Journal of Interdisciplinary Mathematics
ISSN: 0972-0502 (Print), ISSN: 2169-012X (Online)
Vol. 28 (2025), No. 3-B, pp. 1185–1196
DOI : 10.47974/JIM-2207

An innovation analytical application of the proposed system of equations on Hilbert space

Salim Dawood Mohsen *
Ihsan Abdulsattar Awadh †
Department of Mathematics
College of Education
Mustensiriyah University
Baghdad
Iraq

Abstract

In the realm of Hilbert space, the applicability of the system of modeling equations in Operator Theory is investigated and considered. Recently, the interest in studying this type of systems has increased greatly. This article, employing the operators' equations applicable to Hilbert space, deals with two systems of equations. It attempts to examine the required stipulations for the presence of a solution to this posed system. The first system comprises of two equations. For Λ and F that achieve regularity with non-invertibility, the general solution formula is investigated based on the Moore-Penrose inverse. Furthermore, the second system includes four equations having adjoint operators. Among the required stipulations, the operator attained self-adjoint attribute. The general solution formula is established based on the are regular of the operator when both Λ and F are regular.

Subject Classification (2020): *Primary 46S40, 47A05, 47B02; Secondary 47B15.*

Keywords: *Linear operator, Operator equation, Hilbert space, General solution.*

1. Introduction

In mathematics, the problem of resolving equations of operators is playing a fruitful and significant role in control theory and other allied disciplines, blurring of boundaries between purely scientific disciplines and reviving interest in applied mathematics. It has attracted and aroused the interest of numerous scientists, see [1-8]. In this regard, the application of operators existing in

* *E-mail:* `dr_salim2015@yahoo.com` **(Corresponding Author)**
† *E-mail:* `ahsan.123.amani1989@gmail.com`

Hilbert space to describe and resolve mathematical equations models is one of the most significant realms of Functional Analysis (FA). Following this line of analysis, remarkable contributions have been made to its evolvement by several investigators. Interestingly, in 1976, Khatri and Mitra [9] obtained expressions for the general, non-negative, and constrained Hermitian solutions of the three-equation system $\Lambda X = \Omega = \Lambda X\mathrm{F}$ and $X\mathrm{F} = D$, after specifying the essential and required stipulations for the existence of these solutions. In 2007, Dajić and Koliha [10] presented the stipulations for the presence of the Hermitian and positive solution and the general solution formula for equation $\Lambda X = \Omega$. Then they discussed the system of equations $\Lambda X = \Omega\,, X\mathrm{F} = D$ and provided representations of the solutions. In 2007, Wang, Chang, and Lin, [11] looked at the explicit and general resolution to the system of equations: $\Lambda_1 X_1 = \Omega_1$, $\Lambda_2 X_2 = \Omega_2$, $\Lambda_3 X_3 = \Omega_3$, $\Lambda_4 X_4 = \Omega_4$ and $\Lambda_5 X_5 \mathrm{F}_5 = \Omega_5$, $\Lambda_6 X_6 \mathrm{F}_6 = \Omega_6$ by discussing the substantial and required stipulations for the existence of the solution. Thereafter, in 2012, Farid et al. [12] explored some leading and required stipulations criteria for the presence of a resolution to the system of equations: $\Lambda_2 Y = \Omega_2, Y\mathrm{F}_2 = D_2, \Lambda_3 Z = \Omega_3, Z\mathrm{F}_3 = D_3, C_3 Y \Omega_3^* = D_3 Z\, \mathrm{D}_3^* = \Lambda_3$. Subsequent, in 2013, Deng [13] investigated and gave explicit resolutions to equation $\mathrm{F}\Lambda X = \mathrm{F} = X\Lambda\mathrm{F}$ for linear bounded operators on Hilbert spaces, specifying uniqueness requirements and term the unknown operator X as the inverse of Λ along F. Both Vosough, M. and M. S. Moslehian established certain essential and Enough prerequisites to the presence of an answer to the equation system F$X\Lambda$= F= ΛXF in 2017, They then demonstrated that, under certain circumstances, x represents a solution to the system before presenting the system's general solution, [14-24]. Several aspects of this theme are also addressed in [15, 30]. Where a section of this reference addresses the definitions and characteristics of certain types of influences, while another section discusses the solutions to some operator equations under specific circumstances.

In this study, the symbols H, $B(H)$, $R(\Lambda)$ and $N(\Lambda)$ represent the complex Hilbert space, all bounded linear operators from H to H in the space, the range Λ and the null space of Λ, respectively. In the next section, we will review some definitions and theories that will be useful in proving the claims that will be presented in the third section of this work, which will take the form of theories.

2. Preliminaries

This section addresses some definitions and relevant theories utilized in this work are enveloped, needed by the reader of this article to be familiar with most of its basics, and we also refer to a set of sources that were useful in presenting this information.

Definition 2.1: [25] If $\Lambda: H \longrightarrow D$ such that $H\,, D$ are normed space, then Λ is called bounded if there exist $C > 0$ such that $\| \Lambda x \| \leq C \| x \|$.

Definition 2.2: [25] Let $H\ and\ D$ are vectors space then the operator $\Lambda: H \longrightarrow D$. It is called invertibility. If there exist an operator $\mathrm{F}: D \longrightarrow F$ where $\Lambda\mathrm{F} = \mathrm{F}\Lambda = I$, where I is identity operator

Definition 2.3: [26] Let H and D are Hilbert spaces and $\Lambda: H \longrightarrow D$ be liner operator then the operator $\Lambda^*: D \longrightarrow F$ is called to be adjoint operator of Λ if $< \Lambda x, y >=< x, \Lambda^* y >$ for each $x \in H$, $y \in D$.

Definition 2.4: [27] An operator $\Lambda: H \longrightarrow D$ where $H\ and\ D$ are Hilbert space. It is called self-adjoint if $\Lambda = \Lambda^*$.

Definition 2.5: [26] Let $\Lambda \in B(H)$, then the Moore-Penrose invers of Λ, symbolized by Λ^+ and identified as the unique solution X to the system operator equation: $\Lambda X\Lambda = \Lambda$, $X\Lambda X = X$, $(\Lambda X)^* = \Lambda X$, $(X\Lambda)^* = X\Lambda$.

