binom{\kappa+\mu\nu+\alpha-1}{\sigma+\mu\nu+\alpha-1}\binom{\kappa}{\kappa-\sigma}^{-1}a_{\nu}z^{\mu\nu+\alpha-1}\right|}<\sigma,\quad z\in\mathcal{D}. \tag{9}$$

For all $z$, we know that $\Re(z)\leq|\Re(z)|\leq|z|$, then from (9) we get

$$\Re\left[\left(\sum_{\nu=2}^{\infty}\binom{\kappa+\mu\nu+\alpha-1}{\sigma+\mu\nu+\alpha-1}\binom{\kappa}{\kappa-\sigma}^{-1}a_{\nu}z^{\mu\nu+\alpha-1}\right)/\right.$$

$$\left.2(1-\varpi)-\sum_{\nu=2}^{\infty}\binom{\kappa+\mu\nu+\alpha-1}{\sigma+\mu\nu+\alpha-1}\binom{\kappa}{\kappa-\sigma}^{-1}a_{\nu}z^{\mu\nu+\alpha-1}\right]$$

$$<\sigma,\quad z\in\mathcal{D}. \tag{10}$$

Given a value for $z$ on the real axis, is real accordingly. After the denominator (10) is cleared and $z^{-1}\chi_{z,\mu}^{\kappa,\sigma}f(z)$ is allowed to pass through real value, we have

$$\binom{\kappa}{\kappa-\sigma}^{-1}\sum_{\nu=2}^{\infty}\binom{\kappa+\mu\nu+\alpha-1}{\sigma+\mu\nu+\alpha-1}|a_{\nu}|\leq 2\sigma(1-\varpi)-\sigma\binom{\kappa}{\kappa-\sigma}^{-1}\sum_{\nu=2}^{\infty}\binom{\kappa+\mu\nu+\alpha-1}{\sigma+\mu\nu+\alpha-1}|a_{\nu}|,$$

which provides the necessary claim (8).

Finally, we observe that (8) of Theorem 1 is sharp and the extremal function is

$$f(z) = z - \sum_{\nu=2}^{\infty} \frac{2\sigma(1-\varpi)\Gamma(\kappa+1)\Gamma(\sigma+\mu\nu+\alpha)}{(1+\sigma)\Gamma(\sigma+1)\Gamma(\kappa+\mu\nu+\alpha)} z^{\mu\nu+\alpha}. \tag{11}$$

**Remark 1:** If $\mu(0) = 1,$ then $f(z)$ given by (6) is the Silverman class $\mathcal{T}^{\bowtie}(\varpi)$ if and only if [6]

$$\sum_{\nu=2}^{\infty} \left[\frac{\nu-\varpi}{1-\varpi}\right] |a_\nu| \leq 1. \tag{12}$$

Therefore, we have

$$\mathbb{P}_{\sigma}^{\kappa}(\mu, \alpha, \varpi, \sigma) \subset \mathcal{T}^{\bowtie}(\varpi), \tag{13}$$

for $0 < \sigma \leq 1,\ 0 < \kappa \leq 1, 0 \leq \alpha \leq 1,\ 0 \leq \varpi < 1$ and

$0 < \sigma \leq \min_{\nu\geq 2} \left\{\frac{1}{\varrho(\nu)-1}\right\}$, where

$$\varrho(\nu) = \frac{2(\mu\nu+\alpha-\varpi)\Gamma(\kappa+1)\Gamma(\sigma+\mu\nu+\alpha)}{(\mu\nu+\alpha)\,\Gamma(\sigma+1)\Gamma(\kappa+\mu\nu+\alpha)} \tag{14}$$

is positive and non-decreasing for $\nu \geq 2$, also

$$\lim_{\nu\to\infty} \varrho(\nu) = \begin{cases} 2, & if\ \mu(0) = 1, \quad \varpi = 0\ and\ \sigma = \kappa, \\ \infty, & if \quad \sigma, \kappa < 1. \end{cases}$$

Consequently, (13) holds true for $0 \leq \varpi < 1$ and $0 < \sigma \leq 1$ if and only if $\mu(0) = 1, \sigma = 1$ and $\sigma = \kappa$.

**Remark 2:** Theorem 1 simplifies to the comparable result due to Gupta and Jain [7] if $\mu(0) = 1,\ \sigma = \kappa$. It follows immediately that $\mathbb{P}_{\sigma}^{\kappa}(1,0,\varpi,\sigma) = \mathbb{P}(\varpi,\sigma)$.

The following observations are now presented, according to Theorem 1.

