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# Analytical solutions for nonlinear differential equations via differential transform technique

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**Abstract**

A relatively novel technique differential equations named Differential Transform Technique (DTM) is proposed. The evaluation of this approach is based on an iterative method in series form. Here, First give several basic definitions and properties of DTM, and applied various examples for the proposed method are tested to show the efficiency and accuracy to get the solutions of non-linear pantograph equation with elegantly computed components. In addition to solving the problems efficiently and rapidly, The advantage of the suggested approach is that it produces an analytical approach with fewer terms in a convergent series form.

***Subject Classification:*** *35R11, 35R60.*

**Keywords:** *Differential equations, Differential transform method, Non-linear pantograph equation, Semi-analytic technique.*

## 1. Introduction

Due to the nonlinear nature of the majority of differential equations, it can be challenging to discover accurate or analytical solutions. Some researchers have combined two or more methods to find analytical or numerical solutions of equations, while others are finding ways to solve some types of equations by changing the conventional methods. Finding the differential equations' Taylor expansion is the goal of the differential transformation approach. For a number

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of functions, the polynomial form can be used. This method can be used to find polynomial solutions for differential equations. In 1986, Zhou published the first DTM, which was used to solve electrical circuit issues. This technique for partial differential equations was created by Chen and Ho, and Ayaz used it to solve a system of differential equations. Through the efforts of Zhou and Chen, the unknown structure of DTM has started to emerge. In order to solve boundary value problems for integro-differential equations, Arikoglu employed a new version of DTM in 2005.Additionally, a new DTM form for differential equation solutions was created. An efficient and potent method known (RDTM) was introduced by other researchers [1–13].

## 2. Some Basic Properties of DTM

**Definition 1:** The differential transformation with one variable of analytic function $¥(\mathrm{X})$ is denoted by:

$$¥(K) = \frac{1}{K!}\left[\frac{d^K}{d\chi^K}¥(\mathrm{X})\right]_{\mathrm{X}=\mathrm{X}_i} \tag{1}$$

Where $¥(\mathrm{X})$ is the default function and $¥(K)$ is the modified function that is named T-function (it is also known the spectrum of the $\mathbf{f}(\mathrm{X})$ at $\mathrm{X} = \mathrm{X}_\mathbf{0}$ in the k domain). The definition of the differential inverse transform of ¥(K) is:

$$¥(\mathrm{X}) = \sum_{k=0}^{\infty}¥(K)\,\chi^K \simeq ¥_\ell(\mathrm{X}) = \sum_{k=0}^{\ell}¥(K)\,\mathrm{X}^K \tag{2}$$

By substituting Eq. (2.1) in (2.2), we get:

$$¥(\mathrm{X}) = \sum_{K=0}^{\infty}\frac{\mathrm{X}^K}{K!}\left[\frac{d^K\,¥(\mathrm{X})}{d\mathrm{X}^k}\right]_{\chi=0} \tag{3}$$

In general, if $¥(\mathrm{X})$ is analytic, then $¥(K)$ is defined as:$¥(\mathrm{X}) = \sum_{k=0}^{\infty}\frac{(\mathrm{X}-\mathrm{X}_i)^K}{K!}¥(K)$ when $\mathrm{X} \neq \mathrm{X}_i$ Then

$$¥(\mathrm{X}) = \sum_{K=0}^{\infty}\frac{(\mathrm{X}-\mathrm{X}_i)^K}{k!}\left[\frac{d^K¥(\mathrm{X})}{d\mathrm{X}^K}\right]_{\mathrm{X}=\mathrm{X}_i}$$

## 3. Results and Discussion

**Example 1:** Consider the nonlinear pantograph equation as follows:

$$2¥''(t) - 2¥(t) + \frac{16}{t^2}¥^2\left(\frac{t}{2}\right) = 0\,,\ 0 \le t \le 1 \tag{4}$$

With the initial conditions $¥(0) = 0\,, ¥'(0) = 1$

**Solution:** First, we can write the Eq. (4) as follows:

$2\,t^2¥''(\mathrm{t}) - 2\mathrm{t}^2¥(t) + 16\,¥^2(\frac{t}{2}) = 0$ By taking the properties of DTM to the following recurrence relations:

$$2\sum_{r=0}^{\vartheta}\delta(r-2)\,(\vartheta-r+1)(\vartheta-r+2)¥(\mathrm{m}-4+2) = 2\sum_{r=0}^{\vartheta}\delta(r-2)\,¥(\vartheta-r) - 16\sum_{r=0}^{\vartheta}\left(\frac{t}{2}\right)^{\vartheta}¥(r)\,¥(\vartheta-r) \tag{5}$$

And from the initial condition, we have: $¥(0) = 0$ , $¥(0)$=1

Substituting the above equation in Eq. (5), recursively we derive the following results:

$$¥(2) = -1\,, ¥(3) = \frac{1}{2!}, ¥(4) = \frac{-1}{3!}, ¥(5) = \frac{1}{4!}, ¥(6) = \frac{-1}{5!}$$