From this definition, we obtain $\Lambda^*\Lambda\Lambda^+ = \Lambda^* = \Lambda^+\Lambda\Lambda^*$, which provides us with the following:$R(\Lambda^+) = R(\Lambda^*) = R(\Lambda^+\Lambda) = R(\Lambda^*\Lambda)$, $R(\Lambda) = R(\Lambda\Lambda^+) = R(\Lambda\Lambda^*)$, $P_{\overline{R(\Lambda)}} = \Lambda\Lambda^+$ and $P_{\overline{R(\Lambda^*)}} = \Lambda^+\Lambda$. For more information, see [28].

If there is F self-adjoint operator such that $\Lambda\mathrm{F} = 0$ and $\Lambda^+\mathrm{F} = \Omega$, So we write $\Lambda \preccurlyeq \mathrm{F}$. It is said that order $\preccurlyeq$ is the logical order on the linear operator space $B(H)$, when all operators in this space is self-adjoint. For $\Lambda, \mathrm{F} \in B(H)$, let $\Lambda \overset{*}{\leq} \mathrm{F}$ mean $\Lambda\Lambda^* = \mathrm{F}\Lambda^*, \Lambda^*\Lambda = \Lambda^*\mathrm{F}$, it is known that, for $\Lambda, \mathrm{F} \in B(H)$, when all operators in this space is self-adjoint, $\Lambda \preccurlyeq \mathrm{F}$ if and only if $\Lambda \overset{*}{\leq} \mathrm{F}$ see [29].

Theorem 2.6: [30] *Let $\Lambda, \mathrm{F} \in B(H)$ and $\overline{\mathcal{W}}$ dented the closure of a space $\mathcal{W}$. (i) $\Lambda\Lambda^* = \mathrm{F}\Lambda^* \Leftrightarrow \Lambda = \mathrm{F}P_{\overline{R(\Lambda^*)}} \Leftrightarrow \Lambda = \mathrm{F}Q$ for some $Q \in \mathcal{G}(H)$, where $\mathcal{G}(H)$ the set of all orthogonal projections; (ii) $\Lambda^*\Lambda = \Lambda^*\mathrm{F} \Leftrightarrow \Lambda = \mathrm{F}P_{\overline{R(\Lambda)}} \Leftrightarrow \Lambda = \mathrm{F}P$ for some $P \in \mathcal{G}(H)$; (iii) $\Lambda \overset{*}{\leq} \mathrm{F} \Leftrightarrow \mathrm{F} = \Lambda + P_{\overline{N(\Lambda^*)}}\mathrm{F}P_{\overline{N(\Lambda)}}$; (iv) $\Lambda \overset{*}{\leq} \mathrm{F} \Leftrightarrow \Lambda = P_{\overline{R(\Lambda)}}\mathrm{F} = \mathrm{F}P_{\overline{R(\Lambda^*)}} = P_{\overline{R(\Lambda)}}\mathrm{F}P_{\overline{R(\Lambda^*)}}$.*

Theorem 2.7: [10] *Let Λ, $\Omega \in B(F)$ and Λ is regular. Then the next requirements are comparable: (i) there is $X \in B(F)$ a solution of the equation $\Lambda X = \Omega$; (ii) $\Lambda\Lambda^+\Omega = \Omega$; (iii) $R(\Omega) \subseteq R(\Lambda)$. The general solution in this instance is $X = \Lambda^+\Omega + (I - \Lambda^+\Lambda)T$, where $T \in B(H)$ random.*

Theorem 2.8: [9] *Let $\Lambda, \mathrm{F}, \Omega \in B(H)$ are regular, then the next requirements are comparable: (i) There is $X \in B(H)$ a solution of the equation $\Lambda X\mathrm{F} = \Omega$; (ii) $R(\Omega) \subseteq R(\Lambda), R(\Omega^*) \subseteq R(\mathrm{F}^*)$; (iii) $\Lambda\Lambda^+\Omega\mathrm{F}^+\mathrm{F} = \Omega$.*

In this case, $X = \Lambda^+\Omega B^+ + U - \Lambda^+\Lambda U B B^+$. Where $U \in B(H)$ is arbitrary.

3. Result discussions

This section gives a statement for the operator equation system's solution

$$\Lambda X\mathrm{F} = \Omega = \mathrm{F}^* X \Lambda^* \tag{1}$$

as well as the prerequisites for the presence of a solution, then we will discuss how to construct its formula. And then we discuss the system's solution

$$\Lambda X\mathrm{F} = \Omega_1 = \mathrm{F}^* X \Lambda^*, \ \ \Lambda^* X \mathrm{F}^* = \Omega_2 = \mathrm{F} X \Lambda \tag{2}$$

where we start with the requirements for the existence of the solution and conclude with constructing its formula.

Theorem 3.1: *If* $\Lambda, \mathrm{F} \in B(H)$ *is regular and* $R(\mathrm{F}) \subseteq R(\Lambda^*)$*, then the sentences that follow are interchangeable*

i) *The operator equation system (1) is solvable.*

ii) $\mathrm{R}(\Omega), \mathrm{R}(\Omega^*) \subseteq \mathrm{R}(\Lambda)$ and the system $\mathrm{F}^*\mathrm{X} = \Omega(\Lambda^*)^+$, $\mathrm{XF} = \Lambda^+\Omega$ is solvable.

Proof: *(i)⇒(ii)* Let the system (1) is solvable, then we have $R(\Omega), R(\Omega^*) \subseteq R(\Lambda)$, taking $\theta \in B(H)$ is the system (1) solution, then we get:

$$\Lambda \theta \mathrm{F} = \Omega = \mathrm{F}^* \theta \Lambda^*.$$

Now we will take the right side of this operator equation system and multiply it from the right by Λ^+ to yield us

$$\Lambda^+ \Lambda \theta \mathrm{F} = \Lambda^+ \Omega. \tag{3}$$

Considering that $R(\mathrm{F}) \subseteq R(\Lambda^*)$, the next equation is the result of Theorem 2.7,

$$\Lambda^* (\Lambda^*)^+ \mathrm{F} = \mathrm{F}.$$

Because of this relationship, (3) takes on the next for

$$\Lambda^+ \Lambda \theta \Lambda^* (\Lambda^*)^+ \mathrm{F} = \Lambda^+ \Omega. \tag{4}$$

