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A semi-analytical approach for solving time-fractional nonlinear KdV equations using fractional multiple transformed iterative method

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Abstract

This paper introduces a novel successful hybrid method of solving different kinds for time-fractional nonlinear partial differential equation (TFr-NLPDEs). A technique combine Shehu, & Sumudu integral transform of fractional order derivative, that described in the Caputo sense, with the variational iteration method (VIM). Time-fractional equations such as the Kortwe-De Veries (KdV) and the Kortweg-De Veries Burgers (KdV-B) equations are solved by the suggested method with results occurred as a convergent series. Some numerical examples confirm the effectiveness of the suggested method under appropriate initial conditions. The resulting series solutions converge to the exact solutions of the equations. The results are compared with the exact solutions and to some existing approaches in the literature to show the efficiency, accuracy, and simplicity of the method.

Subject Classification: *47B49, 44A30.*

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1. Introduction

Notable achievements in the theory and applications of fractional differential equations (FrDEs) have been made during the past century. In a wide range of research domains in engineering, applied mathematics, and physical sciences; these equations are being used extensively for modeling problems **[1-3]**. A fractional calculus (FrC) category for mathematical analysis which generalizes operation for differentiation & integration to be non-integer

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(fractional) order [4]. Many scientists developed various approaches to the (FrC), like Grünwald-Letnikov, Riemann-Liouville, Caputo, and Caputo Fabrizio [5, 6]. They found the ability to describe complex phenomena such as viscoelasticity, diffusion, and fractal geometries, with more accurate results that cannot be handled with classical calculus. Using the idea of (FrC), the (FrDEs) are an extension of standard integer order DEs **[7]**. Though it is commonly known that the operations of differentiation and integration of fractional order are nonlocal, both integer order of these operations are known to be local. This implies that a system's future state depends on all of its past states alongside to its current one. This is actually the primary reason why fractional order differential operator are a great tool for characterizing inherited features and memory of different engineering and other sciences processes. The nonlocal feature of (FrDEs) is the main benefit of employing them in many engineering and other scientific research domains. This feature enables (FrDEs) to be used as a strong alternative mathematical tool to simulate the characteristics of several real-world processes and physical systems. This article examines three widely recognized equation like as nonlinear time-fractional KdV & nonlinear time-fractional KdV-B equations.

The KdV type equations are essential to applied sciences and physics. They are a classic NLPDEs that have drawn increased attention because of their many applications in the theory of quantum fields and the plasma [8]. Fractional KdV equation provides an elegant model for many physical phenomena, such as the evolution of unidirectional nonlinear shallow water waves [9]. On the other hand, the KdV-B equation is an interesting mathematical model of waves on shallow water surfaces. It has received a lot of attention due to the considerable interesting of its mathematical theory that make it the subject of intensive mathematical and physical research [10].

Many researchers paid attention to solving various linear/nonlinear differential equations with fractional orders using various approaches like as the (ADM), the (VIM), the (HPM), and the (HAM) [11-14]. Besides that, many scholars suggested combining integral transforms with iterative techniques to solve such types of equations and integral equations [15-20] and references therein.

The primary objective of this article is to develop and study the use of combining multiple integral transforms of fractional order derivatives in the Caputo sense with an iterative method to solve some nonlinear time-fractional equations. The multiple Shehu-Sumudu integral transform is combined with the VIM for solve nonlinear timefractional KdV & the KdV-B equations. The method is termed the Fractional Multiple Transformed Variational Iteration Method (Fr-MTVIM). The proposed method provides an effective way to solve FrNLPDEs without difficulty and gives highly accurate solution & least amount for calculation, as shown in illustrated examples in the next parts. Algebra is play in applied mathematics see Mohammed, Asaad, and Ahmed [32] and Ahmed, Ebrahim, and Al-Fayadh [33].

2. Basic Useful Priminalaries

Definition 1 [21]: If $f\colon \mathbb{R}^+ \to \mathbb{R}$ and $\varphi \in (k-1,k), k \in \mathbb{N}$, then:

$$ {}_0^C D_x^{\varphi} f(x) = \frac{1}{\Gamma(k-\varphi)} \int_0^x \frac{f^{(k)}(\mu)}{(x-\mu)^{\varphi-k+1}} d\mu \tag{1} $$

called Caputo fractional derivative for order φ, where $\Gamma(.)$ be Gamma map.