**Corollary 1:** Let $f(z)$ given by (6) belong to $\mathbb{P}_{\sigma}^{\kappa}(\mu, \alpha, \varpi, \sigma)$, then

$$a_\nu \leq \frac{2\sigma(1-\varpi)\Gamma(\kappa+1)\Gamma(\sigma+\mu\nu+\alpha)}{(1+\sigma)\Gamma(\sigma+1)\Gamma(\kappa+\mu\nu+\alpha)} \leq \nu, \quad \nu \geq 2. \tag{15}$$

## 3. The Hardy space

In this section, we consider studying the inclusion theorem involving the Hardy space $\mathcal{H}^{p}(0 < p \leq \infty)$ which is the class of all analytic functions $f(z)$ defined in $\mathcal{D}$, with the norm [8]:

$$\|f\|_{p} \coloneqq \lim_{r\to 1^-}\{\mathcal{M}_{p}(r,f)\} < \infty, \tag{16}$$

where

$$\mathcal{M}_{p}(r,f) \coloneqq \begin{cases} \left(\frac{1}{2\pi}\int_0^{2\pi}\left|f\left(re^{i\theta}\right)\right|^{p} d\theta\right)^{1/p}, & (0 < p < \infty) \\ sup_{|z|\leq r}\ |f(z)|, & (p = \infty). \end{cases} \tag{17}$$

For $\nu \geq 2$, we demonstrate the inclusion result by recalling the following:

**Lemma 1: [8]** *If* $f(z) \in \mathfrak{K}$, *then* $f \in \mathcal{H}^{p}(0 < p \leq \infty)$ .

In the following, we prove the inclusion outcome on the Hardy space in $\mathcal{D}$.

**Theorem 2:** *Let $f(z)$ given by (6) belong to the class $\mathfrak{K}(\mu,\alpha,t)$ , then*

$$\chi_{z,\mu}^{\kappa,\sigma} f(z) \in \mathcal{H}^{p},\ (0 < p \leq \infty), \tag{18}$$

and

$$\chi_{z,\mu}^{\kappa,\sigma} f(z) \in \mathcal{H}^{\infty},\ (\sigma - \kappa > 0), \tag{19}$$

***Proof:*** From (3) and (5), we find that

$$\Re\left(\frac{d}{dz}\chi_{z,\mu}^{\kappa,\sigma} f(z)\right) = \frac{\Gamma(\sigma+1)(\sigma-\kappa)}{\Gamma(\kappa+1)\Gamma(1-\kappa+\sigma)}\int_0^1 u^{\kappa-1}(1-u)^{\sigma-\kappa}\Re\big(f'(zu)\big)du. \tag{20}$$

Since $f(z) \in \mathfrak{K}(\mu,\alpha,t)$, it follows that $\chi_{z,\mu}^{\kappa,\sigma} f(z) \in \mathfrak{K}(\mu,\alpha,t)$, by applying lemma 1, yields assentation (18). To prove (19), we need the following

$$\frac{d}{dz}\chi_{z,\mu}^{\kappa,\sigma} f(z) = z^{-1}\{(1+\sigma)\,\chi_{z,\mu}^{\kappa,\sigma-1} f(z) - \sigma\,\chi_{z,\mu}^{\kappa,\sigma} f(z)\}, \tag{21}$$

which meets the following requirements:

$$\left|\frac{d}{dz}\chi_{z,\mu}^{\kappa,\sigma} f(z)\right|^{p} \leq r^{-1}\left\{(1+\sigma)^{p}\left|\chi_{z,\mu}^{\kappa,\sigma-1} f(z)\right|^{p} + \sigma^{p}\left|\chi_{z,\mu}^{\kappa,\sigma} f(z)\right|^{p}\right\} \tag{22}$$

for $|z| = r$. In view of (16) and (17), the inequality (22) with $p = 1$gives

$$\mathcal{M}_1\left(r,\frac{d}{dz}\chi_{z,\mu}^{\kappa,\sigma} f(z)\right) \leq$$

$$r^{-1}\left\{(1+\sigma)\mathcal{M}_1\left(r,\chi_{z,\mu}^{\kappa,\sigma-1} f(z)\right) + \sigma\mathcal{M}_1\left(r,\chi_{z,\mu}^{\kappa,\sigma} f(z)\right)\right\} \tag{23}$$

and

$$\left\|\frac{d}{dz}\chi_{z,\mu}^{\kappa,\sigma} f(z)\right\|_1 \leq (1+\sigma)\left\|\chi_{z,\mu}^{\kappa,\sigma-1} f(z)\right\|_1 + \sigma\left\|\chi_{z,\mu}^{\kappa,\sigma} f(z)\right\|_1. \tag{24}$$

By considering (24), we get $\chi_{z,\mu}^{\kappa,\sigma-1} f(z) \in \mathcal{H}^1$and $\chi_{z,\mu}^{\kappa,\sigma} f(z) \in \mathcal{H}^1(\sigma \leq 1)$. Hence, $\frac{d}{dz}\chi_{z,\mu}^{\kappa,\sigma} f(z) \in \mathcal{H}^1$. We conclude from the above results that $\chi_{z,\mu}^{\kappa,\sigma} f(z)$ is continuous in $|z| \leq 1$. And, since $|z| \leq 1$ is compact then $\chi_{z,\mu}^{\kappa,\sigma} f(z)$ is bounded analytic function in $|z| < 1$, and implies (19).

## 4. Conclusion

The definition of a novel subclass of fractional functions in terms of fractional differential operators is thoroughly investigated. Various aspects of this subclass, such as coefficient inequality and application to the Hardy space of analytic functions, are the main focus of the study.

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