Now we will take the left side of this operator equation system and multiply it from the left by $(\Lambda^*)^+$ to yield us

$$\Omega (\Lambda^*)^+ = \mathrm{F}^* \theta \Lambda^* (\Lambda^*)^+, \tag{5}$$

and our possession of $\Lambda^* (\Lambda^*)^+ \mathrm{F} = \mathrm{F}$ provides us with $\mathrm{F}^* = \mathrm{F}^* \Lambda^+ \Lambda$. and by taking the relationship that has recently developed for us and compensating it in (5), we will obtain:

$$\Omega (\Lambda^*)^+ = \mathrm{F}^* \Lambda^+ \Lambda \theta \Lambda^* (\Lambda^*)^+. \tag{6}$$

Through (4) and (6), it is clear to us that $\Lambda^+ \Lambda \theta \Lambda^* (\Lambda^*)^+$ is a solution for the system

$$\mathrm{F}^* X = \Omega (\Lambda^*)^+ , X\mathrm{F} = \Lambda^+ \Omega.$$

(ii) ⇒(i): Taking $R(\Omega), R(\Omega^*) \subseteq R(\Lambda)$ and the system $\mathrm{F}^* X = \Omega (\Lambda^*)^+ , X\mathrm{F} = \Lambda^+ \Omega$ is solvable. By utilizing this data and applying Theorem 2.7, we will obtain the following information:

$$\Omega = \Lambda \Lambda^+ \Omega \ , \Omega^* = \Lambda \Lambda^+ \Omega^* \tag{7}$$

And taking θ is solution of the system $\mathrm{F}^*X = \Omega(\Lambda^*)^+ , X\mathrm{F} = \Lambda^+\Omega$ we get:

$$\theta\mathrm{F} = \Lambda^+\Omega \, , \mathrm{F}^*\theta = \Omega(\Lambda^*)^+.$$

Now, after we multiply the last two equations consecutively by Λ from the left and Λ^* from the right, and then substitute Ω and Ω^* with the values they equal in (7), we will obtain new equations in the following form

$$\Lambda\theta\mathrm{F} = \Lambda\Lambda^+\Omega = \Omega \ \ , \mathrm{F}^*\theta\Lambda^* = \Omega(\Lambda^*)^+\Lambda^* = \Omega,$$

here we can clearly observe that θ is a solution to the system (1).

In the following theorem, it will be proven that Ω^+ satisfies the system (1) under certain conditions.

Theorem 3.2: *If $\Lambda, \mathrm{F} \in B(H)$, if $\Omega \in B(H)$ is regular and $\Omega \overset{*}{\leq} \mathrm{F}^*, \mathrm{F}, \Lambda, \Lambda^*$. Then the system of operator equations (1) is solvable such that the solution of this system is Ω^+.*

Proof: From the presumption we have $\Omega \overset{*}{\leq} \Lambda$ and $\Omega \overset{*}{\leq} \mathrm{F}$. Therefore, using these data respectively with Theorem 2.6, we obtain $\Omega = \Omega\Omega^+\Lambda = \Lambda\Omega^+\Omega$ and $\Omega = \Omega\Omega^+\mathrm{F} = \mathrm{F}\Omega^+\Omega$ consecutively. Using these last two relationships, we can build a new relationship represented by

$$\Omega = \Lambda\Omega^+\Omega = \Lambda\Omega^+\Omega\Omega^+\mathrm{F} = \Lambda\Omega^+\mathrm{F}. \tag{8}$$

Also, from the presumption we have $\Omega \overset{*}{\leq} \Lambda^*$ and $\Omega \overset{*}{\leq} \mathrm{F}^*$. Therefore, using these data respectively with Theorem 2.6, We acquire $\Omega = \Omega\Omega^+\Lambda^*$ and $\Omega = \mathrm{F}^*\Omega^+\Omega$ consecutively. Using these last two relationships, we can build a new relationship represented as follows

$$\mathrm{F}^*\Omega^+\Lambda^* = \mathrm{F}^*\Omega^+\Omega\Omega^+\Lambda^* = \mathrm{F}^*\Omega^+\Omega = \Omega. \tag{9}$$

From relationships (8) and (9), we can clearly observe that Ω^+ represents a solution to the system of operator equations (1).

In the subsequent theory, the solution formula for system (1) will be constructed after addressing its existence requirements.

Theorem 3.3: *If $\Lambda, \mathrm{F} \in B(H)$ is regular, $\Omega \in B(H)$ and $R(\Omega) \subseteq R(\Lambda), R(\Omega^*) \subseteq R(\mathrm{F}^*), \mathrm{F} \overset{*}{\leq} \Lambda^*$ then the system (1)'s general solution is*

$$X = \Lambda^+\Omega\mathrm{F}^+ + (\mathrm{F}^*)^+(\Omega - \mathrm{F}^*\Lambda^+\Omega\mathrm{F}^+\Lambda^*)\mathcal{S}^+ + W - (\mathrm{F}^*)^+\mathrm{F}^*W\mathcal{S}\mathcal{S}^+$$
$$-\Lambda^+\Lambda[(\mathrm{F}^*)^+(\Omega - \mathrm{F}^*\Lambda^+\Omega\mathrm{F}^+\Lambda^*)\mathcal{S}^+ + W - (\mathrm{F}^*)^+\mathrm{F}^*W\mathcal{S}\mathcal{S}^+]\mathrm{F}\mathrm{F}^+,$$

Where $W \in B(H)$ is arbitrary, $\mathcal{S} = \Lambda^* - \mathrm{F}$.

