Journal of Interdisciplinary Mathematics
ISSN: 0972-0502 (Print), ISSN: 2169-012X (Online)
Vol. 28 (2025), No. 3-B, pp. 1197–1206
DOI : 10.47974/JIM-2208

A new general model version of quasinormal operator based on a fuzzy soft Hilbert domain

Muna Abuljabbar Ahmed *
Department of Mathematics
Al-Khark Education Directorate
Ministry of Education
Baghdad
Iraq

Ines Ellouze †
Department of Mathematics
College of faculty of Sciences of Sfax
University of Sfax
Sfax
Tunisia

Salim Dawood Mohsen §
Department of Mathematics
College of Education
Mustensiriyah University
Baghdad
Iraq

Abstract

In fuzzy functional analysis, the soft fuzzy Hilbert space is a remarkable space. Compared to soft fuzzy sets, this space is more general and flexible. Most of the challenges in describing the fuzziness of the problem parameters are explained. This offers effective and interesting contributions to the field of operator theory. The idea of the fuzzy soft Hilbert space is useful in proposing a new operator of the general quasi-normal type associated with two parameters. It contains a number of known operators, including the soft quasi-normal operator, the quasi-normal operator, and the normal operator. In addition, the algebraic properties and various accompanying analyses are analyzed and presented. Multiplication

* *E-mail:* **munaabdoljabar@gmail.com (Corresponding Author)**
† *E-mail:* **inesllouze@yahoo.fr**
§ *E-mail:* **dr_salim2015@yahoo.com**

and addition are the basic abilities. In addition, the relationships between those categories are also addressed.

Subject Classification (2020): *Primary 46S40, 47A05, 47B02; Secondary 47B15.*

Keywords: *Normal operator, General quasi-normal operator, Fuzzy set, Fuzzy soft Hilbert space, Fuzzy soft functional analysis.*

1. Introduction

One important area of mathematical analysis is the Theory of Functional Analysis (FAT), which has gained notoriety because of its strong ties to and uses in Operator Theory (OT). see [1, 2]. Algebra and topology are combined in this theory. In fact, FAT covers algebraic methods and classical analysis. Additionally, solving mathematical problems actively contributes to the development of FAT. Fuzzy functional analysis (FFA) is one intriguing area. In the actual world, ambiguity caused by uncertainty typically leads to complexity. Because of this, FFA frequently tackles difficult issues in a variety of domains, including as business management, engineering, medicine, sociology, and the environment. In many cases, traditional mathematical methods are unable to effectively tackle assumptions issues. The membership function from set X to codomain [0,1] is a unique kind of function that may be used to ascertain the domain area of fuzzy sets, according to a 1965 mathematical concept introduced by Zadeh [3] as a subfield of set theory. Molodtsov [4] identified the notion of soft sets in 1999 and talked about uncertainty and challenging circumstances that can't be resolved using traditional methods. in several fields, including as computer science, numerical analysis, functional analysis, control theory, and approximation theory, among others. Soft sets are a specific family of subsets from a standardized set that mathematicians use as a model to add a set and collection of parameters to mimic uncertainty. Soft points, soft metric spaces, soft normed spaces, soft pre-Hilbert product spaces, and soft Hilbert spaces are just a few of the characteristics that have been recognized by additional scholars as they have developed new and enlarged concepts based on soft sets, illustrated them, and described them, [5-9]. The fuzzy soft inner product (Fs-IP) space and fuzzy soft Hilbert (Fs-H) space were proposed by Faried Ali, and Sakr, [10] in 2020. The characteristics of operators posed on Fs-H space were also compared to those of other common kinds of operators posed on Hilbert space. Besides, Faried Ali, and Sakr, [11] in 2020 provided the fuzzy soft linear operators in FS-H spaces which combine OT and the Fs-H spaces. They also looked into discoveries that were more thorough and in-depth. The following academics have also shown how Fs-H space may be used to different fields, ([12], [13]). Subsequently, the concept of fuzzy soft self-adjoint operators is defined and its attributes are analyzed, [14]. On the other hand, in relation to FAT, Mohsen [15–17] provided a variety of operators posed on Hilbert space and examined a

Inspired by the aforementioned Mohsen studies, this research introduces a new class of generalized quasi-normal type operators based on Fs-H space that are connected to two-parameter θ *and* k. These operators are called fuzzy soft quasi-normal operators, and they are denoted to $fS\kappa QN$ $-$operator. Additionally, several results pertaining to various analyses and algebraic properties are examined and displayed. Multiplication, addition, and restriction are important characteristics. Furthermore, the relationships among such categories are also investigated.

2. Relevant Principle on Fuzzy Soft Operators

This section provides several basic information of fuzzy soft vector spaces with structures upon these concepts, in order to complete this work.

Definitions 2.1:

i) **(Fuzzy set):** The fuzzy set (FS) is the set has each element form $(x, \mu_{\hat{A}}(x))$, where $x \in X$, is a universal set and $\mu_{\hat{A}}: X \longrightarrow [0,1]$ be a mapping call membership mapping. Through this article can be denoted as $\widehat{F}$. [1]

ii) **(Soft set):** The soft set (SS) is the set has each element form $\big(x, S(x)\big)$, where $x \in U \subset E$, such that E the set of all parameters and $S: U \to P(X)$ is a mapping from subset of parameter U into power set of universal set X, and $S(x) \in P(X)$. Through this article can be denoted as (S, U). [2]

The concept of fuzzy soft set (FSS), in fuzzy soft theory have been recalled thought the following definition, submitted first one by Maji in 2002, see [3].

Definition 2.2

i) **(Fuzzy soft sets):** A soft set (S, U) to be fuzzy soft set (FSS) in case has each element from $\big(x, \widetilde{S(x)}\big)$, where $\wp: U \to [0,1]^X$ such that $[0,1]^X$ the collection of each fuzzy subset on universal set X. Through this article can be denoted as $\widetilde{(S, U)}$. [3]

ii) **(Special case fuzzy soft points):** The fuzzy soft point (FS $-$point) of $\widetilde{(S, U)}$ over X, shortly as $\widetilde{w}_{\mu_{S(x)}}$, if $e \in U$ and $w \in X$, $\widetilde{w}_{\mu_{S(x)}}(x) = \{\delta, if\ w = w_o\ and\ x = w_0,\ 0\ if\ w \in X - \{w_0\}\ or\ e \in U - \{x_0\}$, such as $\delta \in (0,1]$. [4]

in source [4], Molodtsov, give the symbolize of the family of each fuzzy soft real numbers and fuzzy soft complex numbers via $R\big(\hat{A}\big)$ and $C\big(\hat{A}\big)$, respectively.

