Journal of Interdisciplinary Mathematics
ISSN: 0972-0502 (Print), ISSN: 2169-012X (Online)
Vol. 28 (2025), No. 3-B, pp. 1059–1064
DOI : 10.47974/JIM-2079

ℕ_b- paracompactness in nano topological spaces

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Abstract

A New type of nano topological spaces (N.T.S.), termed nano b-open sets(briefly ℕ_b-open), is established. explored of this work. Since a nano topology includes at most five set, so developing novel structures for nano- open set, as well as then new requirements concept were discuss like ℕ_b-open cover. This work explores the notion of ℕ_b- Paracompactness with the help of nano ℕ_b-open sets in an N.T.S. Discuss their basic characteristics using nano locally finite, locally refinement covers in nano topological spaces.

Subject Classification: *Primary 93A30, Secondary 49K15.*

Keywords: *Nano topology, Paracompact space, Refinement cover, b-open sets locally finite.*

1. Introduction

The pair (*U,R*) containing equivalence relation *R* of *U* is called space for approximation. In the same equivalence class, elements are called indistinguishable from each other. Nano topological spaces (briefly, *N.T.S.*) have significant advancement in the field of topology, introducing new methods and concepts that extend traditional topological ideas. A nano topological space $(\mathcal{U}, \mathcal{T}_R(\mathcal{X}))$ non-empty finite $\mathcal{U}$ for objects (a universe) with an equivalence relation R on a subset X of $\mathcal{U}$, that is carried by a topology (nano topology), defined by a lower and upper approximation $L_R(\mathcal{X})$, $\mathcal{U}_R(\mathcal{X})$ and boundary region $B_R(\mathcal{X}) = \mathcal{U}_R(\mathcal{X}) - L_R(\mathcal{X})$ of*X.*

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The lower approximation $L_R(\mathcal{X})$ of$\mathcal{X}$ represented a set for each object, that is classifies by $\mathcal{X}$ on R and its equal to the union $\cup_{x\in\mathcal{U}}\{R(x): R(x)\subseteq \mathcal{X}\}$ while,the upper approximation $\mathcal{U}_R(\mathcal{X})$ of $\mathcal{X}$ determined as set for each object, that can classified as $\mathcal{X}$ concerning R with its defined as the union $\cup_{x\in\mathcal{U}}\{R(x): R(x)\cap\mathcal{X}\neq\emptyset\}$. The set of all objects, that, are classified neither as $\mathcal{X}$ nor as not in $\mathcal{X}$ is called the boundary region $B_R(\mathcal{X})$and it is represented the difference among an upper approximation & a lower approximation. The topic of paracompact spaces generalizes both compact and metrizable spaces, it is studied for more general spaces like topological linear spaces, nano topologies, and infinite manifolds [2], [3], [4], [5], [7], [8] and [9]. This work, we study forms for nano $b-open$ sets to study the precompactness for nano topology space. A subset S for *N.T.S.* $(\mathcal{U},\mathcal{T}_R(\mathcal{X}))$ is said to be the nano $b-open$ (briefly $\mathsf{N}_\mathrm{b}-$open) set if $S\subseteq \mathsf{N}_{\mathrm{cl}}\left(\mathsf{N}_{\mathrm{int}}(S)\right)\cup\mathsf{N}_{\mathrm{int}}\left(\mathsf{N}_{\mathrm{cl}}(S)\right)$. A complement for nano $b-open$ named nano $b-closed$ set (briefly $\mathsf{N}_\mathrm{b}-$ closed). The $\mathsf{N}_\mathrm{b}-closure$ of S, denoted by $\mathsf{N}_{\mathrm{cl}_\mathrm{b}}(S)$, is the intersection of $\mathsf{N}_\mathrm{b}-closed$ sets including S. The $\mathsf{N}_\mathrm{b}-interior$ of S, denoted by$\mathsf{N}_{\mathrm{int}_\mathrm{b}}(S)$, is the union of $\mathsf{N}_\mathrm{b}-open$ sets included in S. Since the nano topology can support many applications in some medical and physics models, used to solve real-world problems [1] we tried to modify these topologies by adding a new class of them hopping to get new results that can be applicable in medicine and life in the future.

Many authors are studied some types algebra for family of set see, [10, 11] and [12].