Proof: From the assumption $R(\Omega) \subseteq R(\Lambda), R(\Omega^*) \subseteq R(\mathrm{F}^*)$. This provides us with the following data:

$$\Omega = \Lambda\Lambda^+\Omega \ and \ \Omega = \Omega\mathrm{F}^+\mathrm{F},$$

these data provide us with the following result

$$\varOmega = \varLambda\varLambda^{+}\varOmega = \varLambda\varLambda^{+}\varOmega\mathrm{F}^{+}\mathrm{F}.$$

We notice here that the condition of Theorem 2.8, which requires the existence of a solution for Equation $\varLambda X\mathrm{F} = \varOmega$, has been satisfied. Therefore, we can use this Theorem 2.8 to construct the solution for Equation $\varLambda X\mathrm{F} = \varOmega$ in the following manner:

$$X = \varLambda^{+}\varOmega\mathrm{F}^{+} + Z - \varLambda^{+}\varLambda Z\mathrm{F}\mathrm{F}^{+}, \tag{10}$$

where Z is arbitrary. Now we will consider X as a solution to the equation

$$\varOmega = \mathrm{F}^{*}X\varLambda^{*},$$

So, we will substitute it into this equation to obtain

$$\varOmega = \mathrm{F}^{*}(\varLambda^{+}\varOmega\mathrm{F}^{+} + Z - \varLambda^{+}\varLambda Z\mathrm{F}\mathrm{F}^{+})\varLambda^{*},$$

from him, and after opening the bracket it contains, provides us with the following equation.

$$\varOmega = \mathrm{F}^{*}\varLambda^{+}\varOmega\mathrm{F}^{+}\varLambda^{*} + \mathrm{F}^{*}Z\varLambda^{*} - \mathrm{F}^{*}\varLambda^{+}\varLambda Z\mathrm{F}\mathrm{F}^{+}\varLambda^{*}. \tag{11}$$

By the assumption we possess $\mathrm{F} \overset{*}{\leq} \varLambda^{*}$, and by using Theorem 2.7, This brings us to the next point

$$\mathrm{F} = \varLambda^{*}\mathrm{F}^{+}\mathrm{F}\,, \mathrm{F} = \mathrm{F}\mathrm{F}^{+}\varLambda^{*}, \tag{12}$$

and this provides us with $R(\mathrm{F}) \subseteq R(\varLambda^{*})$. Based on this recent relationship that has formed, and using Theory 2.7, we will obtain $\varLambda^{*}(\varLambda^{*})^{+}\mathrm{F} = \mathrm{F}$ And this generates

$$\mathrm{F}^{*} = \mathrm{F}^{*}\varLambda^{+}\varLambda. \tag{13}$$

Using relation (12) and relation (13), the formula of equation (11) will be transformed into this form

$$\mathrm{F}^{*}\varLambda^{+}\varOmega\mathrm{F}^{+}\varLambda^{*} + \mathrm{F}^{*}Z\varLambda^{*} - \mathrm{F}^{*}Z\mathrm{F} = \varOmega,$$

After rearranging this equation, we derive the next equation

$$\mathrm{F}^{*}Z\mathcal{S} = \varOmega - \mathrm{F}^{*}\varLambda^{+}\varOmega\mathrm{F}^{+}\varLambda^{*}. \tag{14}$$

The equation (14) that we recently formed represents the formula mentioned in Theorem 2.8, where Z is its solution. Therefore, using Theorem 2.8, it will be Z in the form of

$$Z = (\mathrm{F}^{*})^{+}(\varOmega - \mathrm{F}^{*}\varLambda^{+}\varOmega\mathrm{F}^{+}\varLambda^{*})\mathcal{S}^{+} + W - (\mathrm{F}^{*})^{+}\mathrm{F}^{*}W\mathcal{S}\mathcal{S}^{+},$$

where W is arbitrary. Now we will replace Z that appeared in (10) with its equivalent to obtain

$$\begin{aligned} X = {} & \varLambda^{+}\varOmega\mathrm{F}^{+} + (\mathrm{F}^{*})^{+}(\varOmega - \mathrm{F}^{*}\varLambda^{+}\varOmega\mathrm{F}^{+}\varLambda^{*})\mathcal{S}^{+} + W - (\mathrm{F}^{*})^{+}\mathrm{F}^{*}W\mathcal{S}\mathcal{S}^{+} \\ & - \varLambda^{+}\varLambda[(\mathrm{F}^{*})^{+}(\varOmega - \mathrm{F}^{*}\varLambda^{+}\varOmega\mathrm{F}^{+}\varLambda^{*})\mathcal{S}^{+} + W - (\mathrm{F}^{*})^{+}\mathrm{F}^{*}W\mathcal{S}\mathcal{S}^{+}]\mathrm{F}\mathrm{F}^{+}. \end{aligned}$$

Now we will present a solution formula for System 1 after addressing the requirements for the presence of this solution, and this is a result based on Theorem 3.3.

Corollary 3.4: *Let* $\Lambda, \mathrm{F} \in B(H)$*, if* Λ, F *is regular,* Λ *is self-adjoint and* $\mathrm{F} \overset{*}{\leq} \Lambda$*, Consequently, the generic solution to the operator system equations* $\Lambda X \mathrm{F} = \mathrm{F} = \mathrm{F}^* X \Lambda^*$*, is*

$$X = \Lambda^+ \mathrm{F}\mathrm{F}^+ + \mathrm{F}^+(\mathrm{F} - \mathrm{F}\Lambda^+ \mathrm{F}\mathrm{F}^+ \Lambda^*)\mathcal{S}^+ + W - \mathrm{F}^+\mathrm{F}^* W \mathcal{S}\mathcal{S}^+$$
$$-\Lambda^+\Lambda[\mathrm{F}^+(\mathrm{F} - \mathrm{F}\Lambda^+\mathrm{F}\mathrm{F}^+\Lambda^*)\mathcal{S}^+ + W - \mathrm{F}^+\mathrm{F} W\mathcal{S}\mathcal{S}^+]\mathrm{F}\mathrm{F}^+,$$

Where $W \in B(H)$ is arbitrary, $\mathcal{S} = \Lambda^* - \mathrm{F}$.