In order to complete the background of this article must give the next proposition, which is appeared in source [5].

Proposition 2.3: A fuzzy soft vector space on the fuzzy soft set $(\widetilde{U})$ symbolized FSV$(\widetilde{U})$ has linearity if requirement the down conditions

i) $\widetilde{(w+v)}_{\left(\mu_{1S(x_1)}+\mu_{2S(x_2)}\right)} = \tilde{w}_{\mu_{1S(x_1)}} + \tilde{v}_{\mu_{2S(x_2)}}$

ii) $\widetilde{(a\cdot w)}_{\mu_{S(x)}} = \tilde{a}\cdot\tilde{w}_{\mu_{S(x)}} =, \tilde{a}\in\tilde{R}(A)$, for all $\tilde{w}_{\mu_{1S(x_1)}}, \tilde{v}_{\mu_{2S(x_2)}} \in \mathrm{FSV}(\tilde{U})$

Now, using the following formulation, we will review the generalization of the norm on fuzzy soft vector.

Definition 2.5: [6] A mapping with the formula $\widetilde{\|.\|}: \tilde{U} \to R(\hat{A})$ designated fuzzy soft norm on $\tilde{U}$ if the following propositions are true.

i) $\|\widetilde{\tilde{w}_{\mu_{S(x)}}}\| \geq \tilde{0}$, for each $\tilde{w}_{\mu_{S(x)}} \in \tilde{U}$ (In general),and $\tilde{w}_{\mu_{S(x)}} = \tilde{\theta}$ if and only if $\|\widetilde{\tilde{w}_{\mu_{S(x)}}}\| = \tilde{0}$

ii) $\|\tilde{\alpha}\widetilde{\tilde{w}_{\mu_{S(x)}}}\| = |\tilde{\alpha}|\|\widetilde{\tilde{w}_{\mu_{S(x)}}}\|$, for each $\tilde{w}_{\mu_{S(x)}} \in \tilde{U}$, and the fuzzy soft scalar $\tilde{\alpha}\in C(\hat{A})$.

iii) $\|\tilde{w}_{\mu_{1S(x_1)}} \widetilde{+\, \tilde{v}_{\mu_{2S(x_2)}}}\| \leq \|\widetilde{\tilde{w}_{\mu_{1S(x_1)}}}\| + \|\widetilde{\tilde{v}_{\mu_{2S(x_2)}}}\|$, for each $\tilde{w}_{\mu_{1S(x_1)}}$, $\tilde{v}_{\mu_{2S(x_2)}} \in \tilde{U}$.

So, we have the fuzzy soft norm space symbolized $\left(\tilde{U}, \widetilde{\|.\|}\right)$ with Abbreviation as FSN- space.

In the following part, review the definition of "generalization of inner product space on fuzzy soft vector.

Definition 2.6: [7] A mapping with the formula $\widetilde{\langle .,.\rangle}: \tilde{U}\times\tilde{U} \to C(\hat{A})$ designated fuzzy soft inner on $\tilde{U}\times\tilde{U}$ if the following requirements are fulfilled.

i) $\langle \tilde{w}_{\mu_{S(x)}}\widetilde{,\, \tilde{w}_{\mu_{S(x)}}}\rangle \geq \tilde{0}$, for each $\tilde{w}_{\mu_{S(x)}} \in \tilde{U}$ with $\tilde{a}_{\mu_{S(x)}} = \tilde{\theta}$ if and only if $\langle \tilde{w}_{\mu_{S(x)}}\widetilde{,\, \tilde{w}_{\mu_{S(x)}}}\rangle = \tilde{0}$

ii) $\langle \tilde{w}_{\mu_{1S(x_1)}}\widetilde{,\, \tilde{v}_{\mu_{2S(x_2)}}}\rangle = \underline{\langle \tilde{v}_{\mu_{2S(x_2)}}\widetilde{,\, \tilde{w}_{\mu_{1S(x_1)}}}\rangle}$, for each $\tilde{w}_{\mu_{1S(x_1)}}, \tilde{v}_{\mu_{2S(x_2)}} \in \tilde{U}$

iii) $\langle \tilde{\alpha}\tilde{w}_{\mu_{1S(x_1)}}\widetilde{,\, \tilde{v}_{\mu_{2\wp(x_2)}}}\rangle = \tilde{\alpha}\langle \tilde{w}_{\mu_{1S(x_1)}}\widetilde{,\, \tilde{v}_{\mu_{2S(x_2)}}}\rangle$, for each $\tilde{w}_{\mu_{1S(x_1)}}, \tilde{v}_{\mu_{2S(x_2)}} \in \tilde{U}$, for all $\tilde{\alpha}\in C(\hat{A})$.

iv) $\tilde{w}_{\mu_{1S(x_1)}}, \tilde{v}_{\mu_{2S(x_2)}}, \tilde{u}_{\mu_{3S(x_3)}} \in \tilde{U}, \langle \tilde{w}_{\mu_{1S(x_1)}} + \widetilde{\tilde{v}_{\mu_{2S(x_2)}}}, \tilde{u}_{\mu_{3S(x_3)}}\rangle =$ $\langle \tilde{w}_{\mu_{1S(x_1)}}\widetilde{,\, \tilde{u}_{\mu_{3S(x_3)}}}\rangle + \langle \tilde{v}_{\mu_{2S(x_2)}}\widetilde{,\, \tilde{u}_{\mu_{3S(x_3)}}}\rangle$

So we have the fuzzy soft norm space symbolized $\left(\tilde{U}, \widetilde{\langle .,.\rangle}\right)$ with Abbreviation as FSI- space.