Preliminaries:

Definition 1.1: [1] Let $\mathcal{U}$ be the universe and$\mathcal{X}\subseteq\mathcal{U}$, Rbe a relationship on U that is equivalent to U. Then$\tau_R(\mathcal{X})=\{\mathcal{U},\emptyset, L_R(\mathcal{X}),\mathcal{U}_R(\mathcal{X}),B_R(\mathcal{X})\}$.

To achieve this, we will define the Nano topology on U in terms of X. $(\mathcal{U},\tau_R(\mathcal{X}))$ called a nano topological space and it is denoted by NTS and its element named nano- open.

Proposition 1.2: [6]

1. $L_R(\mathcal{X})\subseteq\mathcal{X}\subseteq\mathcal{U}_R(\mathcal{X})$;
2. $L_R(\varphi)=\mathcal{U}_R(\varphi)=\varphi\ and\ L_R(\mathcal{U})=\mathcal{U}_R(\mathcal{U})=\mathcal{U}$;
3. $\mathcal{U}_R(\mathcal{X}\cup\mathcal{Y})=\mathcal{U}_R(\mathcal{X})\cup L_R(\mathcal{Y})$;
4. $\mathcal{U}_R(\mathcal{X}\cap\mathcal{Y})\subseteq\mathcal{U}_R(\mathcal{X})\cap\mathcal{U}_R(\mathcal{Y})$;
5. $L_R(\mathcal{X}\cup\mathcal{Y})\supseteq L_R(\mathcal{X})\cup L_R(\mathcal{Y})$;
6. $L_R(\mathcal{X}\cap \mathrm{Y})\subseteq L_R(\mathcal{X})\cap L_R(\mathcal{Y})$;
7. $L_R(\mathcal{X})\subseteq L_R(\mathcal{Y})\ and\ \mathcal{U}_R(\mathcal{X})\subseteq\mathcal{U}_R(\mathcal{Y})\ whenever\ \mathcal{X}\subseteq\mathcal{Y}$;

Example 1.3: Let $\mathcal{U} = \{a, b, c, d\}$ with $\mathcal{U}/R = \{a\}, \{c\}, \{b, d\}$ *and* $\mathcal{X} = \{a, b\}$. Then the nano topology $\mathcal{T}_R(\mathcal{X}) = \{\mathcal{U}, \emptyset, \{a\}, \{a, b, d\}, \{b, d\}\}$and $\mathsf{N}_{\mathrm{b}} - open$ sets are $\mathcal{U}, \emptyset, \{a\], \{b\}, \{d\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, d\}, \{a, b, c\}, \{a, b, d\}, \{b, c, d\}$.

$$int\{a\} = \{a\}, cl\{a\} = \{a, c\}.$$
$$int\{b\} = \emptyset, cl\{b\} = \{b, c, d\}.$$
$$int\{c\} = \emptyset, cl\{c\} = \{c\}.$$
$$int\{d\} = \emptyset, cl\{d\} = \{b, c, d\}.$$
$$int\{a, b\} = \{a\}, cl\{a, b\} = \mathcal{U}.$$
$$int\{a, c\} = \{a\}, cl\{a, c\} = \{a, c\}.$$
$$int\{a, d\} = \{a\}, cl\{a, d\} = \mathcal{U}.$$
$$int\{b, c\} = \emptyset, cl\{b, c\} = \{b, c, d\}.$$
$$int\{b, d\} = \{b, d\}, cl\{b, d\} = \{b, c, d\}.$$
$$int\{c, d\} = \emptyset, cl\{c, d\} = \{b, c, d\}.$$
$$int\{a, b, c\} = \{a\}, cl\{a, b, c\} = \mathcal{U}.$$
$$int\{a, b, d\} = \{a, b, d\}, cl\{a, b, d\} = \mathcal{U}.$$
$$int\{b, c, d\} = \{b, d\}, \quad cl\{b, c, d\} = \{b, c, d\}.$$
$$int\{a, c, d\} = \{a\}, cl\{a, c, d\} = \mathcal{U}.$$
$$int(cl\{a\}) = \{a\}, cl(int\{a\}) = \{a, c\} \implies \{a\}\ is\ N_b - open.$$
$$int(cl\{b\}) = \{b, d\}, cl(int\{b\}) = \emptyset \implies \{b\} is\ N_b - open.$$
$$int(cl\{c\}) = \emptyset, cl(int\{c\}) = \emptyset \implies \{c\} \notin N_b - open.$$
$$int(cl\{d\}) = \emptyset, cl(int\{d\}) = \emptyset \implies \{d\} \notin N_b - open.$$
$$int(cl\{a, b\}) = \mathcal{U}, cl(int\{a, b\}) = \{a, c\} \implies \{a, b\} \in N_b - open.$$