Proof: From the presumption, we have Λ is self-adjoint and $\mathrm{F} \overset{*}{\leq} \Lambda$, and these two circumstances provide us with

$$\mathrm{F} \overset{*}{\leq} \Lambda^*, \tag{15}$$

and since we have $\mathrm{F} \overset{*}{\leq} \Lambda$, and by referring to Theorem 2.6, we will get:

$$R(\mathrm{F}) \subseteq R(\Lambda). \tag{16}$$

Obviously, we have

$$R(\mathrm{F}^*) \subseteq R(\mathrm{F}^*). \tag{17}$$

We note that relations (15), (16), and (17) represent the conditions of Theorem 3.3, when we express Ω as F, so their fulfillment allows us to use that theorem to construct the general solution formula for system $\Lambda X\mathrm{F} = \mathrm{F} = \mathrm{F}^* X \Lambda^*$, in the form

$$X = \Lambda^+ \mathrm{F}\mathrm{F}^+ + \mathrm{F}^+(\mathrm{F} - \mathrm{F}\Lambda^+ \mathrm{F}\mathrm{F}^+ \Lambda^*)\mathcal{S}^+ + W - \mathrm{F}^+\mathrm{F}^* W \mathcal{S}\mathcal{S}^+$$
$$-\Lambda^+\Lambda[\mathrm{F}^+(\mathrm{F} - \mathrm{F}\Lambda^+\mathrm{F}\mathrm{F}^+\Lambda^*)\mathcal{S}^+ + W - \mathrm{F}^+\mathrm{F} W\mathcal{S}\mathcal{S}^+]\mathrm{F}\mathrm{F}^+.$$

In the subsequent theorem, we will present the general solution formula for the system of operator equations (2), after mentioning the prerequisites for the solution's presence, and this will be based on a set of theories that have been previously mentioned, including our latest theory.

Theorem 3.5: *Let* $\Lambda, \mathrm{F}, \Omega_1, \Omega_2 \in B(H)$*, if* Λ, F *is regular and* $\mathcal{S}$ *is invertible,* $R(\Omega_1) \subseteq R(\Lambda), R({\Omega_1}^*) \subseteq R(\mathrm{F}^*)$*,* $\Lambda \overset{*}{\leq} \mathrm{F}^*, \mathrm{F} \overset{*}{\leq} \Lambda^*$*, then the general solution of the system of operator equation (2) is*

$$X = \Lambda^+\Omega_1\mathrm{F}^+ + ((\mathrm{F}^*)^+\Omega_1 - N)\mathcal{S}^{-1} + M^+[K - L - \mathrm{F}((\mathrm{F}^*)^+\Omega_1 - N)\mathcal{S}^{-1}\Lambda$$
$$+\mathrm{F}\Lambda^+\Lambda((\mathrm{F}^*)^+\Omega_1 - N)\mathcal{S}^+\mathrm{F}\mathrm{F}^+\Lambda]\Lambda^+ + W - M^+M\,W\Lambda\Lambda^+) - (\mathrm{F}^*)^+\mathrm{F}^*(M^+[K - L$$
$$-\mathrm{F}((\mathrm{F}^*)^+\Omega_1 - N)\mathcal{S}^{-1}\Lambda + \mathrm{F}\Lambda^+\Lambda((\mathrm{F}^*)^+\Omega_1 - N)\mathcal{S}^{-1}\mathrm{F}\mathrm{F}^+\Lambda]\Lambda^+$$
$$+W - M^+M\,W\Lambda\Lambda^+)) - \Lambda^+\Lambda((\mathrm{F}^*)^+\Omega_1 + N)\mathcal{S}^{-1}\mathrm{F}\mathrm{F}^+,$$

Where $W \in B(H)$ is arbitrary, $N = (\mathrm{F}^*)^+\mathrm{F}^*\Lambda^+\Omega_1\mathrm{F}^+\Lambda^*$, $K = \mathrm{F}\mathrm{F}^+\Omega_2\,\Lambda^+\Lambda$, $L = \mathrm{F}\Lambda^+\Omega_1\mathrm{F}^+\Lambda$, and $M = \mathrm{F}(I - \mathrm{F}\mathrm{F}^+)$, $\mathcal{S} = \Lambda^* - \mathrm{F}$,

Proof: If Λ, F is regular and $\mathcal{S}$ is invertible, and since $R(\Omega_1) \subseteq R(\Lambda), R({\Omega_1}^*) \subseteq R(\mathrm{F}^*)$, $\mathrm{F} \overset{*}{\leq} \Lambda^*$, These requirements represent the requirements for the existence of the solution for the system of equations (1) mentioned in the previous theorem 3.3, then the general solution for the system of operator equation $\Lambda X\mathrm{F} = \Omega_1 = \mathrm{F}^* X \Lambda^*$ is

$$X = \Lambda^+\Omega_1\mathrm{F}^+ + (\mathrm{F}^*)^+(\Omega_1 - \mathrm{F}^*\Lambda^+\Omega_1\mathrm{F}^+\Lambda^*)\mathcal{S}^{-1} + Z - (\mathrm{F}^*)^+\mathrm{F}^*Z$$
$$-\Lambda^+\Lambda[(\mathrm{F}^*)^+(\Omega_1 - \mathrm{F}^*\Lambda^+\Omega_1\mathrm{F}^+\Lambda^*)\mathcal{S}^{-1} + Z - (\mathrm{F}^*)^+\mathrm{F}^*Z]\mathrm{F}\mathrm{F}^+. \tag{18}$$

where Z is random operator. We will consider that X is a solution for the system, $\Lambda^* X\mathrm{F}^* = \Omega_2 = \mathrm{F}X\Lambda$, so we will substitute it into them

$$\Lambda^*\Lambda^+\Omega_1\mathrm{F}^+\mathrm{F}^* + \Lambda^*(\mathrm{F}^*)^+(\Omega_1 - \mathrm{F}^*\Lambda^+\Omega_1\mathrm{F}^+\Lambda^*)\mathcal{S}^{-1}\mathrm{F}^* + \Lambda^*Z\mathrm{F}^* - \Lambda^*(\mathrm{F}^*)^+\mathrm{F}^*Z\mathrm{F}^*$$
$$-\Lambda^*\Lambda^+\Lambda[(\mathrm{F}^*)^+(\Omega_1 - \mathrm{F}^*\Lambda^+\Omega_1\mathrm{F}^+\Lambda^*)\mathcal{S}^{-1} + Z - (\mathrm{F}^*)^+\mathrm{F}^*Z]\mathrm{F}\mathrm{F}^+\mathrm{F}^*$$
$$= \Omega_2$$
$$= \mathrm{F}\Lambda^+\Omega_1\mathrm{F}^+\Lambda + \mathrm{F}(\mathrm{F}^*)^+(\Omega_1 - \mathrm{F}^*\Lambda^+\Omega_1\mathrm{F}^+\Lambda^*)\mathcal{S}^{-1}\Lambda + \mathrm{F}Z\Lambda - \mathrm{F}(\mathrm{F}^*)^+\mathrm{F}^*Z\Lambda$$
$$-\mathrm{F}\Lambda^+\Lambda[(\mathrm{F}^*)^+(\Omega_1 - \mathrm{F}^*\Lambda^+\Omega_1\mathrm{F}^+\Lambda^*)\mathcal{S}^{-1} + Z - (\mathrm{F}^*)^+\mathrm{F}^*Z]\mathrm{F}\mathrm{F}^+\Lambda.$$

