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Build new operator using cluster system

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Abstract

The new $\mathbb{C}_\int$-operator construction procedure is centered on the cluster system. Some features of the $\mathbb{C}_\int$-operator without conditions have been examined in this paper. Specific requirements must be met in order to gain specific qualities. Additionally, we introduced a few theories that connect the ideals of $\mathbb{C}_\int$-operator and $\int$-operator. Additionally, we demonstrated the degree to which the πnet-work influenced the $\mathbb{C}_\int$-operator's characteristics and established an equivalency between them. This study refers to a space $(\mathbb{V}, \mathfrak{I})$ that has a cluster system $\int$ on $\mathbb{V}$ as $\int$ topological space, which is represented as $CS\mathfrak{I}S$ $\mathbb{V}_\int^{\mathfrak{I}}$.

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1. Introduction

Many researchers and those interested in the field of mathematics have tried to develop new and advanced concepts of operators through which they provide a broader topology than the usual one and compare them in terms of construction and properties, with or without any conditions. Among these scientists is Kuratowski [1], who introduced the definition of ideal space in 1966. In 1947, Choquet [2], introduced the idea of grill space, and each of them developed a deeper and broader topology by building families for these definitions. A new idea for grouping sets in topological spaces was presented by Matejdes [3] in 1984. He named it a cluster system which is a non-empty system of the form $\int \subseteq 2^{\mathbb{V}}/\{\emptyset\}$, and $2^{\mathbb{V}}$ is the power set. He examined some of its

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features and used it to define an operator on the set ζ, which he named the $\int$-operator. Using the $\int$-operator under specific circumstances, Thangamariappan and Renukadevi [4] provided a more comprehensive explanation of topology in 2015. If $\zeta = Int_{\mathfrak{I}}(CL_{\mathfrak{I}}(\zeta))$ is called regular open [5]. By $CL_{\mathfrak{I}}(\zeta)$ and $Int_{\mathfrak{I}}(\zeta)$, represent the closure and interior of subset ζ in $(\mathbb{V}, \mathfrak{I})$, respectively.

There are three portions to the work: The analysis of the main traits of the$\int$-operator is covered in section 2. The most significant aspects of the newly defined operator, which we named the $\not\subset_{\int}$-operator, were presented in section 3. In section 4, we use a condition πnet-work in $\mathbb{V}$ to show a few characteristics.

2. Preliminaries

First, we need some concepts and basics in terms of theories, definitions, and ideas, which are considered the basic foundation from which we will start our research, and we will address them briefly.

Definition 2.1: [4] Let $\mathbb{V}_{\int}^{\mathfrak{I}}$ be a $CS\mathfrak{I}S$. So $\int$ satisfy the:

i) Satisfy I, if $\mu_1, \mu_2 \in \int$,so $\mu_1 \cap \mu_2 \in \int$.

ii) Satisfy H, if for any $\maltese \in \mathfrak{I}_{(v)}$ and $\zeta, B \subseteq \mathbb{V} \exists\, \mu \in \int \ni \mu \subseteq \maltese \cap (\zeta \cup B)$, so $\exists\, \mu_1, \mu_2 \in \int \ni \mu_1 \subseteq \maltese \cap \zeta$ or $\mu_2 \subseteq \maltese \cap B$.

Definition 2.2: Let $\mathbb{V}_{\int}^{\mathfrak{I}}$ be a $CS\mathfrak{I}S$. So $\int$ satisfy the property C whenever $\emptyset \neq \mathrm{P} \subseteq \zeta$ and $\zeta \in \int$ so $\mathrm{P} \in \int$.

Definition 2.3: [2] Let $\mathbb{V}_{\int}^{\mathfrak{I}}$ be a $CS\mathfrak{I}S$. So $\int$ satisfy the property stack whenever $\zeta \in \int$ and $\zeta \subseteq \mathrm{P}$, so $\mathrm{P} \in \int$.

Definition 2.4: [6] Let $\mathbb{V}_{\int}^{\mathfrak{I}}$ be a $CS\mathfrak{I}S$. And $\zeta \subseteq \mathbb{V}$, we define $\int(\zeta) = \{v \in \mathbb{V}: \forall \maltese \in \mathfrak{I}_{(v)}\ \exists \mu \in \int \ni \mu \subseteq \maltese \cap \zeta\}$, is called $\int$-operator of ζ. In other word:

1) $v \in \int(\zeta)$ iff $\forall \maltese \in \mathfrak{I}_{(v)}\ \exists\, \mu \in \int \ni \mu \cap \left(\mathbb{V} - (\maltese \cap \zeta)\right) = \emptyset$.