It is clear that any nano open set is $\mathsf{N}_{\mathrm{b}} - open$ sets and the converse is not true, in example above, the sets $\{b\}, \{d\}, \{a, b\}, \{a, c\}, \{a, d\}\{a, b, c\}$and $\{b, c, d\}$are $\mathsf{N}_{\mathrm{b}} - open$ but not nano open.

Theorem 1.3:

(1) Arbitrary union of two $\mathsf{N}_{\mathrm{b}} - open$ sets is not a $\mathsf{N}_{\mathrm{b}} - open$.

(2) Finite intersection of two $\mathsf{N}_{\mathrm{b}} - open$ sets may fail to be $\mathsf{N}_{\mathrm{b}} - open$

Proof:

(1) Let $\mathcal{M}\ and\ \mathcal{N}$ be two nano $b - open$ sets. Then, $\mathcal{M} \subseteq \mathsf{N}_{\mathrm{cl_b}}(\mathsf{N}_{\mathrm{int_b}}(\mathcal{M})) \cup \mathsf{N}_{\mathrm{int_b}}(\mathsf{N}_{\mathrm{cl_b}}(\mathcal{M}))\ and\ \mathcal{N} \subseteq \mathsf{N}_{\mathrm{cl_b}}(\mathsf{N}_{\mathrm{int_b}}(\mathcal{M})) \cup \mathsf{N}_{\mathrm{int_b}}(\mathsf{N}_{\mathrm{cl_b}}(\mathcal{M}))$. Then $\mathcal{M} \cup \mathcal{N} \subseteq [\mathsf{N}_{\mathrm{cl_b}}(\mathsf{N}_{\mathrm{int_b}}(\mathcal{M})) \cup \mathsf{N}_{\mathrm{int_b}}(\mathsf{N}_{\mathrm{cl_b}}(\mathcal{M}))] \cup [\mathsf{N}_{\mathrm{cl_b}}(\mathsf{N}_{\mathrm{cl_b}}(\mathcal{M})) \cup \mathsf{N}_{\mathrm{int_b}}(\mathsf{N}_{\mathrm{cl_b}}(\mathcal{M}))] \subseteq \mathsf{N}_{\mathrm{cl_b}}(\mathsf{N}_{\mathrm{int_b}}(\mathsf{N}_{\mathrm{cl_b}}(\mathcal{M}))) \cup \mathsf{N}_{\mathrm{int_b}}(\mathsf{N}_{\mathrm{cl_b}}(\mathcal{N})) \subseteq \mathsf{N}_{\mathrm{cl_b}}(\mathsf{N}_{\mathrm{int_b}}(\mathsf{N}_{\mathrm{cl_b}}(\mathcal{M}) \cup \mathsf{N}_{\mathrm{cl_b}}(\mathcal{N})))) \subseteq \mathsf{N}_{\mathrm{cl_b}}(\mathsf{N}_{\mathrm{int_b}}(\mathsf{N}_{\mathrm{cl_b}}(\mathcal{M} \cup \mathcal{N})))$, Therefore $\mathcal{M} \cup \mathcal{N}$ is nano $b - open$.

(2) It is a obvious, using the same assumption of (1) and Demorgas laws in set theory for the intersection.

2. $\mathcal{N}_{\mathrm{b}}$ − Paracompact spaces.