Now we will multiply the sides of this system of equations by $\Lambda^+\Lambda$ from the right and by $\mathrm{F}\mathrm{F}^+$ from the left to obtain

$$\mathrm{F}\mathrm{F}^+\Lambda^*\Lambda^+\Omega_1\mathrm{F}^+\mathrm{F}^*\,\Lambda^+\Lambda + \mathrm{F}\mathrm{F}^+\Lambda^*(\mathrm{F}^*)^+(\Omega_1 - \mathrm{F}^*\Lambda^+\Omega_1\mathrm{F}^+\Lambda^*)\mathcal{S}^{-1}\mathrm{F}^*\,\Lambda^+\Lambda$$
$$+\mathrm{F}\mathrm{F}^+\Lambda^*Z\mathrm{F}^*\,\Lambda^+\Lambda - \mathrm{F}\mathrm{F}^+\Lambda^*(\mathrm{F}^*)^+\mathrm{F}^*Z\mathrm{F}^*\,\Lambda^+\Lambda$$
$$-\mathrm{F}\mathrm{F}^+\Lambda^*\Lambda^+\Lambda[(\mathrm{F}^*)^+(\Omega_1 - \mathrm{F}^*\Lambda^+\Omega_1\mathrm{F}^+\Lambda^*)\mathcal{S}^{-1} + Z - (\mathrm{F}^*)^+\mathrm{F}^*Z]\mathrm{F}\mathrm{F}^+\mathrm{F}^*\,\Lambda^+\Lambda$$
$$= \mathrm{F}\mathrm{F}^+\Omega_2\,\Lambda^+\Lambda$$
$$= \mathrm{F}\mathrm{F}^+\mathrm{F}\Lambda^+\Omega_1\mathrm{F}^+\Lambda\,\Lambda^+\Lambda + \mathrm{F}\mathrm{F}^+\mathrm{F}(\mathrm{F}^*)^+(\Omega_1 - \mathrm{F}^*\Lambda^+\Omega_1\mathrm{F}^+\Lambda^*)\mathcal{S}^{-1}\Lambda\,\Lambda^+\Lambda$$
$$+\mathrm{F}\mathrm{F}^+\mathrm{F}Z\Lambda\,\Lambda^+\Lambda - \mathrm{F}\mathrm{F}^+\mathrm{F}(\mathrm{F}^*)^+\mathrm{F}^*Z\Lambda\,\Lambda^+\Lambda$$
$$-\mathrm{F}\mathrm{F}^+\mathrm{F}\Lambda^+\Lambda[(\mathrm{F}^*)^+(\Omega_1 - \mathrm{F}^*\Lambda^+\Omega_1\mathrm{F}^+\Lambda^*)\mathcal{S}^{-1} + Z - (\mathrm{F}^*)^+\mathrm{F}^*Z]\mathrm{F}\mathrm{F}^+\Lambda\,\Lambda^+\Lambda. \tag{19}$$

From the assumption we have $\Lambda \overset{*}{\leq} \mathrm{F}^*, \mathrm{F} \overset{*}{\leq} \Lambda^*$, this condition, leads us to the following data

$$\mathrm{F}^*\Lambda^+\Lambda = \Lambda, \quad \mathrm{F}\mathrm{F}^+\Lambda^* = \mathrm{F}.$$

These data, when used in the system of equations (19) and after making some simplifications, will transform this system into a single equation of the next form.

$$\mathrm{F}\Lambda^+\Omega_1\mathrm{F}^+\Lambda + \mathrm{F}(\mathrm{F}^*)^+(\Omega_1 - \mathrm{F}^*\Lambda^+\Omega_1\mathrm{F}^+\Lambda^*)\mathcal{S}^{-1}\Lambda + \mathrm{F}Z\Lambda - \mathrm{F}(\mathrm{F}^*)^+\mathrm{F}^*Z\Lambda$$
$$-\mathrm{F}\Lambda^+\Lambda[(\mathrm{F}^*)^+(\Omega_1 - \mathrm{F}^*\Lambda^+\Omega_1\mathrm{F}^+\Lambda^*)\mathcal{S}^{-1} + Z - (\mathrm{F}^*)^+\mathrm{F}^*Z]\mathrm{F}\mathrm{F}^+\Lambda$$
$$= \mathrm{F}\mathrm{F}^+\Omega_2\,\Lambda^+\Lambda,$$

after simplifying this equation, the next equation is produced

$$\mathrm{F}\Lambda^+\Omega_1\mathrm{F}^+\Lambda + \mathrm{F}(\mathrm{F}^*)^+(\Omega_1 - \mathrm{F}^*\Lambda^+\Omega_1\mathrm{F}^+\Lambda^*)\mathcal{S}^{-1}\Lambda + \mathrm{F}Z\Lambda - \mathrm{F}(\mathrm{F}^*)^+\mathrm{F}^*Z\Lambda$$
$$-\mathrm{F}\Lambda^+\Lambda(\mathrm{F}^*)^+(\Omega_1 - \mathrm{F}^*\Lambda^+\Omega_1\mathrm{F}^+\Lambda^*)\mathcal{S}^{-1}\mathrm{F}\mathrm{F}^+\Lambda - \mathrm{F}\Lambda^+\Lambda Z\mathrm{F}\mathrm{F}^+\Lambda$$
$$+ \mathrm{F}\Lambda^+\Lambda(\mathrm{F}^*)^+\mathrm{F}^*Z\mathrm{F}\mathrm{F}^+\Lambda$$
$$= \mathrm{F}\mathrm{F}^+\Omega_2\,\Lambda^+\Lambda. \tag{20}$$