Definitions 2.7:

i) (Complete space) A fuzzy soft vector space FSV ($\tilde{U}$), to be fuzzy soft complete FSC($\tilde{U}$) when to be for each fuzzy soft Cauchy sequence is fuzzy soft Convergent sequence in this vector, [10].

ii) (Fuzzy soft Hilbert space) A fuzzy soft inner product space FSI on fuzzy soft vector space FSV to be fuzzy soft Hilbert space FS-H, when it is FSC$(\tilde{U})$, [10].

Also, we recall the definition of FSL -operator depended on FS-H space, up on fuzzy vector space

Definitions 2.8: [11], [14] Considering $\widetilde{H_S}$ is Fs-H space with $\tilde{\Delta}: \widetilde{H_S} \to \widetilde{H_S}$ is operator, so have:

i) $\tilde{\Delta}$ is namely fuzzy soft linear operator with shortly FSL-operator if: $\tilde{\Delta}\left(\tilde{\alpha}\widetilde{w}_{\mu_{1S(x_1)}} + \tilde{\beta}\tilde{v}_{\mu_{2S(x_2)}}\right) = \tilde{\alpha}\tilde{\Delta}\left(\widetilde{w}_{\mu_{1S(x_1)}}\right) + \tilde{\beta}\tilde{\Delta}\left(\tilde{v}_{\mu_{2S(x_2)}}\right)$, where $\widetilde{w}_{\mu_{1S(x_1)}}, \tilde{v}_{\mu_{2S(x_2)}} \in \widetilde{H_S}$ and $\tilde{\alpha}, \tilde{\beta} \in C(\hat{A})$

ii) $\tilde{\Delta}$ which is namely fuzzy soft bounded operator with shortly FSB-operator if:

$\exists\ \widetilde{m} \in R(\hat{A})^+$ with property $\|\tilde{\Delta}\left(\widetilde{\widetilde{w}_{\mu_{1S(x_1)}}}\right)\| \leq \widetilde{m}\|\widetilde{\widetilde{w}_{\mu_{1S(x_1)}}}\|$, for all $\widetilde{w}_{\mu_{1S(x_1)}} \in \widetilde{H_S}$.

iii) Assumes that $\tilde{\Delta}: \widetilde{H_S} \to \widetilde{H_S}$. be FSB-operator define on $\widetilde{H_S}$. so, namely fuzzy soft adjoint of operator symbolize by $\tilde{\Delta}^*$ defined as $\langle \widetilde{\tilde{\Delta}\widetilde{w}_{\mu_{1S(x_1)}}, \tilde{v}_{\mu_{2S(x_2)}}} \rangle = \langle \widetilde{\widetilde{w}_{\mu_{1S(x_1)}}, \tilde{\Delta}^*\, \tilde{v}_{\mu_{2S(x_2)}}} \rangle$, where $\widetilde{w}_{\mu_{1S(x_1)}}, \tilde{v}_{\mu_{2S(x_2)}} \in \widetilde{H_S}$.

Next, some properties of fuzzy soft adjoint of FSB-linear operator, have been given in the following theorem

Theorem 2.10: [14] *Assumes that $\tilde{\Delta}$ and $\widetilde{\wp}$ are FSLB-operators, define on $\widetilde{H_S}$ is FS-H space and $\tilde{k} \in C(\hat{A})$, then*

i) $\tilde{\Delta}^{**} = \tilde{\Delta}$

ii) $\left(\tilde{k}\tilde{\Delta}\right)^* = \underline{\tilde{k}}\tilde{\Delta}^* \quad k$

iii) $\left(\tilde{\Delta} + \widetilde{\wp}\right)^* = \tilde{\Delta}^* + \widetilde{\wp}^*$ and $\left(\tilde{\Delta}\widetilde{\wp}\right)^* = \widetilde{\wp}^*\tilde{\Delta}^*$.

Definition 2.11: [14] The $FS-$ operator define on $\widetilde{H_S}$ which namely FS self-adjoint if satisfy the property $\tilde{\Delta} = \tilde{\Delta}^*$.

3. Concerning $fS\kappa QN$ −operator

This section presents a new generalized a $fSQN$ −operator, which is represented by $fS\kappa QN$ −operator. Some characteristics of this idea will be discussed, along with some features that have been provided, such as some essential properties illustrated in this section

Definition 3.1: Assume that $\widetilde{\wp}$ $fSB-$ operator from $\widetilde{H}$ into $\widetilde{H}$, then $\widetilde{\wp}$ is named fuzzy soft κ Quasi-Normal operator, presently, $fS\kappa QN-$operator if $(\widetilde{\wp}\,\widetilde{\wp}^*)^\kappa\,(\widetilde{\wp}^\kappa+\widetilde{\wp}^*)=(\widetilde{\wp}^\kappa+\widetilde{\wp}^*)(\widetilde{\wp}\,\widetilde{\wp}^*)^\kappa$. For some positive integer κ**.**

Next, some of the qualities of this concept utilizing the subsequent remakes are presented.

Remakes 3.2:

i) If $\kappa=1$, then $\widetilde{\wp}$ coincides $fS\kappa QN-$operator.

ii) Verifying that each Fs- self a djoint is a $fS\kappa QN-$operator is a simple process.

Corollary 3.3: *Assume that* $\widetilde{\wp}$ $fSB-$ *operator from* $\widetilde{H}$ *into* $\widetilde{H}$, *then*

i) $\widetilde{\wp}+\widetilde{\wp}^*$ is $fS\kappa QN-$operator.

ii) $\widetilde{\wp}\widetilde{\wp}^*$ is $fS\kappa QN-$operator

iii) $\widetilde{\wp}^*\widetilde{\wp}$ is $fS\kappa QN-$operator

iv) $\tilde{I}+\widetilde{\wp}^*\widetilde{\wp}$ is $fS\kappa QN-$operator

v) $\tilde{I}+\widetilde{\wp}\widetilde{\wp}^*$ is $fS\kappa QN-$operator

vi) $(\widetilde{\wp}+\widetilde{\wp}^*)+(\widetilde{\wp}\,\widetilde{\wp}^*)+(\widetilde{\wp}^*\widetilde{\wp})+(\tilde{I}+\widetilde{\wp}^*\widetilde{\wp})+(\tilde{I}+\widetilde{\wp}\,\widetilde{\wp}^*)$ is $fS\kappa QN-$operator.