Definition 2 [22, 23]: Consider the following set

$A = \left\{ w(\varkappa) \mid \exists\, K, \varepsilon_1, \varepsilon_2 > 0, |w(\varkappa)| < P * exp\left(\frac{|\varkappa|}{\varepsilon_i}\right),\ \text{if } \varkappa \in (-1)^i \times [0,\infty) \right\}$, then Sumudu integral transform for map $f(\varkappa) \in A$ is,

$$ S_{\varkappa}[f(\varkappa)] = \mathcal{F}(\mathrm{r}) = \frac{1}{\mathrm{r}} \int_0^{\infty} exp\left(\frac{-\varkappa}{\mathrm{r}}\right) f(\varkappa)\, d\varkappa, \quad \varkappa \geq 0 \tag{2} $$

Shehu integral transform for a map. $z(s) \in A$ is,

$$ U_s\,[z(s)] = Z(b,v) = \int_0^{\infty} exp\left(-\frac{b}{v}\, s\right) z(s) ds, \quad s \geq 0 \tag{3} $$

Definition 3 [24, 25]: If $f(\varkappa)$ and $u(x)$ are two non-negative functions in the set A, then the Sumudu transform of fractional order φ_1 of the non-negative function $f(\varkappa)$ is:

$$ S_{\varkappa}^{\varphi_1}[f(\varkappa)] = F_{\varphi_1}(\mathrm{r}) = \int_0^{\infty} E_{\varphi_1}\left(\frac{-\varkappa^{\varphi_1}}{\mathrm{r}^{\varphi_1}}\right) f(\varkappa)(d\varkappa)^{\varphi_1}, k-1 < \varphi_1 < k, k \in Z^+ \tag{4} $$

the Shehu transform of fractional order φ_2 of the non-negative function $g(s)$ is:

$$ U_s^{\varphi_2}[g(s)] = G_{\varphi_2}(b,v) = \int_0^{\infty} E_{\varphi_2}\left(\frac{-b^{\varphi_2} s^{\varphi_2}}{v^{\varphi_2}}\right) g(s)(ds)^{\varphi_2}, \quad p-1 < \varphi_2 < p, p \in Z^+ \tag{5} $$

Definition 4: Consider the following set

$B = \left\{ f(s,\varkappa) \mid \exists\, M > 0,\ \text{s.t}\, |f(s,\varkappa)| \leq M \exp(ks + c\varkappa),\ \forall s, \varkappa \geq 0, \right.$
$\left. M \text{ is constant and } \lim_{s,\varkappa \to \infty} \exp\left(-\left(\frac{b}{v}\, s + \frac{\varkappa}{\mathrm{r}}\right)\right) |f(s,\varkappa)| = 0 \right\}$

then the fractional multiple Shehu-Sumudu transform of fractional order φ_2, φ_1 (Fr-MSST) of a non-negative function $g(s,\varkappa) \in B$, for $p-1 < \varphi_2 < p$ and $k-1 < \varphi_1 < k$ and $k, p \in Z^+$, is:

$$ U_s^{\varphi_2} S_{\varkappa}^{\varphi_1}[g(s,\varkappa)] = G_{\varphi_2,\varphi_1}\big((b,v),\mathrm{r}\big) $$
$$ = \int_0^{\infty}\int_0^{\infty} E_{\varphi_2,\varphi_1}\left(-\frac{b^{\varphi_2} s^{\varphi_2}}{v^{\varphi_2}} - \frac{\varkappa^{\varphi_1}}{\mathrm{r}^{\varphi_1}}\right) g(s,\varkappa)(d\varkappa)^{\varphi_1}(ds)^{\varphi_2} \tag{6} $$

where $b, \mathrm{r} \in \mathbb{C}$ (complex) and $E_{\varphi_2,\varphi_1}(s,\varkappa) = \sum_{\varphi_2,\varphi_1=0}^{\infty} \frac{s^{\varphi_2}}{\Gamma(\varphi_2 p+1)} \frac{\varkappa^{\varphi_1}}{\Gamma(\varphi_1 k+1)}$ is the Mittag-Leffler with the Gamma functions $\Gamma(\varphi_2 p + 1), \Gamma(\varphi_1 k + 1)$ **[26, 27]**.