From the assumption, we have $\Lambda \overset{*}{\leq} \mathrm{F}^*$. This relationship, along with Theorem 2.6, provides us with relationship $P_{\overline{R(\Lambda)}}\mathrm{F}^* = \Lambda$. From this relationship, we obtain $R(\Lambda^*) \subseteq R(\mathrm{F})$. According to Theorem 2.7, and the final relationship, we arrive at $\mathrm{FF}^+\Lambda^* = \Lambda^*$. This leads us to the following relationship

$$\Lambda = \Lambda(\mathrm{F}^*)^+\mathrm{F}^*.$$

Now we will use these last two relationships to simplify equation (20) to take the form

$$\begin{gathered}\mathrm{F}\Lambda^+\Omega_1\mathrm{F}^+\Lambda + \mathrm{F}(\mathrm{F}^*)^+(\Omega_1 - \mathrm{F}^*\Lambda^+\Omega_1\mathrm{F}^+\Lambda^*)\mathcal{S}^{-1}\Lambda + \mathrm{F}Z\Lambda - \mathrm{F}(\mathrm{F}^*)^+\mathrm{F}^*Z\Lambda \\ -\mathrm{F}\Lambda^+\Lambda(\mathrm{F}^*)^+(\Omega_1 - \mathrm{F}^*\Lambda^+\Omega_1\mathrm{F}^+\Lambda^*)\mathcal{S}^{-1}\mathrm{FF}^+\Lambda \\ = \mathrm{FF}^+\Omega_2\,\Lambda^+\Lambda.\end{gathered}$$

After rearranging the equation and placing the constants on one side and the variables on the other, the equation takes the following form

$$\begin{gathered}\mathrm{F}(I - \mathrm{FF}^+)Z\Lambda = \mathrm{FF}^+\Omega_2\,\Lambda^+\Lambda - \mathrm{F}\Lambda^+\Omega_1\mathrm{F}^+\Lambda - \mathrm{F}(\mathrm{F}^*)^+(\Omega_1 - \mathrm{F}^*\Lambda^+\Omega_1\mathrm{F}^+\Lambda^*)\mathcal{S}^{-1}\Lambda \\ +\mathrm{F}\Lambda^+\Lambda(\mathrm{F}^*)^+(\Omega_1 - \mathrm{F}^*\Lambda^+\Omega_1\mathrm{F}^+\Lambda^*)\mathcal{S}^{-1}\mathrm{FF}^+\Lambda.\end{gathered}$$

here we notice that Z represents a solution to the last equation, so by using Theorem 2.8, we obtain

$$\begin{aligned}&Z = \left(\mathrm{F}(I - \mathrm{FF}^+)\right)^+[\mathrm{FF}^+\Omega_2\Lambda^+\Lambda - \mathrm{F}\Lambda^+\Omega_1\mathrm{F}^+\Lambda - \mathrm{F}(\mathrm{F}^*)^+(\Omega_1 - \mathrm{F}^*\Lambda^+\Omega_1\mathrm{F}^+\Lambda^*)\mathcal{S}^{-1}\Lambda \\ &+\mathrm{F}\Lambda^+\Lambda(\mathrm{F}^*)^+(\Omega_1 - \mathrm{F}^*\Lambda^+\Omega_1\mathrm{F}^+\Lambda^*)\mathcal{S}^{-1}\mathrm{FF}^+\Lambda]\Lambda^+ + W - (\mathrm{F}(I - \mathrm{FF}^+))^+(\mathrm{F}(I - \\ &\mathrm{FF}^+))W\Lambda\Lambda^+,\end{aligned}$$

where W is arbitrary. We substitute the value of Z into (18), At the same time, we will express $(\mathrm{F}^*)^+\mathrm{F}^*\Lambda^+\Omega_1\mathrm{F}^+\Lambda^*$ as H, $\mathrm{FF}^+\Omega_2\,\Lambda^+\Lambda$ *as* K, $\mathrm{F}\Lambda^+\Omega_1\mathrm{F}^+\Lambda$ as L, and $\mathrm{F}(I - \mathrm{FF}^+)$ as M to obtain

$$\begin{gathered}\mathrm{X} = \Lambda^+\Omega_1\mathrm{F}^+ + ((\mathrm{F}^*)^+\Omega_1 - \mathrm{N})\mathcal{S}^{-1} + \mathrm{M}^+[\mathrm{K} - \mathrm{L} - \mathrm{F}((\mathrm{F}^*)^+\Omega_1 - \mathrm{H})\mathcal{S}^{-1}\Lambda \\ +\mathrm{F}\Lambda^+\Lambda((\mathrm{F}^*)^+\Omega_1 - N)\mathcal{S}^{-1}\mathrm{FF}^+\Lambda]\Lambda^+ + W - M^+M\,W\Lambda\Lambda^+) - \\ (\mathrm{F}^*)^+\mathrm{F}^*(M^+[K - L - \mathrm{F}((\mathrm{F}^*)^+\Omega_1 - \mathrm{N})\mathcal{S}^{-1}\Lambda + \mathrm{F}\Lambda^+\Lambda((\mathrm{F}^*)^+\Omega_1 - \mathrm{N}) \\ \mathcal{S}^{-1}\mathrm{FF}^+\Lambda]\Lambda^+ + \mathrm{W} - \mathrm{M}^+\mathrm{M}\,\mathrm{W}\Lambda\Lambda^+)) - \Lambda^+\Lambda((\mathrm{F}^*)^+\Omega_1 + \mathrm{N})\mathcal{S}^{-1}\mathrm{FF}^+.\end{gathered}$$

4. Discussion and Conclusion

In this paper, the general solution formula for the system of equations $\Lambda X\mathrm{F} = \Omega = \mathrm{F}^*X\Lambda^*$ is discussed by examining the conditions that must be met for this solution to exist. Initially, the essential conditions for the presence of the solution represented by θ were proposed, in addition to other cases. Afterwards, the general solution for this system was proposed based on the Moore-Penrose inverse. Furthermore, the general solution formula for the system $\Lambda\mathrm{XF} = \Omega_1 = \mathrm{F}^*\mathrm{X}\Lambda^*$, $\Lambda^*\mathrm{XF}^* = \Omega_2 = \mathrm{FX}\Lambda$ was also proposed based on the Moore-Penrose inverse. Therefore, the method used to construct these solutions might be suitable for constructing solution formulas for other systems, such as the system $\Lambda\mathrm{XF} + \mathrm{FX}\Lambda = \Omega = \mathrm{AXB} + \mathrm{BXA}$, for example.