Another property for $fS\kappa QN-$operator has been provided with the subsequent proposition

Proposition 3.4: Assume $\widetilde{\wp}:\widetilde{H}\to\widetilde{H}$ is $fS\kappa QN-$ operator, then $[(\widetilde{\wp}^\kappa\,\widetilde{\wp}^*)(\widetilde{\wp}+\widetilde{\wp}^*)^\kappa]^m=[(\widetilde{\wp}+\widetilde{\wp}^*)^\kappa(\widetilde{\wp}^\kappa\,\widetilde{\wp}^*)]^m$.

Proof: This theorem can be demonstrated using mathematical induction, which $[(\widetilde{\wp}^\kappa\,\widetilde{\wp}^*)(\widetilde{\wp}+\widetilde{\wp}^*)^\kappa]^m=[(\widetilde{\wp}+\widetilde{\wp}^*)^\kappa(\widetilde{\wp}^\kappa\,\widetilde{\wp}^*)]^m$. and utilizing the subsequent procedures.

i) We demonstrate that one is possible when $m=1$. $(\widetilde{\wp}^\kappa\,\widetilde{\wp}^*)(\widetilde{\wp}+\widetilde{\wp}^*)^\kappa=(\widetilde{\wp}+\widetilde{\wp}^*)^\kappa(\widetilde{\wp}^\kappa\,\widetilde{\wp}^*)$, That's accurate since $\widetilde{\wp}$ be $fS\kappa QN-$operator

ii) Suppose that true when $m=\nu$ such that $[(\widetilde{\wp}^\kappa\,\widetilde{\wp}^*)(\widetilde{\wp}+\widetilde{\wp}^*)^\kappa]^q=[(\widetilde{\wp}+\widetilde{\wp}^*)^\kappa(\widetilde{\wp}^\kappa\,\widetilde{\wp}^*)]^q$

iii) We ought to demonstrate when $m=\nu+1$
$[(\widetilde{\wp}^\kappa\,\widetilde{\wp}^*)(\widetilde{\wp}+\widetilde{\wp}^*)^\kappa]^{\nu+1}=[(\widetilde{\wp}^\kappa\,\widetilde{\wp}^*)(\widetilde{\wp}+\widetilde{\wp}^*)^\kappa]^\nu\,[(\widetilde{\wp}^\kappa\,\widetilde{\wp}^*)(\widetilde{\wp}+\widetilde{\wp}^*)^\kappa]$,
Utilizing (i) and (ii), one is able to
$[(\widetilde{\wp}^\kappa\,\widetilde{\wp}^*)(\widetilde{\wp}+\widetilde{\wp}^*)^\kappa]^\nu\,[(\widetilde{\wp}^\kappa\,\widetilde{\wp}^*)(\widetilde{\wp}+\widetilde{\wp}^*)^\kappa]=$
$[(\widetilde{\wp}+\widetilde{\wp}^*)^\kappa(\widetilde{\wp}^\kappa\,\widetilde{\wp}^*)]^\nu[(\widetilde{\wp}+\widetilde{\wp}^*)^\kappa(\widetilde{\wp}^\kappa\,\widetilde{\wp}^*)]=[(\widetilde{\wp}+\widetilde{\wp}^*)^\kappa(\widetilde{\wp}^\kappa\,\widetilde{\wp}^*)]^{\nu+1}$.

Theorem 3.5: *Assume* $\widetilde{\wp},\widetilde{\omega}:\widetilde{H}\to\widetilde{H}$ *to be* $fS\kappa QN-$*operator meets the requirements* $\widetilde{\wp}\,\widetilde{\omega}=\widetilde{\omega}\,\widetilde{\wp}^*=\widetilde{\wp}\,\widetilde{\omega}^*=\widetilde{\wp}^*\widetilde{\omega}^*=\tilde{0}$, *then* $\widetilde{\wp}+\widetilde{\omega}$ *is* $fS\kappa QN-$ *operator.*

Proof:

$$(\widetilde{\wp}+\widetilde{\omega})^{\kappa}\,(\widetilde{\wp}+\widetilde{\omega})^{*}[(\widetilde{\wp}+\widetilde{\omega})+(\widetilde{\wp}+\widetilde{\omega})^{*}]^{\kappa}$$

$= (\widetilde{\wp}^{\kappa}+\kappa\,(\widetilde{\wp}^{\kappa-1})\widetilde{\omega}+\cdots+\widetilde{\omega}^{\kappa})(\widetilde{\wp}^{*}+\widetilde{\omega}^{*})[(\widetilde{\wp}+\widetilde{\omega})^{\kappa}+\kappa\,(\widetilde{\wp}+\widetilde{\omega})^{\kappa-1}(\widetilde{\wp}+\widetilde{\omega})^{*}+..+(\widetilde{\wp}+\widetilde{\omega})^{*\kappa}$

$= (\widetilde{\wp}^{\kappa}+\widetilde{\omega}^{\kappa})(\widetilde{\wp}^{*}+\widetilde{\omega}^{*})[(\widetilde{\wp}^{\kappa}+\kappa\,(\widetilde{\wp}^{\kappa-1})\widetilde{\omega}+\cdots+\widetilde{\omega}^{\kappa})+\kappa\,[(\widetilde{\wp}^{\kappa-1}+(\kappa-1)(\widetilde{\wp}^{\kappa-2})\,\widetilde{\omega}+\cdots+\widetilde{\omega}^{\kappa-1})(\widetilde{\wp}^{*}+\widetilde{\omega}^{*})+\cdots+(\widetilde{\wp}^{*^{\kappa}}+\kappa\,(\widetilde{\wp}^{*\kappa-1})\widetilde{\omega}^{*}+\cdots+\widetilde{\omega}^{*\kappa})$

$= (\widetilde{\wp}^{\kappa}+\widetilde{\omega}^{\kappa})(\widetilde{\wp}^{*}+\widetilde{\omega}^{*})[(\widetilde{\wp}^{\kappa}+\widetilde{\omega}^{\kappa})+\kappa\,(\widetilde{\wp}^{\kappa-1}+\widetilde{\omega}^{\kappa-1})\,(\widetilde{\wp}^{*}+\widetilde{\omega}^{*})+\cdots+(\widetilde{\wp}^{*\kappa}+\widetilde{\omega}^{*\kappa})$