Theorem 1 [28]: *Let* $\varphi_1 > 0,\ k = \varphi_1 + 1\ \&\ g(\mathcal{s},\varkappa), \frac{\partial}{\partial\varkappa} g(\mathcal{s},\varkappa), \frac{\partial^2}{\partial\varkappa^2} g(\mathcal{s},\varkappa)$, $\ldots, \frac{\partial^k}{\partial\varkappa^k} g(\mathcal{s},\varkappa), {}_0^C D_\varkappa^{\varphi_1} g(\mathcal{s},\varkappa) \in B$, *then:*

$$S\left[{}_0^C D_\varkappa^{\varphi_1} g(\mathcal{s},\varkappa)\right] = \left(\frac{1}{\mathrm{r}}\right)^{\varphi_1} S[g(\mathcal{s},\varkappa)] - \sum_{i=0}^{k-1} \left(\frac{1}{\mathrm{r}}\right)^{\varphi_1 - i - 1} g^{(i)}(\mathcal{s},0), \quad k \in \mathbb{N}$$

Theorem 2 [25]: *If* $\varphi_2 > 0,\ p = \varphi_2 + 1\ \&\ g(\mathcal{s},\varkappa), \frac{\partial}{\partial\mathcal{s}} g(\mathcal{s},\varkappa), \frac{\partial^2}{\partial\mathcal{s}^2} g(\mathcal{s},\varkappa), \ldots,$ $\frac{\partial^p}{\partial\mathcal{s}^p} g(\mathcal{s},\varkappa), {}_0^C D_\mathcal{s}^{\varphi_2} g(\mathcal{s},\varkappa) \in B$, then:

$$U\left[{}_0^C D_\mathcal{s}^{\varphi_2} g(\mathcal{s},\varkappa)\right] = \left(\frac{b}{v}\right)^{\varphi_2} U[g(\mathcal{s},\varkappa)] - \sum_{j=0}^{p-1} \left(\frac{b}{v}\right)^{\varphi_2 - j - 1} g^{(j)}(0,\varkappa), \quad p \in \mathbb{N}.$$

The following theorems provide some properties of the fractional multiple Shehu-Sumudu transform (Fr-MSST).

Theorem 3: *Existence of the Fractional Multiple Shehu-Sumudu Transform*

Let the function $g(\mathcal{s},\varkappa) \in B$ for $0 \le \mathcal{s} \le \alpha, \& \ 0 \le \varkappa \le \beta, and$ is for exponential order $(a^{\varphi_2}\mathcal{s}^{\varphi_2} + w^{\varphi_1}\varkappa^{\varphi_1})$, then it's fractional Shehu-Sumudu transform $G_{\varphi_2,\varphi_1}\big((b,v),\mathrm{r}\big)$ exists.

Proof: By using definition 4, we have

$$U_\mathcal{s}^{\varphi_2} S_\varkappa^{\varphi_1}[g(\mathcal{s},\varkappa)] = \int_0^\infty \int_0^\infty E_{\varphi_2,\varphi_1}\left(-\frac{b^{\varphi_2}\mathcal{s}^{\varphi_2}}{v^{\varphi_2}} - \frac{\varkappa^{\varphi_1}}{\mathrm{r}^{\varphi_1}}\right) g(\mathcal{s},\varkappa)\,(d\varkappa)^{\varphi_1}(d\mathcal{s})^{\varphi_2}$$

$$= \underbrace{\int_0^\beta \int_0^\alpha E_{\varphi_2,\varphi_1}\left(-\frac{b^{\varphi_2}\mathcal{s}^{\varphi_2}}{v^{\varphi_2}} - \frac{\varkappa^{\varphi_1}}{\mathrm{r}^{\varphi_1}}\right) g(\mathcal{s},\varkappa)\,(d\varkappa)^{\varphi_1}(d\mathcal{s})^{\varphi_2}}_{\text{exists, since the function } g(\mathcal{s},\varkappa) \text{is piecewise continuous}} +$$

$$\underbrace{\int_\beta^\infty \int_\alpha^\infty E_{\varphi_2,\varphi_1}\left(-\frac{b^{\varphi_2}\mathcal{s}^{\varphi_2}}{v^{\varphi_2}} - \frac{\varkappa^{\varphi_1}}{\mathrm{r}^{\varphi_1}}\right) g(\mathcal{s},\varkappa)\,(d\varkappa)^{\varphi_1}(d\mathcal{s})^{\varphi_2}}_{*}$$

For the part *, we have

$$\left|\int_\beta^\infty \int_\alpha^\infty E_{\varphi_2,\varphi_1}\left(-\frac{b^{\varphi_2}\mathcal{s}^{\varphi_2}}{v^{\varphi_2}} - \frac{\varkappa^{\varphi_1}}{\mathrm{r}^{\varphi_1}}\right) g(\mathcal{s},\varkappa)\,(d\varkappa)^{\varphi_1}(d\mathcal{s})^{\varphi_2}\right| \le$$