2) $v \notin \int(\zeta)$ iff $\exists\, \maltese \in \mathfrak{I}_{(v)}\ \forall \mu \in \int \ni \mu \cap \left(\mathbb{V} - (\maltese \cap \zeta)\right) \neq \emptyset$.

Example 2.5: Let $\mathbb{V} = (-\infty, 0)$ with $\mathfrak{I} = \{\mathbb{V}, \emptyset, (-\infty, s): s \in \mathbb{V}\}$ and $\int = \{(q-1, w): q \in Z^- \cup \{0\}\}$. If $\zeta = [-5.5, -4.5] \cup (-2, -1)$. So $\int(\zeta) = [-1,0]$. Where Z^- is a set negative integer.

Proposition 2.6: [6] Let $\mathbb{V}_{\int}^{\mathfrak{I}}$ be a $CS\mathfrak{I}S$. The following hold. For $\zeta, \maltese \subseteq \mathbb{V}$

a) $\int(\emptyset) = \emptyset$.

b) $\int(\zeta)$ is closed.

c) $CL_{\mathfrak{I}}(\zeta) \supseteq \int(\zeta)$.

d) If $\zeta \subseteq \maltese$, so $\int(\zeta) \subseteq \int(\maltese)$.

Proposition 2.7: [4] Let $\mathbb{V}_{\int}^{\mathfrak{J}}$ be a $CS\mathfrak{J}S$. Any $Ȥ \subseteq \mathbb{V}$. The following hold:

1) $\int(Ȥ) \supseteq \int(\int(Ȥ))$.
2) If $\int_1$and $\int_2$ are a cluster system on $\mathbb{V}$ and $\int_1 \subseteq \int_2$, so $\int_1(Ȥ) \subseteq \int_2(Ȥ)$.
3) $₦ \cap \int(₦ \cap Ȥ) = ₦ \cap \int(Ȥ) \subseteq \int(Ȥ \cap ₦)$, if $₦ \in \mathfrak{J}$.
4) For any $E \in \int$, $Int_{\mathfrak{J}}(E) \neq \emptyset$, , so $CL_{\mathfrak{J}}\big(Int_{\mathfrak{J}}(Ȥ)\big) \supseteq \int(Ȥ)$.

Proposition 2.8: [4] Let $\mathbb{V}_{\int}^{\mathfrak{J}}$ be a $CS\mathfrak{J}S$. The properties hold, for $Ȥ, B \subseteq \mathbb{V}$.

i) If $\int$ is satisfy H. So
   1) $\int(Ȥ \cup Ȩ) = \int(Ȥ) \cup \int(Ȩ)$.
   2) $\int(Ȥ - Ȩ) - \int(Ȩ) = \int(Ȥ) - \int(Ȩ) \subseteq \int(Ȥ - Ȩ)$.
ii) If $\int$ is satisfy C, so $\int(Ȥ) = CL_{\mathfrak{J}}(Ȥ)$.
iii) If $\int$ is satisfy I, so $\int(Ȥ \cap Ȩ) = \int(Ȥ) \cap \int(Ȩ)$.

Definition 2.9: [6] Let $\mathbb{V}_{\int}^{\mathfrak{J}}$ be a $CS\mathfrak{J}S$. For any $Q, Ķ \subseteq \mathbb{V}$ if every $\emptyset \neq Q \in \mathfrak{J}$, and $\emptyset \neq Ķ \in \mathfrak{J} \ \exists \ Ȩ \in \int \ni Ȩ \subseteq Q \subseteq Ķ$, so $\int$ πnet-work in $Ķ$.

Proposition 2.10: [4] Let $\mathbb{V}_{\int}^{\mathfrak{J}}$ be a $CS\mathfrak{J}S$. Any $Ȥ \subseteq \mathbb{V}$ and $Q \in \mathfrak{J}$ then:

1) $\int$ a πnet-work in Q if and only if $\int(Q) = \int\big(CL_{\mathfrak{J}}(Q)\big) = CL_{\mathfrak{J}}(Q)$.
2) $CL_{\mathfrak{J}}(Int_{\mathfrak{J}}(Ȥ)) \subseteq \int(Ȥ)$, whenever $\int$ a πnet-work in $\mathbb{V}$.
3) $\int(\int(Ȥ)) = \int(Ȥ)$, whenever $\int$ πnet-work and $Int_{\mathfrak{J}}(ӈ) \neq \emptyset, \forall ӈ \in \int$.
4) $\int(\mathbb{V}) = \mathbb{V}$ iff $\int$ πnet-work in $\mathbb{V}$.