For easy, instead of $(\mathcal{U}, \tau_R(\mathcal{X}))$ we use $\mathcal{U}$ with nano topology$\tau_R(\mathcal{X})$

Definition 2.1: Let $\mathcal{U}$ be a $N.T.S$. A nano open cover is a family $\mathfrak{I} = \{\mathfrak{I}_i \in \tau_R(\mathcal{X})\}_{i\in\lambda}$ of $\mathcal{N}_{\mathrm{bo}}$ subsets of $\mathcal{U}$. $\mathfrak{I}$ is called a nano locally finite if and only if for each point $x \in \mathcal{X}$ there exists a nano neighbourhood V such that the set $\{\mathfrak{I}_i, V \cap \mathfrak{I}_i \neq \emptyset\ ;\ i \in \lambda\}$ is a finite set.

Theorem 2.2: *For every nano locally finite family* $\mathfrak{I} = \{\mathfrak{I}_i\}_{i\in\lambda}$ *we have,* $\mathcal{N}_{cl_b}(\cup_{i\in\lambda}\mathfrak{I}_i) = \cup_{i\in\lambda}\mathcal{N}_{cl_b}(\mathfrak{I}_i)$.

Proof: If $\mathfrak{I}_i \subset \cup_{i\in\lambda}\mathfrak{I}_i$ then $\mathcal{N}_{\mathrm{cl_b}}(\mathfrak{I}_i) \subset \mathcal{N}_{\mathrm{cl_b}}(\cup_{i\in\lambda}\mathfrak{I}_i)$, for every $i \in \lambda$; therefore we have $\mathcal{N}_{\mathrm{cl_b}}(\mathfrak{I}_i) \subset \mathcal{N}_{\mathrm{cl_b}}(\cup_{i\in\lambda}\mathfrak{I}_i)$.To prove the reverse inclusion, let us note that, by nano local finiteness of $\mathfrak{I}$, for every $x \in \mathcal{N}_{\mathrm{cl_b}}(\cup_{i\in\lambda}\mathfrak{I}_i)$, there exists a nano neighbourhood V such that the set $\mathcal{S}_0 = \{\mathfrak{I}_i, V \cap \mathfrak{I}_i \neq \emptyset\ ;\ i \in \lambda\}$ is a nano finite. From it followes that $x \notin \mathcal{N}_{\mathrm{cl_b}}(\cup_{i\in\lambda\setminus\mathcal{S}_0}\mathfrak{I}_i)$, since

$$x \in \mathcal{N}_{\mathrm{cl_b}}(\cup_{i\in\lambda}\mathfrak{I}_i) = \mathcal{N}_{\mathrm{cl_b}}(\cup_{i\in\mathcal{S}_0}\mathfrak{I}_i) \cup \mathcal{N}_{\mathrm{cl_b}}(\cup_{i\in\lambda\setminus\mathcal{S}_0}\mathfrak{I}_i),$$

Corollary 2.3: *A nano locally finite family* $\mathfrak{I}$ *is* $\mathcal{N}_{bo} - set(resp.\mathcal{N}_{bc} - set)$ *if* $\mathfrak{I}$ *covered by* $N_{bo} - sets(resp.\mathcal{N}_{bc} - set)$.

Theorem 2.4: *If* $\{\mathfrak{I}_i\}_{i\in\lambda}$ *is a nano locally finite system of sets in* $\mathcal{X}$*, then the set* $\{\mathcal{N}_{cl_b}(\mathfrak{I}_i)\}_{i\in\lambda}$ *is also a nano locally finite.*

Proof: Given $h \in \mathcal{X}$ and find an open nano neighbourhood V of h such that $V \cap \mathfrak{I}_i = \emptyset$, except for finitely is a many nano locally finite system of sets many i.

Since $\{\mathfrak{I}_i\}_{i\in\lambda}$ is a nano locally finite system of sets.
$\Rightarrow V \cap \mathcal{N}_{\mathrm{cl_b}}(\mathfrak{I}_i) = \emptyset$ for every $i \Rightarrow V \cap \mathcal{N}_{\mathrm{cl_b}}(\mathfrak{I}_i) \neq \emptyset$ for every i.
$\{\mathcal{N}_{\mathrm{cl_b}}(\mathfrak{I}_i)\}_{i\in\lambda}$is a nano locally finite.