$$\int_\beta^\infty \int_\alpha^\infty E_{\varphi_2,\varphi_1}\left(-\frac{b^{\varphi_2}\mathcal{s}^{\varphi_2}}{v^{\varphi_2}} - \frac{\varkappa^{\varphi_1}}{\mathrm{r}^{\varphi_1}}\right) |g(\mathcal{s},\varkappa)|(d\varkappa)^{\varphi_1}(d\mathcal{s})^{\varphi_2}$$

$$\le \int_\beta^\infty \int_\alpha^\infty E_{\varphi_2,\varphi_1}\left(-\frac{b^{\varphi_2}\mathcal{s}^{\varphi_2}}{v^{\varphi_2}} - \frac{\varkappa^{\varphi_1}}{\mathrm{r}^{\varphi_1}}\right).K\, E_{\varphi_2,\varphi_1}(a^{\varphi_2}\mathcal{s}^{\varphi_2} + w^{\varphi_1}\varkappa^{\varphi_1})\,(d\varkappa)^{\varphi_1}(dx)^{\varphi_2}$$

$$= K \int_\beta^\infty \int_\alpha^\infty E_{\varphi_2,\varphi_1}\left(-\left(\frac{b^{\varphi_2} - v^{\varphi_2}a^{\varphi_2}}{v^{\varphi_2}}\right)\mathcal{s}^{\varphi_2} - \left(\frac{1 - \mathrm{r}^{\varphi_1}w^{\varphi_1}}{\mathrm{r}^{\varphi_1}}\right)\varkappa^{\varphi_1}\right)(d\varkappa)^{\varphi_1}(dx)^{\varphi_2}$$

$$= -K\left\{\frac{v^{\varphi_2}}{(b^{\varphi_2} - v^{\varphi_2}a^{\varphi_2})} + \frac{\mathrm{r}^{\varphi_1}}{(1 - \mathrm{r}^{\varphi_1}w^{\varphi_1})}\right\} \lim_{p,k\to\infty} E_{\varphi_2,\varphi_1}\left(\left(\frac{b^{\varphi_2} - v^{\varphi_2}a^{\varphi_2}}{v^{\varphi_2}}\right)\mathcal{s}^{\varphi_2} + \right.$$
$$\left.\left(\frac{1 - \mathrm{r}^{\varphi_1}w^{\varphi_1}}{\mathrm{r}^{\varphi_1}}\right)\varkappa^{\varphi_1}\right)\Big|_{\beta,\alpha}^{p,k}$$

$$= -K\left\{\frac{v^{\varphi_2}}{(b^{\varphi_2} - v^{\varphi_2}a^{\varphi_2})} + \frac{\mathrm{r}^{\varphi_1}}{(1 - \mathrm{r}^{\varphi_1}w^{\varphi_1})}\right\}$$

Theorem 4: *Linearity of the Fractional Multiple Shehu-Sumudu Transform*

Let $g(\mathscr{s},\varkappa)$ and $y(\mathscr{s},\varkappa)$ be functions in the set B, and $\vartheta g(\mathscr{s},\varkappa) + \sigma y(\mathscr{s},\varkappa) \in B$, where ϑ, ω are nonzero constants, then:

$$U_{\mathscr{s}}^{\varphi_2} S_{\varkappa}^{\varphi_1}[\vartheta g(\mathscr{s},\varkappa) + \omega y(\mathscr{s},\varkappa)] = \vartheta U_{\mathscr{s}}^{\varphi_2} S_{\varkappa}^{\varphi_1}[g(\mathscr{s},\varkappa)] + \sigma\, U_{\mathscr{s}}^{\varphi_2} S_{\varkappa}^{\varphi_1}[y(\mathscr{s},\varkappa)]$$

Proof: Applying definition 4, we get

$$U_{\mathscr{s}}^{\varphi_2} S_{\varkappa}^{\varphi_1}[\vartheta g(\mathscr{s},\varkappa) + \omega y(\mathscr{s},\varkappa)] =$$

$$\iint_0^\infty E_{\varphi_2,\varphi_1}\left(-\frac{b^{\varphi_2}\mathscr{s}^{\varphi_2}}{\upsilon^{\varphi_2}} - \frac{\varkappa^{\varphi_1}}{\mathrm{r}^{\varphi_1}}\right)\{\vartheta g(\mathscr{s},\varkappa) + \sigma y(\mathscr{s},\varkappa)\}\,(d\varkappa)^{\varphi_1}(d\mathscr{s})^{\varphi_2}$$

$$= \vartheta \iint_0^\infty E_{\varphi_2,\varphi_1}\left(-\frac{b^{\varphi_2}\mathscr{s}^{\varphi_2}}{\upsilon^{\varphi_2}} - \frac{\varkappa^{\varphi_1}}{\mathrm{r}^{\varphi_1}}\right) g(\mathscr{s},\varkappa)(d\varkappa)^{\varphi_1}(d\mathscr{s})^{\varphi_2}$$