3. The New $\not\subset_{\int}$-Operator

Our investigation will require some crucial properties associated with the study of the concepts $\int$-operator. Through which we will build a new topology that is broader than the original topology.

Theorem 3.1: *Let $\mathbb{V}_{\int}^{\mathfrak{J}}$ be a $CS\mathfrak{J}S$, so $\partial - \int(\partial) \cap \int\big(\partial - \int(\partial)\big) = \emptyset$. For $\partial \subseteq \mathbb{V}$.*

Proof: Let $v \in [\partial - \int(\partial) \cap \int(\partial - \int(\partial))]$, so $x \in [\partial - \int(\partial)] \wedge x \in \int[\partial - \int(\partial)]$, we have $v \in \int(\partial)$, so $\exists U \in \mathfrak{J}_v \forall ӈ_1 \in \int \ni [\mathbb{V} - (U \cap \partial)] \cap ӈ_1 \neq \emptyset$...(1). Now, $v \in \int(\partial - \int(\partial))$, we get $\forall V \in \mathfrak{J}_{(v)} \ \exists ӈ_2 \in \int \ni ӈ_2 \cap [\mathbb{V} - (V \cap (\partial - \int(\partial)))] = \emptyset$, we have $ӈ_2 \cap \big[\mathbb{V} - \big(V \cap (\partial \cap \mathbb{V} - (\int(\partial)))\big)\big] = \emptyset$, so $ӈ_2 \cap \big(\mathbb{V} - (U \cap \partial)\big) \subseteq ӈ_2 \cap [\mathbb{V} - (U \cap \partial \cap (\mathbb{V} - \int(\partial)))] = \emptyset$, therefore $ӈ_2 \cap (\mathbb{V} - (U \cap \partial)) = \emptyset$... (2). Contradiction by (1).

Theorem 3.2: *For every $v \in \mathbb{V}, ₦ \subseteq \mathbb{V}$ and $\int_v = \{Ȥ \subseteq \mathbb{V}: v \in Ȥ\}$ is a $CS\mathfrak{J}S$ on $\mathbb{V}$, so* $\int_v(₦) = \begin{cases} \emptyset & If\ v \notin ₦ \\ CL_{\mathfrak{J}}(\{v\}) & If\ v \in ₦ \end{cases}$

Proof: For any $v \in \mathbb{V}$, if $v \notin ₦$, and it was $\int_v(₦) \neq \emptyset$,i.e. there $p \in \int_v(₦)$, therefore, for any $Ꮟ \in \Im_{(v)}$, there $E \in \int_v \ni v \in E \subseteq Ꮟ \cap ₦$, then $v \in ₦$, which runs counter to the hypothesis. Suppose that, for $v \in ₦$, $p \in \int_v(₦)$, then for any $V \in \Im_{(v)}$, there, $E \in \int_v \ni v \in E \subseteq \mathbb{V} \cap ₦$ therefore $v \in \mathbb{V}$, in a different sense $p \in CL_{\Im}(\{ v\})$. Conversely if $p \in CL_T(\{v\})$, then every $\mathbb{V} \in \Im_{(p)}$ and $\mathbb{V} \cap \{v\}$ non empty, so $v \in \mathbb{V}$, and $\{v\} \subseteq \mathbb{V} \cap ₦$,this is accurate, providing us with $p \in \int_v(₦)$.

Theorem 3.3: *Let* $\mathbb{V}_{\int}^{\Im}$ *be a* $CS\Im S$. *If* $\int$ *is satisfy the property stack and any* $ꞑ \notin \int$, *so* $\int(ꞑ) = \emptyset$.

Theorem 3.4: *Let* $\mathbb{V}_{\int}^{\Im}$ *be* $CS\Im S$. *If* $\int$ *satisfy the property* H. *And* $\partial \subseteq \mathbb{V}$ *so, the two that follow can be compared.*

1) If $\partial \cap \int(\partial) = \emptyset$. Then, $\int(\partial) = \emptyset$.
2) $\int\big(\partial - \int(\partial)\big) = \emptyset$.
3) $\int(\partial) = \int\big(\partial \cap \int(\partial)\big)$.