References

[1] Y. Tian, "The solvability of two linear matrix equations," *Linear Multilinear Algebra*, vol. 48, no. 2, pp. 123–147 (2000).

[2] D. C. Sorensen and A. C. Antoulas, "The Sylvester equation and approximate balanced reduction," *Linear Algebra Appl.*, vol. 351, pp. 671–700 (2002).

[3] F. Piao, Q. Zhang, and Z. Wang, "The solution to matrix equation AX+XTC=BAX+XTC=B," *J. Franklin Inst.*, vol. 344, no. 8, pp. 1056–1062 (2007).

[4] Q. W. Wang and C. K. Li, "Ranks and the least-norm of the general solution to a system of quaternion matrix equations," *Linear Algebra Appl.*, vol. 430, no. 5–6, pp. 1626–1640 (2009).

[5] M. Dehghan and M. Hajarian, "The general coupled matrix equations over generalized bisymmetric matrices," *Linear Algebra Appl.*, vol. 432, no. 6, pp. 1531–1552 (2010).

[6] Q. W. Wang, X. Zhang, and J. W. van der Woude, "A new simultaneous decomposition of a matrix quaternity over an arbitrary division ring with applications," *Commun. Algebra*, vol. 40, no. 7, pp. 2309–2342 (2012).

[7] S. D. Mohsen, "New generalizations for MM-hyponormal operators," *Iraqi J. Sci.*, pp. 6477–6482 (2023).

[8] S. D. Mohsen, "Solvability of (λ,μ)(λ,μ)-commuting operator equations for bounded generalization of hyponormal operators," *Iraqi J. Sci.*, vol. 63, no. 9, pp. 3854–3860 (2022).

[9] C. G. Khatri and S. K. Mitra, "Hermitian and nonnegative definite solutions of linear matrix equations," *SIAM J. Appl. Math.*, vol. 31, no. 4, pp. 579–585 (1976).

[10] A. Dajić and J. J. Koliha, "Positive solutions to the equations AX=CAX=C and XB=DXB=D for Hilbert space operators," *J. Math. Anal. Appl.*, vol. 333, no. 2, pp. 567–576 (2007).

[11] Q. Wang, H. Chang, and C. Lin, "On the centro-symmetric solution of a system of matrix equations over a regular ring with identity," *Algebra Colloq.*, pp. 555–570 (2007).

[12] F. O. Farid, M. S. Moslehian, Q. W. Wang, and Z. C. Wu, "On the Hermitian solutions to a system of adjointable operator equations," *Linear Algebra Appl.*, vol. 437, no. 7, pp. 1854–1891 (2012).

[13] C. Deng, “On the solutions of operator equation CAX=C=XACCAX =C=XAC,” *J. Math. Anal. Appl.*, vol. 398, no. 2, pp. 664–670 (2013).

[14] M. Vosough and M. S. Moslehian, “Solutions of the system of operator equations BXA=B=AXBBXA=B=AXB via ∗∗-order,” arXiv Prepr., arXiv:1705.07037 (2017).

[15] P. Bhimasankaram, “Common solutions to the linear matrix equations AX=CAX=C, XB=DXB=D, and FXG=HFXG=H,” Sankhyā, Indian J. Stat., Ser. A, pp. 404–409 (1976).

[16] X. Zhang and J. Guoxing, “Solutions to the system of operator equations AXB=C=BXAAXB=C=BXA,” Acta Math. Sci., vol. 38, no. 4, pp. 1143–1150 (2018).

[17] S. D. Mohsen and N. M. Atheab, “Some results on (ℵ,k)(ℵ,k)-hyponormal operators,” *J. Phys. Conf. Ser*., vol. 1591, no. 1, p. 012064 (2020).

[18] Q. W. Wang, “The general solution to a system of real quaternion matrix equations,” *Comput. Math. Appl.*, vol. 49, no. 5–6, pp. 665–675 (2005).

[19] S. D. Mohsen and Y. H. Thiyab, “Some characteristics of completeness property in fuzzy soft bb-metric space,” *J. Appl. Sci. Eng.*, vol. 27, no. 3, pp. 2227–2232 (2023).

[20] Y. X. Peng, X. Y. Hu, and L. Zhang, “An iteration method for the symmetric solutions and the optimal approximation solution of the matrix equation AXB=CAXB=C,” *Appl. Math. Comput.*, vol. 160, no. 3, pp. 763–777א2005).

[21] S. D. Mohsen, “Novel results of KK-quasi (Γ−M)-hyponormal operator,” *Iraqi J. Sci.*, pp. 374–380א2024).

[22] S. D. Mohsen, “On−n(k,m)−n-paranormal operators,” *Iraqi J. Sci.*, pp. 3087–3092 (2023).

[23] S. D. Mohsen, “On(n,D)-quasi operators,” *Iraqi J. Comput. Sci. Math.*, vol. 5, no. 1, pp. 175–180 (2024).

[24] S. D. Mohsen, “Some generalizations of fuzzy soft(k∗−A)-quasinormal operators in fuzzy soft Hilbert spaces,” *J. Interdiscip. Math.*, vol. 26, no. 6, pp. 1133–1143 (2023).

[25] E. Kreyszig, Introductory Functional Analysis with Applications. New York: Wiley (1991).

[26] V. Rakočević, “On continuity of the Moore-Penrose and Drazin inverses,” *Mat. Vesn.*, vol. 49, pp. 163–172 (1997).

[27] K. K. Vasishtha, Teacher Education in India: A Study in New Dimensions. New Delhi: Concept Publishing (1979).

[28] C. R. Saranya and K. C. Sivakumar, "Generalized inverses of an invertible infinite matrix over a finite field," *Linear Algebra Appl.*, vol. 418, no. 2–3, pp. 468–479 (2006).

[29] C. Deng and A. Yu, "Some relations of projection and star order in Hilbert space," *Linear Algebra Appl.*, vol. 474, pp. 158–168 (2015).

[30] X. M. Xu, H. K. Du, X. Fang, and Y. Li, "The supremum of linear operators for the $**$-order," Linear Algebra Appl., vol. 433, no. 11–12, pp. 2198–2207 (2010).

Received December, 2024