$$+ \sigma \iint_0^\infty E_{\varphi_2,\varphi_1}\left(-\frac{b^{\varphi_2}\mathscr{s}^{\varphi_2}}{\upsilon^{\varphi_2}} - \frac{\varkappa^{\varphi_1}}{\mathrm{r}^{\varphi_1}}\right) y(\mathscr{s},\varkappa)\,(d\varkappa)^{\varphi_1}(d\mathscr{s})^{\varphi_2}$$

$$= \vartheta\, G_{\varphi_2,\varphi_1}\big((b,\upsilon),\mathrm{r}\big) + \sigma\, Y_{\varphi_2,\varphi_1}\big((b,\upsilon),\mathrm{r}\big)$$

$$= \vartheta U_{\mathscr{s}}^{\varphi_2} S_{\varkappa}^{\varphi_1}[g(\mathscr{s},\varkappa)] + \sigma\, U_{\mathscr{s}}^{\varphi_2} S_{\varkappa}^{\varphi_1}[y(\mathscr{s},\varkappa)]$$

3. Structure of the Fractional Multiple Transformed Variational Iteration Method (Fr-MTVIM)

Put general inhomogeneous nonlinear differential equation:

$${}_0^C D_{\varkappa}^{\varphi_1}\, g(\mathscr{s},\varkappa) + {}_0^C D_{x}^{\varphi_2}\, g(\mathscr{s},\varkappa) + Ng(\mathscr{s},\varkappa) = \mathfrak{f}(\mathscr{s}), \quad n-1 < \varphi_2 < n,\ m-1 < \varphi_1 < mn,\ m \in Z^+ \tag{7}$$

with the initial conditions

$$g(\mathscr{s},0) = \mathfrak{f}(\mathscr{s}),\ g_{\varkappa}(\mathscr{s},0) = \mathfrak{f}_1(\mathscr{s}), \dots, g_{m\varkappa}(\mathscr{s},0) = \mathfrak{f}_m(\mathscr{s}) \tag{8}$$

and the boundary conditions

$$g(0,\varkappa) = \sigma(\varkappa),\ g_{\mathscr{s}}(0,\varkappa) = \sigma_1(\varkappa), \dots, g_{n\mathscr{s}}(0,\varkappa) = \sigma_n(\varkappa) \tag{9}$$

${}_0^C D_{\varkappa}^{\varphi_1}$ and ${}_0^C D_{x}^{\varphi_2}$ are denoted the time-fractional derivative, A space-fractional derivative for non-negative continuous map. $g(\mathscr{s},\varkappa)$, respectively, N the nonlinear operator & $f(\mathscr{s})$ inhomogeneous term.

Apply Fr-MSST, with its properties, on both sides of the equations (7-9),

$$U_{\mathscr{s}}^{\varphi_2} S_{\varkappa}^{\varphi_1}\left[{}_0^C D_{\varkappa}^{\varphi_1}\, g(\mathscr{s},\varkappa)\right] + U_{\mathscr{s}}^{\varphi_2} S_{\varkappa}^{\varphi_1}\left[{}_0^C D_{x}^{\varphi_2}\, g(\mathscr{s},\varkappa)\right] + U_{\mathscr{s}}^{\varphi_2} S_{\varkappa}^{\varphi_1}[Ng(\mathscr{s},\varkappa)] = U_{\mathscr{s}}^{\varphi_2} S_{\varkappa}^{\varphi_1}[f(\mathscr{s})] \tag{10}$$

$$\left(\frac{1}{\mathrm{r}}\right)^{\varphi_1} S[g(\mathscr{s},\varkappa)] - \sum_{i=0}^{m-1}\left(\frac{1}{\mathrm{r}}\right)^{\varphi_1-i-1} g^{(i)}(\mathscr{s},0) + \left(\frac{b}{\upsilon}\right)^{\varphi_2} U[g(\mathscr{s},\varkappa)] - \sum_{j=0}^{n-1}\left(\frac{b}{\upsilon}\right)^{\varphi_2-j-1} g^{(j)}(0,\varkappa) + U_{\mathscr{s}}^{\varphi_2} S_{\varkappa}^{\varphi_1}[Ng(\mathscr{s},\varkappa)] = U_{\mathscr{s}}^{\varphi_2} S_{\varkappa}^{\varphi_1}[f(\mathscr{s})] \tag{11}$$