Theorem 3.5: *Let* $\mathbb{V}_{\int}^{\Im}$ *be a* $CS\Im S$ *and any* $\partial \subseteq \mathbb{V}$. *If* $\int$ *satisfy the satisfy H, the stack property and* $\forall\, ꞑ \in \int$, *then* $\int(\partial \cup ꞑ) = \int(\partial) = \int(\partial - ꞑ)$.

In this theorem we will introduce the equivalent definition of the π net-work which we will use to prove many fundamental facts and theorems.

Theorem 3.6: *Let* $\mathbb{V}_{\int}^{\Im}$ *be a* $CS\Im S$. *Equivalent sentences.*

1) $\int$ is πnet-work in G, for $Ɱ \in \Im/\{\emptyset\}$.
2) Every $Ɱ \in \Im/\{\emptyset\}$, then $Ɱ \subseteq \int(Ɱ)$.
3) $\int(\mathbb{V}) = \mathbb{V}$.

Definition 3.7: Let $\mathbb{V}_{\int}^{\Im}$ be a $CS\Im S$. An operator $\not\subset_{\int}: 2^{\mathbb{V}} \to \Im$ is defined, any $Ⱬ \subseteq \mathbb{V}$, $\not\subset_{\int}(Ⱬ) = \{v \in \mathbb{V}: \exists U \in \Im_{(v)}\, \forall ꞑ \in \int \ni ꞑ \not\subseteq U \cap (\mathbb{V} - Ⱬ)\}$.

Remark 3.8:

i) $v \in \not\subset_{\int}(Ⱬ)$ iff $\exists U \in \Im_{(v)}\, \forall ꞑ \in \int \ni ꞑ \cap (\mathbb{V} - (U \cap (\mathbb{V} - Ⱬ))) \neq \emptyset$.

ii) $v \notin \not\subset_{\int}(Ⱬ)$ iff $\forall U \in \Im_{(v)}\, \exists ꞑ \in \int \ni ꞑ \cap \Big(\mathbb{V} - \big(U \cap (\mathbb{V} - Ⱬ)\big)\Big) = \emptyset$.

iii) $\not\subset_{\int}(Ⱬ) = \mathbb{V} - \int(\mathbb{V} - Ⱬ)$.

iv) $\int(Ⱬ) = \mathbb{V} - \not\subset_{\int}(\mathbb{V} - Ⱬ)$.

Example 3.9: Let $\mathbb{V} = \{¢_1, ¢_2, ¢_3\}$ with $\int = \{\{¢_1\}, \{¢_3\}\}$ and $\Im = \{\mathbb{V}, \emptyset, \{¢_1\}, \{¢_2\}, \{¢_1, ¢_2\}\}$. Then the operator $\not\subset_{\int}(Ⱬ)$, for $Ⱬ \subseteq \mathbb{V}$ is: $\not\subset_{\int}(\{¢_1\}) = \{¢_1, ¢_2\}$,

$\int(\{¢_2\}) = \{¢_2\}$, $\not{C}_\int(\{¢_3\}) = \{¢_2\}, \not{C}_\int(\{¢_1, ¢_2\}) = \{¢_1, ¢_2\}$, $\not{C}_\int(\{¢_1, d\}) = \mathbb{V}, \not{C}_\int(\{¢_2, ¢_3\}) = \{¢_2\}, \not{C}_\int(\mathbb{V}) = \mathbb{V}$.

Now we will review the important characteristics of the research in the theory below. $\not{C}_\int$-operator that work unconditionally and which in turn will be essential in the proof of the fundamental theorems.

Theorem 3.10: Let $\Im_\int^\Im$ be a $CS\Im S$, for any $Ꝣ \subseteq \mathbb{V}$. Hold the following:

1) $\not{C}_\int(Ꝣ) = \not{C}_\int(\not{C}_\int(Ꝣ))$ iff $\not{C}_\int(\mathbb{V} - Ꝣ) = \int(\int(\mathbb{V} - Ꝣ))$.
2) $\not{C}_\int(\mathbb{V}) = \mathbb{V}$.
3) If $Ꝣ \subseteq S$, so $\not{C}_\int(Ꝣ) \subseteq \not{C}_\int(S)$.
4) If $V \in \Im$, so $V \subseteq \not{C}_\int(V)$.
5) $\not{C}_\int(Ꝣ)$ is open.
6) $\not{C}_\int(Ꝣ) \supseteq \mathrm{Int}_\Im(Ꝣ)$.
7) $\not{C}_{\int_2}(Ꝣ) \subseteq \not{C}_{\int_1}(Ꝣ)$, whenever $\int_1 \subseteq \int_2$.
8) If $\emptyset \neq Ꝏ \subseteq X$, then $\not{C}_\int(Ꝏ \cup Ꝣ) \supseteq Ꝏ \cup \not{C}_\int(Ꝣ) = Ꝏ \cup \not{C}_\int(Ꝏ \cup Ꝣ)$.
9) $\not{C}_\int(Ꝣ) \subseteq \not{C}_\int(\not{C}_\int(Ꝣ))$.
10) If $\forall E \in \int$, $Int_\Im(E) \neq \emptyset$, , then $Int_\Im(CL_\Im(Ꝣ)) \subseteq \not{C}_\int(Ꝣ)$.

In the following note, we will present the properties that work on certain basic satisfies, and we will also show that without these satisfies these properties do not work by creating some examples.

Remark 3.11:

1) By theorem 3.1 show that. $Ꝣ \cup \left(\mathbb{V} - \not{C}_\int(Ꝣ)\right) \cup \not{C}_\int\left(Ꝣ \cup \left(\mathbb{V} - \not{C}_\int(Ꝣ)\right)\right) = \mathbb{V}$. Any $Ꝣ \subseteq \mathbb{V}$.
2) By remark 3.8. It is clear that. Then $\not{C}_\int(\emptyset) \neq \emptyset$. By example 3.9 and $\not{C}_\int(\emptyset) = \{l\}$.
3) If $\int$ satisfy H, so $\not{C}_\int(Ꝣ \cap Ẹ) = \not{C}_\int(Ꝣ) \cap \not{C}_\int(Ẹ)$. Every $Ꝣ, Ẹ \subseteq \mathbb{V}$. The condition cannot be removed. Let $\mathbb{V} = \{¢_1, ¢_2, ¢_3\}$ and $\Im = \{X, \emptyset, \{¢_1, ¢_3\}\}$ with $\int = \{\{¢_1, ¢_3\}, \{¢_2, ¢_3\}\}$. If $Ꝣ = \{¢_1\}, Ẹ = \{¢_1, ¢_3\}$, where $\not{C}_\int(Ꝣ) = \{¢_1, ¢_3\}$, $\not{C}_\int(Ẹ) = \mathbb{V}$. Then $\not{C}_\int(Ꝣ \cap Ẹ) = \not{C}_\int(Ꝣ) \cap \not{C}_\int(Ẹ) = \{¢_1, ¢_3\}$. But cluster system $\int$ does not achieve the satisfy H, because if $M = \{¢_2\}$, $K = \{¢_3\}$ and $E = \{¢_2, ¢_3\}$, then $E \subseteq \mathbb{V} \cap (M \cup K)$, but $\mathbb{V} \cap M$ and $\mathbb{V} \cap K$ does not contain any of the elements $\int$.
4) If $\int$ with satisfy I, so $\not{C}_\int(Ꝣ \cup Ẹ) = \not{C}_\int(Ꝣ) \cup \not{C}_\int(Ẹ)$. Every $Ꝣ, Ẹ \subseteq \mathbb{V}$. The condition cannot be removed. Let $\mathbb{V} = \{¢_1, ¢_2, ¢_3\}$, $\Im = \{\mathbb{V}, \emptyset, \{¢_2, ¢_3\}, \{¢_3\}\}$ with $\int = \{\{¢_1, ¢_2\}, \{¢_2, ¢_3\}\}$. If $Ꝣ = \{¢_1\}$ and $Ẹ = \{¢_2\}$ where $\not{C}_\int(Ꝣ) = \{¢_3\}$,

$\not\subset_{\int}(Ꞡ) = \mathbb{V}$ and $\not\subset_{\int}(Ꝣ \cup Ꞡ) = \mathbb{V}$. Then $\not\subset_{\int}(Ꝣ \cup Ꞡ) = \not\subset_{\int}(Ꝣ) \cup \not\subset_{\int}(Ꞡ) = \mathbb{V}$. But I is not satisfied by cluster system $\int$.