Then, $¥(t) = \sum_{\vartheta=0}^{\infty} ¥(\vartheta)\ t^{\vartheta} = t - t^2 + \frac{t^3}{2\,!} - \frac{t^4}{3\,!} + \frac{t^5}{4\,!} - \cdots$

$$t\left(1 - t + \frac{t^2}{2\,!} - \frac{t^3}{3\,!} + \cdots\right) = t\, e^{-t}$$

**Example 2:** Consider the nonlinear Van Der Pol differential equations:

$$\frac{d^2¥}{d\,t^2} - ¥ + ¥^2 + \left(\frac{d¥}{dt}\right)^2 - 4 = 0 \tag{6}$$

With $¥(0) = 3$, $¥'(0) = 0$ with exact solution:

$$¥(\mathrm{t}) = 1 + 2\cos(\mathrm{t})$$

**Solution:** By using the DTM to solve the above Eq. (6), we get:

$$(\vartheta + 1)(\vartheta + 2)¥(\vartheta + 2) - ¥(\vartheta) + \sum_{i=0}^{\vartheta} ¥(i)\, ¥(\vartheta - i)$$
$$+ \sum_{\Gamma=0}^{\vartheta}(\Gamma + 1)\, ¥(\Gamma + 1)(\vartheta - \Gamma + 1)¥(\vartheta - \Gamma + 1) - 4\delta(\vartheta) = 0$$

Then,

$$¥(\vartheta + 2) = \frac{1}{(\vartheta+1)(\vartheta+2)}\left[¥(\vartheta) - \sum_{i=0}^{\vartheta} ¥(i)¥(\vartheta - i) - \sum_{\Gamma=0}^{\vartheta}(\Gamma + 1)\, ¥(\Gamma + 1)\right.$$
$$\left.(\vartheta - \Gamma + 1)¥\,(\vartheta - \Gamma + 1) + 4\delta(\vartheta)\,\right]$$

From the initial conditions, we get: $¥(0) = 3$ , $¥(1) = 0$ . Then:

$$¥(2) = \frac{1}{2}\,[¥(0) - ¥(0)¥(0) - (¥(1))^2 + 4\delta(0)] = -1$$

$$¥(3) = \frac{1}{3.2}\,[¥(1) - ¥(0)¥(1) - 4¥(1)¥(2)] = 0\,,$$

$$¥(4) = \frac{1}{3.4} = \frac{1}{12}\,, ¥(5) = 0\ \ , ¥(6) = \frac{1}{360}$$

Then $¥(\mathrm{t}) = 3 - \mathrm{t}^2 + \frac{\mathrm{t}^4}{12} - \frac{\mathrm{t}^6}{360} + \cdots \simeq 1 + 2\cos\mathrm{x}$

**Example 3:** Consider the nonlinear pantograph equation as follows:

$$\frac{\partial^2¥}{\partial\,t^2} - 4¥\,(t) + \frac{16}{t^2}¥^2(\tfrac{t}{2}) = 0 \tag{7}$$

With $¥(0) = 0$, $¥'(0) = 1$ and $¥(\mathrm{t}) = \mathrm{te}^{-2\mathrm{t}}$ is the exact solution

**Solution:** By Applying the DTM, rewrite the Eq. (7) as follows:

$$t^2\,\frac{\partial^2¥}{\partial\,t^2} - 4t^2¥(t)\ -\ 16¥^2(\tfrac{t}{2}) = 0$$

By using some theorems of the DTM, we get

$$\sum_{\vartheta=0}^{r}\ \delta(\vartheta - 2)\,(r - \vartheta + 1)(r - \vartheta + 2)¥(r - \vartheta + 2)$$
$$= 4\sum_{\vartheta=0}^{r}\ \delta(\vartheta - 2)\,¥(r - \vartheta) - 16\ \sum_{\vartheta=0}^{r}\ (\tfrac{1}{2})^2\,¥(\vartheta)¥(r - \vartheta) \tag{2.8}$$

We write: $¥(0) = 0, \quad ¥'(0) = 1$
By substituting the above equation in Eq. (2.8), we get

$$¥(2) = -2\,, ¥(3) = 2, ¥(4) = \frac{-4}{3}, ¥(5) = \frac{-4}{6}$$

Then, we have: $¥(\mathrm{t}) = \sum_{r=0}^{\infty} ¥(\mathrm{r})\,\mathrm{t}^{\mathrm{r}} = \mathrm{t} - 2\mathrm{t}^2 + \frac{(2\mathrm{t})^3}{2\,!} - \frac{(2\mathrm{t})^4}{3\,!} + \frac{(2\mathrm{t})^5}{4\,!} + \cdots$
The solution approximates to $\mathrm{y}(\mathrm{t}) = \mathrm{te}^{-2\mathrm{t}}$

## 4. Conclusion

The major goal of this work solving some types of non-linear differential equations iterative method like DTM. This purpose has been obtained by implementing this iterative method, which is so-called the Semi-Analytic method. DTM has the benefit of providing an analytical approximation in the form of a convergent series with fewer terms. Some examples for the proposed methods are tested to show the efficiency and accuracy for obtaining the solution of non-linear differential equations with elegantly computed components.

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