$$S[g(\mathscr{s},\varkappa)] - \sum_{i=0}^{m-1}\left(\frac{1}{\mathrm{r}}\right)^{i+1} g^{(i)}(\mathscr{s},0) + U[g(\mathscr{s},\varkappa)] - \sum_{j=0}^{n-1}\left(\frac{b}{\upsilon}\right)^{j+1} g^{(j)}(0,\varkappa) + (\mathrm{r})^{\varphi_1}\left(\frac{\upsilon}{b}\right)^{\varphi_2} U_{\mathscr{s}}^{\varphi_2} S_{\varkappa}^{\varphi_1}[Ng(\mathscr{s},\varkappa)] = (\mathrm{r})^{\varphi_1}\left(\frac{\upsilon}{b}\right)^{\varphi_2} U_{\mathscr{s}}^{\varphi_2} S_{\varkappa}^{\varphi_1}[f(\mathscr{s})] \tag{12}$$

put inverse for Fr-MSST, both sides in eq. (12)

$$g(\mathcal{s},\varkappa)-\ g_{m\varkappa}(\mathcal{s},0)-g_{n\mathcal{s}}(0,\varkappa)+U_{\mathcal{s}}^{\varphi_2}S_{\varkappa}^{\varphi_1^{-1}}\left[(\mathrm{r})^{\varphi_1}\left(\frac{v}{b}\right)^{\varphi_2}U_{\mathcal{s}}^{\varphi_2}S_{\varkappa}^{\varphi_1}[Ng(\mathcal{s},\varkappa)]\right]$$
$$=U_{\mathcal{s}}^{\varphi_2}S_{\varkappa}^{\varphi_1^{-1}}\left[(\mathrm{r})^{\varphi_1}\left(\frac{v}{b}\right)^{\varphi_2}U_{\mathcal{s}}^{\varphi_2}S_{\varkappa}^{\varphi_1}[f(\mathcal{s})]\right] \tag{13}$$

Put $\frac{\partial^{\varphi_1}}{\partial\varkappa^{\varphi_1}}$ on both sides for eq. (13)

$$\frac{\partial^{\varphi_1}}{\partial\varkappa^{\varphi_1}}g(\mathcal{s},\varkappa)+\frac{\partial^{\varphi_1}}{\partial\varkappa^{\varphi_1}}U_{\mathcal{s}}^{\varphi_2}S_{\varkappa}^{\varphi_1^{-1}}\left[(\mathrm{r})^{\varphi_1}\left(\frac{v}{b}\right)^{\varphi_2}U_{\mathcal{s}}^{\varphi_2}S_{\varkappa}^{\varphi_1}[Ng(\mathcal{s},\varkappa)]\right]-$$
$$\frac{\partial^{\varphi_1}}{\partial\varkappa^{\varphi_1}}U_{\mathcal{s}}^{\varphi_2}S_{\varkappa}^{\varphi_1^{-1}}\left[(\mathrm{r})^{\varphi_1}\left(\frac{v}{b}\right)^{\varphi_2}U_{\mathcal{s}}^{\varphi_2}S_{\varkappa}^{\varphi_1}[f(\mathcal{s})]\right]=0 \tag{14}$$

Now, use Fr-VIM **[29]** for solve eq. (14), where correction functional

$$g_{n+1}(\mathcal{s},\varkappa)=g_n(\mathcal{s},\varkappa)+$$
$$\frac{1}{\Gamma(\varphi_1+1)}\int_0^{\varkappa}\lambda(\xi)\left\{\frac{\partial^{\varphi_1}}{\partial\xi^{\varphi_1}}g_n(\mathcal{s},\xi)+\right.$$
$$\frac{\partial^{\varphi_1}}{\partial\xi^{\varphi_1}}U_{\mathcal{s}}^{\varphi_2}S_{\xi}^{\varphi_1^{-1}}\left[(\mathrm{r})^{\varphi_1}\left(\frac{v}{b}\right)^{\varphi_2}U_{\mathcal{s}}^{\varphi_2}S_{\xi}^{\varphi_1}\left[N\tilde{g}_n(\mathcal{s},\xi)\right]\right]-$$
$$\left.\frac{\partial^{\varphi_1}}{\partial\xi^{\varphi_1}}U_{\mathcal{s}}^{\varphi_2}S_{\xi}^{\varphi_1^{-1}}\left[(\mathrm{r})^{\varphi_1}\left(\frac{v}{b}\right)^{\varphi_2}U_{\mathcal{s}}^{\varphi_2}S_{\xi}^{\varphi_1}[f(\mathcal{s})]\right]\right\}(d\xi)^{\varphi_1} \tag{15}$$