$= (\widetilde{\wp}^{\kappa}\,\widetilde{\wp}^{*}+\widetilde{\omega}^{\kappa}\,\widetilde{\omega}^{*})[(\widetilde{\wp}^{\kappa}+\widetilde{\omega}^{\kappa})+\kappa(\widetilde{\wp}^{\kappa-1}\,\widetilde{\wp}^{*}+\widetilde{\omega}^{\kappa-1}\,\widetilde{\omega}^{*})+\cdots+(\widetilde{\wp}^{*\kappa}+\widetilde{\omega}^{*\kappa})$

$= \widetilde{\wp}^{\kappa}\,\widetilde{\wp}^{*}\,\widetilde{\wp}^{\kappa}+\widetilde{\wp}^{\kappa}\,\widetilde{\wp}^{*}\,\kappa\,(\widetilde{\wp}^{\kappa-1}\,\widetilde{\wp}^{*})+\cdots+(\widetilde{\wp}^{\kappa}\,\widetilde{\wp}^{*}\,\widetilde{\wp}^{*\kappa})+(\widetilde{\omega}^{\kappa}\,\widetilde{\omega}^{*}\,\widetilde{\omega}^{\kappa}+\widetilde{\omega}^{\kappa}\,\widetilde{\omega}^{*}\kappa\,(\widetilde{\omega}^{\kappa-1}\,\widetilde{\omega}^{*})+\cdots+\widetilde{\omega}^{\kappa}\,\widetilde{\omega}^{*}\,\widetilde{\omega}^{*\kappa})$

$$= \widetilde{\wp}^{\kappa}\,\widetilde{\wp}^{*}(\widetilde{\wp}+\widetilde{\wp}^{*})^{\kappa}+\widetilde{\omega}^{\kappa}\,\widetilde{\omega}^{*}(\widetilde{\omega}+\widetilde{\omega}^{*})^{\kappa} \qquad (1)$$

Now, take another side

$$[(\widetilde{\wp}+\widetilde{\omega})+(\widetilde{\wp}+\widetilde{\omega})^{*}]^{\kappa}\,(\widetilde{\wp}+\widetilde{\omega})^{\kappa}\,(\widetilde{\wp}+\widetilde{\omega})^{*}$$

$= [(\widetilde{\wp}+\widetilde{\omega})^{\kappa}+\kappa\,(\widetilde{\wp}+\widetilde{\omega})^{\kappa-1}(\widetilde{\wp}+\widetilde{\omega})^{*}+..+(\widetilde{\wp}+\widetilde{\omega})^{*\kappa}]\,(\widetilde{\wp}^{\kappa}+\kappa\,(\widetilde{\wp}^{\kappa-1})\widetilde{\omega}+\cdots+\widetilde{\omega}^{\kappa})\,(\widetilde{\wp}^{*}+\widetilde{\omega}^{*})$.

$= (\widetilde{\wp}^{\kappa}+\kappa\,(\widetilde{\wp}^{\kappa-1})\widetilde{\omega}+\cdots+\widetilde{\omega}^{\kappa})+\kappa\,\Big[(\widetilde{\wp}^{\kappa-1}+(\kappa-1)(\widetilde{\wp}^{\kappa-2})\,\widetilde{\omega}+\cdots+\widetilde{\omega}^{\kappa-1})(\widetilde{\wp}^{*}+\widetilde{\omega}^{*})+\cdots+\left(\widetilde{\wp}^{*^{\kappa}}+\kappa\left(\widetilde{\wp}^{*^{\kappa-1}}\right)\widetilde{\omega}^{*}+\cdots+\widetilde{\omega}^{*\kappa}\right)\Big]\,(\widetilde{\wp}^{\kappa}+\widetilde{\omega}^{\kappa})\,(\widetilde{\wp}^{*}+\widetilde{\omega}^{*})$

$= [(\widetilde{\wp}^{\kappa}+\widetilde{\omega}^{\kappa})+\kappa\,(\widetilde{\wp}^{\kappa-1}+\widetilde{\omega}^{\kappa-1})(\widetilde{\wp}^{*}+\widetilde{\omega}^{*})+\cdots+(\widetilde{\wp}^{*\kappa}+\widetilde{\omega}^{*\kappa})]\,(\widetilde{\wp}^{\kappa}+\widetilde{\omega}^{\kappa})(\widetilde{\wp}^{*}+\widetilde{\omega}^{*})$

$= [(\widetilde{\wp}^{\kappa}+\widetilde{\omega}^{\kappa})+\kappa(\widetilde{\wp}^{\kappa-1}\widetilde{\wp}^{*}+\widetilde{\omega}^{\kappa-1}\widetilde{\omega}^{*})+\cdots+(\widetilde{\wp}^{*\kappa}+\widetilde{\omega}^{*\kappa})]\,(\widetilde{\wp}^{\kappa}\,\widetilde{\wp}^{*}+\widetilde{\omega}^{\kappa}\,\widetilde{\omega}^{*}) = (\widetilde{\wp}^{\kappa}\,\widetilde{\wp}^{\kappa}\,\widetilde{\wp}^{*}+\kappa\,(\widetilde{\wp}^{\kappa-1}\,\widetilde{\wp}^{*})\,\widetilde{\wp}^{\kappa}\widetilde{\wp}^{*}+\cdots+\widetilde{\wp}^{*\kappa}\,\widetilde{\wp}^{\kappa}\,\widetilde{\wp}^{*})+\widetilde{\omega}^{\kappa}\widetilde{\omega}^{\kappa}\,\widetilde{\omega}^{*}+\kappa\,(\widetilde{\omega}^{\kappa-1}\,\widetilde{\omega}^{*})\,\widetilde{\omega}^{\kappa}\,\widetilde{\omega}^{*}+\cdots+\widetilde{\omega}^{*^{\kappa}}\,\widetilde{\omega}^{\kappa}\,\widetilde{\omega}^{*}$

$$= (\widetilde{\wp}+\widetilde{\wp}^{*})^{\kappa}\,\widetilde{\wp}^{\kappa}\,\widetilde{\wp}^{*}+(\widetilde{\omega}+\widetilde{\omega}^{*})^{\kappa}\,\widetilde{\omega}^{\kappa}\,\widetilde{\omega}^{*} \qquad (2)$$