5) If $\int$ with satisfy H, Then $\not\subset_{\int}(Ꝣ) - \not\subset_{\int}(Ꞡ) = \not\subset_{\int}(Ꝣ) - \not\subset_{\int}\big(\mathbb{V} - (Ꝣ - Ꞡ)\big) \subseteq \mathbb{V} - [\not\subset_{\int}\big(\mathbb{V} - (Ꝣ - Ꞡ)\big)]$. Any $Ꝣ, Ꞡ \subseteq \mathbb{V}$. The condition cannot be removed. Let $\mathbb{V} = \{¢_1, ¢_2, ¢_3\}$ and $\mathfrak{I} = \{\mathbb{V}, \emptyset, \{¢_1, ¢_3\}\}$ with $\int = \{\{¢_1, ¢_3\}, \{¢_2, ¢_3\}\}$. If $Ꝣ = \{¢_1, ¢_2\}$ and $B = \{¢_2\}$, $\not\subset_{\int}(\{¢_1, ¢_2\}) = \mathbb{V}$, $\not\subset_{\int}(\{¢_2\}) = \emptyset$ with $\not\subset_{\int}(\{¢_2, ¢_3\}) = \mathbb{V}$. If $\mathbb{V} - \left[\not\subset_{\int}(\mathbb{V} - \{¢_1\})\right] = \mathbb{V} - [\not\subset_{\int}(\{¢_2, ¢_3\})] = \emptyset$. So $\not\subset_{\int}(Ꝣ) - \not\subset_{\int}(Ꞡ) \not\subseteq \mathbb{V} - [\not\subset_{\int}\big(\mathbb{V} - (Ꝣ - Ꞡ)\big)]$. But H is not satisfied by cluster system $\int$.

6) If $\int$ has satisfy C, so $\mathrm{Int}_{\mathfrak{I}}(Ꝣ) = \not\subset_{\int}(Ꝣ)$, every $Ꝣ \subseteq \mathbb{V}$. The condition cannot be removed. Let $\mathbb{V} = \{¢_1, ¢_2, ¢_3\}, \mathfrak{I} = \{\mathbb{V}, \emptyset, \{¢_2\}\}$ with $\int = \{\{¢_1, ¢_3\}\}$. If $Ꝣ = \{¢_2, ¢_3\}$, also $\not\subset_{\int}(\{¢_2, ¢_3\}) = \mathbb{V}$ and $Int_{\mathfrak{I}}(\{¢_2, ¢_3\}) = \{¢_2\}$. Then $\not\subset_{\int}(Ꝣ) \neq Int_{\mathfrak{I}}(Ꝣ)$. But C is not satisfied by cluster system $\int$.

4. πNet-Work via The $\not\subset_{\int}$-Operator

Through which we show that this satisfy is necessary to obtain new features and without it we cannot obtain them.

Theorem 4.1: *Let $\mathbb{V}_{\int}^{\mathfrak{I}}$ be a $CS\mathfrak{I}S$ and $\emptyset \neq Ꝣ \subseteq \mathbb{V}$, is closed. Then $\int$ πnet-work in $\mathbb{V} - Ꝣ$ iff $\not\subset_{\int}(Ꝣ) = \not\subset_{\int}\big(Int_{\mathfrak{I}}(Ꝣ)\big) = Int_{\mathfrak{I}}(Ꝣ)$.*

Proof: $\Longrightarrow)\not\subset_{\int}\big(Int_{\mathfrak{I}}(Ꝣ)\big) = \mathbb{V} - \int(\mathbb{V} - Int_{\mathfrak{I}}(Ꝣ)) = \mathbb{V} - \int(CL_T(\mathbb{V} - Ꝣ))$, then $\not\subset_{\int}\big(Int_{\mathfrak{I}}(Ꝣ)\big) = \not\subset_{\int}(Ꝣ)$. (From proposition 2.10, 1). Now, $Int_{\mathfrak{I}}(Ꝣ) = \mathbb{V} - CL_{\mathfrak{I}}(\mathbb{V} - Ꝣ) = \mathbb{V} - \int(\mathbb{V} - Ꝣ)$, we have $\not\subset_{\int}\big(Int_{\mathfrak{I}}(Ꝣ)\big) = \not\subset_{\int}(Ꝣ) = Int_{\mathfrak{I}}(Ꝣ)$. (From Proposition 2.10,1).