when λ Lagrange multiplier, can identified from variational with stationary condition $\frac{\partial^{\varphi_1}}{\partial\xi^{\varphi_1}}\tilde{g}=0$, and $1+\lambda(\xi)=0$. Put $\lambda(\xi)=-1$, in eq. (15) yields

$$g_{n+1}(\mathcal{s},\varkappa)=g_n(\mathcal{s},\varkappa)-\frac{1}{\Gamma(\varphi_1+1)}$$
$$\int_0^{\varkappa}\left\{\frac{\partial^{\varphi_1}}{\partial\xi^{\varphi_1}}g_n(\mathcal{s},\xi)+\frac{\partial^{\varphi_1}}{\partial\xi^{\varphi_1}}U_{\mathcal{s}}^{\varphi_2}S_{\xi}^{\varphi_1^{-1}}\left[(\mathrm{r})^{\varphi_1}\left(\frac{v}{b}\right)^{\varphi_2}U_{\mathcal{s}}^{\varphi_2}S_{\xi}^{\varphi_1}\left[N\tilde{g}_n(\mathcal{s},\xi)\right]\right]-\right.$$
$$\left.\frac{\partial^{\varphi_1}}{\partial\xi^{\varphi_1}}U_{\mathcal{s}}^{\varphi_2}S_{\xi}^{\varphi_1^{-1}}\left[(\mathrm{r})^{\varphi_1}\left(\frac{v}{b}\right)^{\varphi_2}U_{\mathcal{s}}^{\varphi_2}S_{\xi}^{\varphi_1}[f(\mathcal{s})]\right]\right\}(d\xi)^{\varphi_1},\quad n\geq 0 \tag{16}$$

Hence, the solution is $g(\mathcal{s},\varkappa)=\lim_{n\to\infty}g_{n+1}(\mathcal{s},\varkappa)$.

4. Applications of the Fr-MTVIM

Example 1 [30]: Put following time-fractional KdV equation in Caputo sense

$${}_0^C D_{\varkappa}^{\varphi_1}g(\mathcal{s},\varkappa)+g(\mathcal{s},\varkappa)g_{\mathcal{s}}(\mathcal{s},\varkappa)+0.5\,g_{\mathcal{sss}}(\mathcal{s},\varkappa)=0,\quad 0<\varphi_1\leq 1 \tag{17}$$

with the initial condition

$$g(\mathcal{s},0)=6\rho^2\mathcal{s}ech^2(\rho\mathcal{s}) \tag{18}$$

Solution: Apply the Fr-MSST on both sides of the eq. (17) and then we put inverse for Fr-MSST over both side for the resulting equation, yields

$$g(\mathcal{s},\varkappa)-6\rho^2\mathcal{s}ech^2(\rho\mathcal{s})+U_{\mathcal{s}}S_{\varkappa}^{\varphi_1^{-1}}\left[\mathrm{r}^{\varphi_1}U_{\mathcal{s}}S_{\varkappa}^{\varphi_1}[\,gg_{\mathcal{s}}+0.5\,g_{\mathcal{sss}}]\right]=0 \tag{19}$$

Put $\frac{\partial^{\varphi_1}}{\partial \varkappa^{\varphi_1}}$ on both sides of eq. (19) and then use the Fr-VIM where $\lambda(\xi) = -1$ and the correct functional is

$$g_{n+1}(s,\varkappa) = g_n(s,\varkappa) - \frac{1}{\Gamma(\varphi_1+1)} \int_0^{\varkappa} \left\{ \frac{\partial^{\varphi_1}}{\partial \xi^{\varphi_1}} g_n(s,\xi) + \frac{\partial^{\varphi_1}}{\partial \xi^{\varphi_1}} U_s^{\varphi_2} S_\xi^{\varphi_1^{-1}} \left[\mathrm{r}^{\varphi_1} U_s S_\xi^{\varphi_1} [\, g_n(s,\xi)(g_n(s,\xi))_s + 0.5\,(g_n(s,\xi))_{sss}] \right] \right\} (d\xi)^{\varphi_1}, \ n \geq 0 \tag{20}$$

$$g_0 = g(s,0) = 6\rho^2 sech^2(\rho s)\,;$$

$$g_1 = 6\rho^2 sech^2(\rho s) + 24\rho^5 \frac{\varkappa^{\varphi_1}}{\Gamma(\varphi_1+1)} sech^2(\rho s) \varkappa anh(\rho s)\,\{\varkappa anh^2(\rho s) + sech^2(\rho s)\}\,;$$