Since $\widetilde{\wp},\widetilde{\omega}:\widetilde{H}\to\widetilde{H}$ be $fS\kappa QN$ −operator, one can have (1)=(2), this lead to

$$(\widetilde{\wp}+\widetilde{\omega})^{\kappa}(\widetilde{\wp}+\widetilde{\omega})^{*}[(\widetilde{\wp}+\widetilde{\omega})+(\widetilde{\wp}+\widetilde{\omega})^{*}]^{\kappa} = [(\widetilde{\wp}+\widetilde{\omega})+(\widetilde{\wp}+\widetilde{\omega})^{*}]^{\kappa}\,(\widetilde{\wp}+\widetilde{\omega})^{\kappa}\,(\widetilde{\wp}+\widetilde{\omega})^{*}$$

Therefore, $\widetilde{\wp}+\widetilde{\omega}$ is $fS\kappa QN$ −operator.

Subsequently, the subsequent theorem provides the prerequisites for $fS\kappa QN$ −operators to be likewise $fS\kappa QN$ −operators when multiplied.

Theorem 3.6: *Assume $\widetilde{\wp},\widetilde{\omega}:\widetilde{H}\to\widetilde{H}$ is $fS\kappa QN$ −operator possess the prerequisites $\widetilde{\wp}\,\widetilde{\omega}=\widetilde{\omega}\,\widetilde{\wp}$, $\widetilde{\wp}\,\widetilde{\omega}^{*}=\widetilde{\omega}^{*}\,\widetilde{\wp}$, then $\widetilde{\wp}\,\widetilde{\omega}$ is $fS\kappa QN$ −operator.*

Proof: $(\widetilde{\wp}\,\widetilde{\omega})^{\kappa}\,(\widetilde{\wp}\,\widetilde{\omega})^{*}\,[(\widetilde{\wp}\,\widetilde{\omega})+(\widetilde{\wp}\,\widetilde{\omega})^{*}]^{\kappa} = (\widetilde{\wp}^{\kappa}\,\widetilde{\omega}^{\kappa})(\widetilde{\omega}^{*}\,\widetilde{\wp}^{*})\,[(\widetilde{\wp}\,\widetilde{\omega})+(\widetilde{\wp}\,\widetilde{\omega})^{*}]^{\kappa}$ by using $\widetilde{\wp}\,\widetilde{\omega}=\widetilde{\omega}\,\widetilde{\wp}$, one can have

$$=(\widetilde{\wp}^{\kappa}\,\widetilde{\wp}^{*})(\widetilde{\omega}^{\kappa}\,\widetilde{\omega}^{*})[(\widetilde{\wp}\,\widetilde{\omega})+(\widetilde{\wp}\,\widetilde{\omega})^{*}]^{\kappa}$$

$$=(\widetilde{\wp}^{\kappa}\,\widetilde{\wp}^{*})(\widetilde{\omega}^{\kappa}\,\widetilde{\omega}^{*})\,[(\widetilde{\wp}\,\widetilde{\omega})^{\kappa}+\kappa\,(\widetilde{\wp}\,\widetilde{\omega})^{\kappa-1}(\widetilde{\wp}\,\widetilde{\omega})^{*}+\cdots+(\widetilde{\wp}\,\widetilde{\omega})^{*\kappa}].$$

$\widetilde{\omega}\,\widetilde{\wp}^{*} = \widetilde{\wp}^{*}\widetilde{\omega}^{\sim}$ may be obtained by finding the adjoint for the two sides of the scenario and applying $\widetilde{\wp}\,\widetilde{\omega}^{*} = \widetilde{\omega}^{*}\widetilde{\wp}$.

$$(\widetilde{\wp}\,\widetilde{\omega})^{*\kappa}\,(\widetilde{\wp}\,\widetilde{\omega})^{\eta}(\widetilde{\wp}\,\widetilde{\omega})^{*} = \left(\widetilde{\wp}^{*^{\kappa}}\,\widetilde{\wp}^{\eta}\,\widetilde{\omega}^{*^{\kappa}}\right)\left(\widetilde{\wp}^{*}\,\widetilde{\omega}^{\eta}\,\widetilde{\omega}^{*}\right)$$
$$= \left(\widetilde{\wp}^{*^{\kappa}}\,\widetilde{\wp}^{\eta}\,\widetilde{\wp}^{*}\right)\left(\widetilde{\omega}^{*^{\kappa}}\widetilde{\omega}^{\eta}\,\widetilde{\omega}^{*}\right)$$

And since $\widetilde{\wp}$ and $\widetilde{\omega}$ are $fS\kappa QN$ −operator, one can have

$$(\widetilde{\wp}\,\widetilde{\omega})^{*\kappa}\,[(\widetilde{\wp}\,\widetilde{\omega})^{\eta}\,(\widetilde{\wp}\,\widetilde{\omega})^{*}] = [(\widetilde{\wp}^{\eta}\,\widetilde{\wp}^{*})\,\widetilde{\wp}^{*\kappa}]\,\widetilde{\eta_2}\,[(\widetilde{\omega}^{\eta}\,\widetilde{\omega}^{*})\,\widetilde{\omega}^{*\kappa}]$$
$$= \widetilde{\eta_1}\,\widetilde{\eta_2}\left[\left((\widetilde{\wp}^{\eta}\,\widetilde{\wp}^{*})\,\widetilde{\wp}^{*\kappa}\right)\left((\widetilde{\omega}^{\eta}\widetilde{\omega}^{*})\,\widetilde{\omega}^{*\kappa}\right)\right]$$

Other one uses the hypothesis, we get,

$$(\widetilde{\wp}\,\widetilde{\omega})^{*\kappa}\,((\widetilde{\wp}\,\widetilde{\omega})^{\eta}(\widetilde{\wp}\,\widetilde{\omega})^{*}) = [(\widetilde{\wp}^{\eta}\,\widetilde{\wp}^{*})\,\widetilde{\omega}^{\eta}\,\widetilde{\wp}^{*\kappa}]\,\widetilde{\omega}^{*}\,\widetilde{\omega}^{*\kappa}$$
$$= \widetilde{\eta_1}\,\widetilde{\eta_2}\,[\widetilde{\wp}^{\eta}\,\widetilde{\omega}^{\eta}\,\widetilde{\wp}^{*}\,\widetilde{\omega}^{*}]\,(\widetilde{\wp}^{*\kappa}\,\widetilde{\omega}^{*\kappa})$$
$$= \tilde{\eta}\,[(\widetilde{\wp}\,\widetilde{\omega})^{\eta}(\widetilde{\wp}\,\widetilde{\omega})^{*}]\,(\widetilde{\omega}\,\widetilde{\wp})^{*\kappa},\ \text{choose}\quad \widetilde{\eta_1}\widetilde{\eta_2} = \tilde{\eta}.$$

Therefore; $\widetilde{\wp}\,\widetilde{\omega}$ is $fS\kappa QN$ −operator.