$\Longleftarrow)$ Since, $\not\subset_{\int}(Ꝣ) = Int_{\mathfrak{I}}(Ꝣ)$ iff $\mathbb{V} - \int(\mathbb{V} - Ꝣ) = \mathbb{V} - CL_{\mathfrak{I}}(\mathbb{V} - Ꝣ)$ iff $\int(\mathbb{V} - Ꝣ) = CL_{\mathfrak{I}}(\mathbb{V} - Ꝣ)$,then $\int$ πnet-work in $\mathbb{V} - Ꝣ$.

Theorem 4.2: *Let $\mathbb{V}_{\int}^{\mathfrak{I}}$ be a $CS\mathfrak{I}S$, then $\not\subset_{\int}(Ꝣ) \subseteq Int_{\mathfrak{I}}\big(CL_{\mathfrak{I}}(Ꝣ)\big)$, Whenever $\int$ πnet-work, for $Ꝣ \subseteq \mathbb{V}$.*

Theorem 4.3: *Let $\mathbb{V}_{\int}^{\mathfrak{I}}$ be a $CS\mathfrak{I}S$, $\not\subset_{\int}\left(\not\subset_{\int}(Ꝣ)\right) = \not\subset_{\int}(Ꝣ)$, whenever $\int$ πnet-work and $Int_{\mathfrak{I}}(E) \neq \emptyset\ \forall E \in \int$ for $Ꝣ \subseteq \mathbb{V}$.*

Remark 4.4: The example demonstrates that Theorem 4.3 fulfill cannot be dropped.

Example 4.5: Let $\mathbb{V} = \{¢_1, ¢_2, ¢_3\}$, with $\int = \{\{¢_1\}, \{¢_3\}\}$, $\mathfrak{I} = \{\mathbb{V}, \emptyset, \{¢_1\}, \{¢_2\}, \{¢_1, ¢_2\}\}$. If $Ꝣ = \{¢_1\}$, then $\int(\mathbb{V}) \neq \mathbb{V}$. And $\not\subset_{\int}\left(\not\subset_{\int}(\{¢_1\})\right) = \not\subset_{\int}(\{¢_1\}) = \{¢_1, ¢_2\}$.

Theorem 4.6: *Let $\mathbb{V}_{\int}^{\mathfrak{I}}$ be a $CS\mathfrak{I}S$. $\not\subset_{\int}(\emptyset) = \emptyset$ iff $\int$ πnet-work in $\mathbb{V}$.*

Theorem 4.7: *Let $\mathbb{V}_{\int}^{\mathfrak{I}}$ be a $CS\mathfrak{I}S$, for any $S \subseteq \mathbb{V}$. If $\int$ is satisfy satisfy H and satisfy stack. Equivalence below in points.*

1) $\int$ is πnet-work in $\mathbb{V}$.
2) If S is closed, then $\not\subset_{\int}(S) - S = \emptyset$.
3) $\not\subset_{\int}(\mathrm{Int}_{\mathfrak{I}}(CL_{\mathfrak{I}}(S))) = \mathrm{Int}_{\mathfrak{I}}(CL_{\mathfrak{I}}(S))$.
4) If S is regular open, then $S = \not\subset_{\int}(S)$.
5) If $\emptyset \neq K \in \mathfrak{I}$, then $\not\subset_{\int}(K) \subseteq \mathrm{Int}_{\mathfrak{I}}(CL_{\mathfrak{I}}(K)) \subseteq \int(K)$.
6) If $E \notin \int$, so $\not\subset_{\int}(E) = \emptyset$.

Proof: $1 \longrightarrow 2)$ Since $S = \not\subset_{\int}(S) \subseteq \int(S) \subseteq CL_{\mathfrak{I}}(S) = S$, then $\not\subset_{\int}(S) \subseteq S$, so $\not\subset_{\int}(S) - S = \emptyset$. (From proposition 2.6, part 3).

$2 \to 3)$ Since $\mathrm{Int}_{\mathfrak{I}}(CL_{\mathfrak{I}}(S))$ is open we have $\mathrm{Int}_{\mathfrak{I}}(CL_{\mathfrak{I}}(S)) \subseteq \not\subset_{\int}(\mathrm{Int}_{\mathfrak{I}}(CL_{\mathfrak{I}}(S)))\ldots(1)$. (From theorem 3.10, part 2).