…. and so on, providing the solution as a convergent series

$$g(s,\varkappa) = \sum_{n=0}^{\infty} g_{n+1}(s,\varkappa).$$

where the exact solution of eq. (17) when $\varphi_1 = 1$, is

$$g(s,\varkappa) = 6\rho^2 sech^2(\rho s - 2\rho^3 \varkappa).$$

Example 2 [30, 31]: Consider the following time-fractional KdV-B equation in Caputo sense

$${}_0^C D_\varkappa^{\varphi_1}\, g(s,\varkappa) + g(s,\varkappa) g_s(s,\varkappa) - \delta g_{ss}(s,\varkappa) + 0.5\, g_{sss}(s,\varkappa) = 0; 0 < \varphi_1 \leq 1 \tag{21}$$

with the initial condition

$$g(s,0) = h\delta^2 \big(2 - 2\tanh(\theta) + sech^2(\theta)\big) \tag{22}$$

where $h = \frac{6}{25}$ and $\theta = \frac{\delta s}{5}$.

Solution: Apply the Fr-MSST on both sides of the eq. (21) and take the inverse of Fr-MSST to the resulting equation, we get

$$g(s,\varkappa) - 6\rho^2 sech^2(\rho s) + U_s S_\varkappa^{\varphi_1^{-1}} \left[\mathrm{r}^{\varphi_1} U_s S_\varkappa^{\varphi_1} [\, g g_s - \delta g_{ss} + 0.5\, g_{sss}] \right] = 0 \tag{23}$$

Put $\frac{\partial^{\varphi_1}}{\partial \varkappa^{\varphi_1}}$ on both sides of eq. (23) and use the Fr-VIM where $\lambda(\xi) = -1$ and the correct functional is

$$g_{n+1}(s,\varkappa) = g_n(s,\varkappa) - \frac{1}{\Gamma(\varphi_1+1)} \int_0^{\varkappa} \left\{ \frac{\partial^{\varphi_1}}{\partial \xi^{\varphi_1}} g_n(s,\xi) + \frac{\partial^{\varphi_1}}{\partial \xi^{\varphi_1}} U_s^{\varphi_2} S_\xi^{\varphi_1^{-1}} \left[\mathrm{r}^{\varphi_1} U_s S_\xi^{\varphi_1} \left[\, g_n(s,\xi)(g_n(s,\xi))_s - \delta\big(g_n(s,\xi)\big)_{ss} + 0.5\,(g_n(s,\xi))_{sss} \right] \right] \right\} (d\xi)^{\varphi_1}, \ n \geq 0 \tag{24}$$

to solve the resulting equation, gives

$$g_0 = g(s,0) = h\delta^2 \big(2 - 2\tanh(\theta) + sech^2(\theta)\big);$$

$g_1 = h\delta^2\big(2 - 2\tanh(\theta) + \mathcal{s}ech^2(\theta)\big) - h\delta^5 \frac{\varkappa^{\varphi_1}}{\Gamma(\varphi_1+1)}$
$\Big\{\frac{76}{125}\mathcal{s}ech^2(\theta)\varkappa anh^2(\theta) - \frac{38}{125}\mathcal{s}ech^2(\theta) - \frac{42}{125}\mathcal{s}ech^4(\theta)\tanh(\theta) -$
$\frac{4}{125}\mathcal{s}ech^2(\theta)\varkappa anh^3(\theta) - \frac{4}{5}\mathcal{s}ech^2(\theta)\Big\}$;

…. and so on, providing the solution as a convergent series

$$g(\mathcal{s},\varkappa) = \sum_{n=0}^{\infty} g_{n+1}(\mathcal{s},\varkappa).$$

Eq. (21) has an exact solution when $\varphi_1 = 1$, as

$$g(\mathcal{s},\varkappa) = 2h\delta^2 - 2h\delta^2\tanh(\omega) + 2a\delta^2\mathcal{s}ech^2(\omega),$$

where $\boldsymbol{\omega} = \frac{-\boldsymbol{\delta s}}{\mathbf{5}} - \frac{\mathbf{2h\delta^3 t}}{\mathbf{5}}$.

5. Conclusion

The present work has offered a method for solving various types of nonlinear fractional differential equations in time-domain in the Caputo sense, utilizing the combination of two integral transforms with the variational iteration method. We have demonstrated the existence of the multiple Shehu-Sumudu transform and other its properties. The obtained findings indicate that the suggested method offers an efficient and accurate solution of nonlinear partial differential equations of fractional order in time-space. The results are compared to the exact solutions and to other existing methods, and show that the proposed method yields good results. The proposed method also has the advantage of providing effective numerical solutions for a range of problems.

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