Corollary 3.8: *Let $\widetilde{\wp}: \widetilde{H} \to \widetilde{H}$ be $fSB -$ operator, then*

i. For $\widetilde{\wp}$ to be a $fSQN$ −operator, we have $\widetilde{\wp}$ is $fS\kappa QN$ −operator.
ii. For $\widetilde{\wp}$ to be a fuzzy soft skew- adjoint operator, we have $\widetilde{\wp}$ is $fS\kappa QN -$ operator.

Theorem 3.9: *Assuming that $\widetilde{\wp}_1 \ldots \widetilde{\wp}_\varrho: \widetilde{H} \to \widetilde{H}$ have fuzzy soft $fS\kappa QN -$ operator property in fuzzy Hilbert space $\widetilde{H}$, then one has $\widetilde{\wp}_1 \,\widetilde{\oplus} \ldots \widetilde{\oplus}\, \widetilde{\wp}_\varrho$ is fuzzy soft $fS\kappa QN$ −operator.*

Proof:

$$= ((\widetilde{\wp}_1 \,\widetilde{\oplus} \ldots \widetilde{\oplus}\, \widetilde{\wp}_\varrho)\,(\widetilde{\wp}_1 \,\widetilde{\oplus} \ldots \widetilde{\oplus}\, \widetilde{\wp}_\varrho)^{*})^{\kappa}\,\left[(\widetilde{\wp}_1 \,\widetilde{\oplus} \ldots \widetilde{\oplus}\, \widetilde{\wp}_\varrho)^{\kappa} +\right.$$
$\left.(\widetilde{\wp}_1 \,\widetilde{\oplus} \ldots \widetilde{\oplus}\, \widetilde{\wp}_\varrho)^{*}\right]$, from conditions of this type of operator have the down step

$$= \left[(\widetilde{\wp}_1 \,\widetilde{\oplus} \ldots \widetilde{\oplus}\, \widetilde{\wp}_\varrho)\,(\widetilde{\wp}_1^{*} \,\widetilde{\oplus} \ldots \widetilde{\oplus}\, \widetilde{\wp}_\varrho^{*})\right]^{\kappa}\,(\widetilde{\wp}_1^{\kappa} \,\widetilde{\oplus} \ldots \widetilde{\oplus}\, \widetilde{\wp}_\varrho^{\kappa})\,(\widetilde{\wp}_1{}^{*} \,\widetilde{\oplus} \ldots \widetilde{\oplus}\, \widetilde{\wp}_\varrho^{*})$$
$$= (\widetilde{\wp}_1\,\widetilde{\wp}_1^{*})^{\kappa}\,(\widetilde{\wp}_1^{\kappa} + \widetilde{\wp}_1^{*})\,\widetilde{\oplus} \ldots \widetilde{\oplus}\,(\widetilde{\wp}_\varrho\,\widetilde{\wp}_\varrho^{*})^{\kappa}\,(\widetilde{\wp}_\varrho^{\kappa} + \widetilde{\wp}_\varrho^{*})$$
$$= (\widetilde{\wp}_1^{\kappa} + \widetilde{\wp}_1^{*})\,(\widetilde{\wp}_1\,\widetilde{\wp}_1^{*})^{\kappa}\,\widetilde{\oplus} \ldots \widetilde{\oplus}\,(\widetilde{\wp}_\varrho^{\kappa} + \widetilde{\wp}_\varrho^{*})\,(\widetilde{\wp}_\varrho\,\widetilde{\wp}_\varrho^{*})^{\kappa}$$
$$= \left[(\widetilde{\wp}_1 \,\widetilde{\oplus} \ldots \widetilde{\oplus}\, \widetilde{\wp}_\varrho)^{\kappa} + (\widetilde{\wp}_1 \,\widetilde{\oplus} \ldots \widetilde{\oplus}\, \widetilde{\wp}_\varrho)^{*}\right]\,\left[((\widetilde{\wp}_1 \,\widetilde{\oplus} \ldots \widetilde{\oplus}\, \widetilde{\wp}_\varrho)\,(\widetilde{\wp}_1 \,\widetilde{\oplus} \ldots \widetilde{\oplus}\, \widetilde{\wp}_\varrho)^{*}\right]^{\kappa}$$

From up step certainly we have, Therefore; $\widetilde{\wp}_1 \,\widetilde{\oplus} \ldots \widetilde{\oplus}\, \widetilde{\wp}_\varrho$ is fuzzy soft $fS\kappa QN -$ operator

Theorem 3.10: *Let $\widetilde{\wp}_1 \ldots \widetilde{\wp}_\varrho: \widetilde{H} \to \widetilde{H}$ be fuzzy soft $fS\kappa QN -$ operator posed within fuzzy Hilbert space $\widetilde{H}$, then $\widetilde{\wp}_1 \otimes \ldots \otimes \widetilde{\wp}_\varrho$ is fuzzy soft $fS\kappa QN -$ operator.*