Definition 2.5: If $\mathcal{A}$ and β are nano open covers (briefly, $\mathcal{N}_{\mathrm{bo}}$ −cover) of the $N.T.S\ (\mathcal{U}, \tau_R(\mathcal{X}))$, then we say that β refines $\mathcal{A}$ and write $\beta \subset \mathcal{A}$ if and only if each $B \in \beta$ is contained in some $A \in \mathcal{A}$ and we say β is a $\mathcal{N}_{\mathrm{b}}$ − refinement of $\mathcal{A}$ and denoted by$\mathcal{N}_{\mathrm{bo}}$ − refinement of $\mathcal{A}$.

Definition 2.6: A $N.T.S.(\mathcal{U}, \tau_R(\mathcal{X}))$ is called a $\mathcal{N}_{\mathrm{b}}$ −Paracompact if every $\mathcal{N}_{\mathrm{bo}}$ − cover $\mathcal{A}$ of $\mathcal{X}$ has a nano locally finite $\mathcal{N}_{\mathrm{bo}}$ −refinement cover.

Example 2.7: Let $\mathcal{U} = \{a, b, c, d\}$with $\mathcal{U}/R = \{\{a\}, \{b, c\}\}$ and $\mathcal{X} = \{a, c\}$. Then

$$L_R(\mathcal{X}) = \{a\},\ \mathcal{U}_R(\mathcal{X}) = \{a, b, c\} \text{ and } B_R(\mathcal{X}) = \{b, c\}$$

$\tau_R(\mathcal{X}) = \{\emptyset, \mathcal{U}, \{a\}, \{a,b,c\}, \{b,c\}\}$, the nano closed sets are

$$\emptyset, \mathcal{U}, \{b,c,d\}, \{d\}, \{a,d\}$$

$$int\{a\} = \{a\}, cl\{a\} = \{a,d\}.$$
$$int\{b\} = \emptyset, cl\{b\} = \{b,c,d\}.$$
$$int\{c\} = \emptyset, cl\{c\} = \{b,c,d\}.$$
$$int\{d\} = \emptyset, cl\{d\} = \{d\}.$$
$$int\{a,b\} = \{a\}, cl\{a,b\} = \mathcal{U}.$$
$$int\{a,c\} = \{a\}, cl\{a,c\} = \{a,c\}.$$
$$int\{a,d\} = \{a\}, cl\{a,d\} = \{a,d\}.$$
$$int\{b,c\} = \{b,c\}, cl\{b,c\} = \{b,c,d\}.$$
$$int\{b,d\} = \emptyset, cl\{b,d\} = \{b,c,d\}.$$
$$int\{c,d\} = \emptyset, cl\{c,d\} = \{b,c,d\}.$$
$$int\{a,b,c\} = \{a,b,c\}, cl\{a,b,c\} = \mathcal{U}.$$
$$int\{a,b,d\} = \{a\}, cl\{a,b,d\} = \{a,d\}.$$
$$int\{b,c,d\} = \{b,c\}, \quad cl\{b,c,d\} = \{b,c,d\}.$$
$$int\{a,c,d\} = \{a\}, cl\{a,c,d\} = \{a,d\}.$$

The $\mathsf{N}_{\mathrm{bo}} -$ sets are the sets

$$\{a\}, \{b\}, \{c\}, \{a,b\}, \{a,d\}, \{b,c\}, \{a,b,c\}, \{a,c,d\}$$

The sets $\mathcal{A}_1 = \{\{b\}, \{c\}, \{a,d\}\}, \mathcal{A}_2 = \{\{a,d\}, \{b,c\}\}, \mathcal{A}_3 = \{\{b\}, \{a,c,d\}\}$, $\mathcal{A}_4 = \{\{a,b\}, \{a,c,d\}\}$, $\mathcal{A}_5 = \{\{a,d\}, \{a,b,c\}\}$, $\mathcal{A}_6 = \{\{b,c\}, \{a,c,d\}\}$ and $\mathcal{A}_7 = \{\{a,b,c\}, \{a,c,d\}\}$ are $\mathsf{N}_{\mathrm{bo}} -$ covers of $(\mathcal{U}, \tau_R(\mathcal{X}))$

The sets, $\beta = \{\{a,d\}, \{b\}, \{c\}\}$ is N_{bo} −refinement locally finite of $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_5, \mathcal{A}_6$, and $\mathcal{A}_7$.

And $(\mathcal{U}, \tau_R(\mathcal{X}))$ is N_{b} −Paracompact.

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Received October, 2024