But $CL_{\mathfrak{I}}(S)$ is closed and $\not\subset_{\int}(CL_{\mathfrak{I}}(S)) \subseteq CL_{\mathfrak{I}}(S)$ and have $\not\subset_{\int}(CL_{\mathfrak{I}}(S)) = \mathrm{Int}_{\mathfrak{I}}\left(\not\subset_{\int}(CL_{\mathfrak{I}}(S))\right) \subseteq \mathrm{Int}_{\mathfrak{I}}(CL_{\mathfrak{I}}(S))$, then $\not\subset_{\int}(\mathrm{Int}_{\mathfrak{I}}(CL_{\mathfrak{I}}(S))) \subseteq \not\subset_{\int}(CL_{\mathfrak{I}}(S)) \subseteq \mathrm{Int}_{\mathfrak{I}}(CL_{\mathfrak{I}}(S))\ldots(2)$. From (1) and (2), therefore $\not\subset_{\int}\left(\mathrm{Int}_{\mathfrak{I}}(CL_{\mathfrak{I}}(S))\right) = \mathrm{Int}_{\mathfrak{I}}(CL_{\mathfrak{I}}(S))$. (From theorem 3.10, part 3).

$3 \to 4)$ Since S is regular open, then $\not\subset_{\int}(S) = \not\subset_{\int}\left(\mathrm{Int}_{\mathfrak{I}}(CL_{\mathfrak{I}}(S))\right)$ $\not\subset_{\int}(S) = \not\subset_{\int}\left(\mathrm{Int}_{\mathfrak{I}}(CL_{\mathfrak{I}}(S))\right) \subseteq \mathrm{Int}_{\mathfrak{I}}(CL_{\mathfrak{I}}(S)) = S$. (From part 2).

$4 \to 5)$ Let $K \in \mathfrak{I}$, then $K \subseteq \int(K)$ and $CL_{\mathfrak{I}}(K) \subseteq CL_{\mathfrak{I}}(\int(K))$. (From proposition 2.7, part 3 and from proposition 2.6, part 2).

Now $K = \mathrm{Int}_{\mathfrak{I}}(K) \subseteq \mathrm{Int}_{\mathfrak{I}}(CL_{\mathfrak{I}}(K)) \subseteq CL_{\mathfrak{I}}(K) \subseteq \int(K)$ …… (1), then $\mathrm{Int}_{\mathfrak{I}}(CL_{\mathfrak{I}}(K))$, then $\not\subset_{\int}(K) \subseteq \not\subset_{\int}(\mathrm{Int}_{\mathfrak{I}}(CL_{\mathfrak{I}}(K))) \subseteq \mathrm{Int}_{\mathfrak{I}}(CL_{\mathfrak{I}}(K)) \subseteq \int(K)$ …… (2), (from (1) and (2), then $\not\subset_{\int}(K) \subseteq \not\subset_{\int}(\mathrm{Int}_{\mathfrak{I}}(CL_{\mathfrak{I}}(K))) \subseteq \mathrm{Int}_{\mathfrak{I}}(CL_{\mathfrak{I}}(K)) \subseteq \int(K)$. Therefore, $\not\subset_{\int}(K) \subseteq \mathrm{Int}_{\mathfrak{I}}(CL_{\mathfrak{I}}(K)) \subseteq \int(K)$.

$5 \to 6)$ If $E \notin \int$, then $\not\subset_{\int}(E) = \mathbb{V} - \int(\mathbb{V})$, but $\mathbb{V} \in \mathfrak{I}$, we get $\not\subset_{\int}(\mathbb{V}) \subseteq \mathrm{Int}_{\mathfrak{I}}(CL_{\mathfrak{I}}(\mathbb{V})) = \mathbb{V} \subseteq \int(\mathbb{V})$, then $\int(\mathbb{V}) = \mathbb{V}$, so $\not\subset_{\int}(E) = \emptyset$.

$6 \to 1)$ Since $\emptyset \notin \int$, then $\not\subset_{\int}(\emptyset) = \mathbb{V} - \int(\mathbb{V}) = \emptyset$.Then, $\mathbb{V} = \int(\mathbb{V})$.

5. Discussion and Conclusion

We see that compared to the open set in the original topology, our new idea produces a broader open set. New definition types of proximity ψ-set can be created with this new operator. As demonstrated in theorem 4.6, it is also evident that if $\int$ πnet-work in $\mathbb{V}$, then $\not\subset_{\int}(\emptyset) = \emptyset$ is satisfied. Concepts in [7-10] can also be developed using the $\not\subset_{\int}$-operator, The new operator can also be generalized in the system neutrosophic.

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