Proof:

$$= \left[(\tilde{\wp}_1 \tilde{\otimes} \dots \tilde{\otimes} \tilde{\wp}_\varrho)(\tilde{\wp}_1 \tilde{\otimes} \dots \tilde{\otimes} \tilde{\wp}_\varrho)^*\right]^\kappa \left[(\tilde{\wp}_1 \tilde{\otimes} \dots \tilde{\otimes} \tilde{\wp}_\varrho)^\kappa + (\tilde{\wp}_1 \tilde{\otimes} \dots \tilde{\otimes} \tilde{\wp}_\varrho)^*\right]$$

$$= \left[(\tilde{\wp}_1 \tilde{\otimes} \dots \tilde{\otimes} \tilde{\wp}_\varrho)(\tilde{\wp}_1^* \tilde{\otimes} \dots \tilde{\otimes} \tilde{\wp}_\varrho^*)\right]^\kappa (\tilde{\wp}_1^\kappa \tilde{\otimes} \dots \tilde{\otimes} \tilde{\wp}_\varrho^\kappa)\ (\tilde{\wp}_1{}^* \tilde{\otimes} \dots \tilde{\otimes} \tilde{\wp}_\varrho^*)$$

$$= (\tilde{\wp}_1 \tilde{\wp}_1^*)^\kappa (\tilde{\wp}_1^\kappa + \tilde{\wp}_1^*) \tilde{\otimes} \dots \tilde{\otimes} (\tilde{\wp}_\varrho \tilde{\wp}_\varrho^*)^\kappa (\tilde{\wp}_\varrho^\kappa + \tilde{\wp}_\varrho^*)$$

$$= (\tilde{\wp}_1^\kappa + \tilde{\wp}_1^*)(\tilde{\wp}_1 \tilde{\wp}_1^*)^\kappa \tilde{\otimes} \dots \tilde{\otimes} (\tilde{\wp}_\varrho^\kappa + \tilde{\wp}_\varrho^*)(\tilde{\wp}_\varrho \tilde{\wp}_\varrho^*)^\kappa$$

$$= \left[(\tilde{\wp}_1 \tilde{\otimes} \dots \tilde{\otimes} \tilde{\wp}_\varrho)^\kappa + (\tilde{\wp}_1 \tilde{\otimes} \dots \tilde{\otimes} \tilde{\wp}_\varrho)^*\right] \left[(\tilde{\wp}_1 \tilde{\otimes} \dots \tilde{\otimes} \tilde{\wp}_\varrho)(\tilde{\wp}_1 \tilde{\otimes} \dots \tilde{\otimes} \tilde{\wp}_\varrho)^*\right]^\kappa$$

This lead to get the tenser product, $\tilde{\wp}_1 \tilde{\otimes} \dots \tilde{\otimes} \tilde{\wp}_\varrho$ is fuzzy soft $fS\kappa QN$ −operator.

4. Conclusions

The generalized quasinormal type operator linked to two-parameters, this is an extension of the most recent studies on fuzzy soft quasinormal operators posed on FS-H space, In accordance with Equality Theory. After that, algebraic and analytical qualities are suggested and examined. The most important qualities are met by providing the necessary circumstances for this operator. It is suggested that this operator be used on soft fuzzy soft Hilbert (SFs-H) space in further research. These quasinormal type operators are also taken into consideration under specific formulations. Additionally, it is possible to examine the benefits of these operators with particular instances, such differential equations.

References

[1] S. D. Mohsen, "Novel results of K quasi (λ - M)-hyponormal operator," *Iraqi J. Sci.*, vol. 65, no. 1, pp. 374–380 (2024).

[2] S. D. Mohsen, "New generalizations for M-hyponormal operators," *Iraqi J. Sci.*, vol. 64, no. 12, pp. 6477–6482 (2023).

[3] L. A. Zadeh, "Fuzzy sets," *Inf. Control*, vol. 8, no. 3, pp. 338–353 (1965).

[4] D. Molodtsov, "Soft set theory—first results," *Comput. Math. Appl.*, vol. 37, no. 4–5, pp. 19–31 (1999).

[5] S. Das and S. Samanta, "Soft metric," *Ann. Fuzzy Math. Inform.*, vol. 6, no. 1, pp. 77–94 (2013).

[6] T. Beaula and M. M. Priyanga, "A new notion for fuzzy soft normed linear space," *Int. J. Fuzzy Math. Arch.*, vol. 9, no. 1, pp. 81–90 (2015).

[7] N. Faried, M. S. S. Ali, and H. H. Sakr, "Fuzzy soft inner product spaces," *Appl. Math. Inform. Sci.,* vol. 14, no. 4, pp. 709–720 (2020).

[8] S. Das and S. K. Samanta, "On soft inner product spaces," *Ann. Fuzzy Math. Inform.,* vol. 6, no. 1, pp. 151–170 (2013).

[9] T. Beaula and C. Gunaseeli, "On fuzzy soft metric spaces," *Malaya J. Math.,* vol. 2, pp. 197–202 (2014).

[10] N. Faried, M. S. S. Ali, and H. H. Sakr, "Fuzzy soft Hilbert spaces," *Math. Stat.*, vol. 8, no. 3 (2020).

[11] N. Faried, M. S. S. Ali, and H. H. Sakr, "On fuzzy soft linear operators in fuzzy soft Hilbert spaces," *Abstr. Appl. Anal.,* vol. 2020, Article ID 2020 (2020).

[12] S. D. Mohsen, "Some generalizations of fuzzy soft (k̂^*–Â)-quasinormal operators in fuzzy soft Hilbert spaces," *J. Interdiscip. Math.,* vol. 26, no. 6, pp. 1133–1143 (2023).

[13] S. D. Mohsen and H. K. Mousa, "Another results related of fuzzy soft quasi normal operator in fuzzy soft Hilbert space," *J. Phys.: Conf. Ser.,* vol. 2322, pp. 1–7 (2022).

[14] N. Faried, M. S. S. Ali, and H. H. Sakr, "Fuzzy soft Hermitian operators," *Adv. Math. Sci. J.,* vol. 9, no. 1, pp. 73–82 (2020).

[15] N. M. Atheab and S. D. Mohsen, "Some results on (n, k)-hyponormal operators," *J. Phys.: Conf. Ser.,* vol. 1591, no. 1, pp. 1–8 (2020).

[16] S. D. Mohsen, "Solvability of (λ, μ)-commuting operator equations for bounded generalization of hyponormal operators," *Iraqi J. Sci.,* vol. 63, no. 9, pp. 3854–3860 (2022).

[17] S. D. Mohsen, "On (k, m)-n-paranormal operators," *Iraqi J. Sci.,* vol. 64, no. 6, pp. 3987–3992 (2023).

Received December, 2024