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Study certain subclasses of holomorphic functions for quasi-subordination

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Abstract

Numerous scholars have shown remarkable interest in geometric function theory, a thriving area of study with roots in complex analysis. The investigation of the geometric properties of analytic (in addition to univalent) functions, with numerous applications in numerous branches of mathematics, mathematical physics, and engineering, is largely responsible for this. Applications in engineering designs, fluid flows in physics, special functions, orthogonal polynomials, and conformal mappings are noteworthy. The article investigates a subclass of quasi-subordination-related analytic and univalent functions established in the domain of logistics sigmoid functions on a unit disk. Using quasi-subordination, the Fekete-Szego functional $|e_3 - \mu e_2^2|$ can be obtained for functions of the classes $S_q^*(\tau, \gamma, \mho)$, and $\mathcal{G}_q(\vartheta, \alpha, \mho)$. We also briefly discuss the subordination results and the improved results for the related classes involving majorization. Some renowned subclasses of analytic and univalent functions, including the classes of starlike functions and convex functions, are included in the definition of the novel class. The description of the class

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involves the application of two fundamental mathematical concepts: quasi-subordination and Taylor's series.

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1. Introduction

Let $\mathcal{A}$ be the class of functions with type:

$$\zeta(t) = t + \sum_{\iota=2}^{\infty} e_\iota t^\iota, \quad t \in D^* = \{t: t \in \mathbb{C}, 0 < |t| < 1\}, \ e_\iota \geq 0. \tag{1}$$

In D^*, if a Schwarz function $\iota(t)$ is present which is analytic with $\iota(0) = 0$ and $|\iota(t)| < 1$, so that $\zeta(t) = h\big(\iota(t)\big)(t \in D^*)$, then we declare that $\zeta(t)$ is subordinate to $h(t)$ in D^* as well as composed $\zeta(t) \prec h(t)$, a functions $\zeta \in S^*(\mathfrak{V})$ is Minda [1],

$$S^*(\mathfrak{V}) = \left\{\frac{t\zeta'(t)}{\zeta(t)} \prec \mathfrak{V}(t)\right\}, \zeta \in \mathcal{A}.$$

The class $C(\mathfrak{V})$ represents the group of functions $\zeta \in \mathcal{A}$ that

$$C(\mathfrak{V}) = \left\{1 + \frac{t\zeta''(t)}{\zeta'(t)} \prec \mathfrak{V}(t)\right\},$$

where $\mathfrak{V}(t)$ is a holomorphic function that maps D^* onto a region that is symmetrical and starlike with respect to 1, satisfying $\mathfrak{V}(0) = 1$ and $\mathfrak{V}'(0) > 0$.

As special cases, the classes $S^*(\mathfrak{V})$ and $C(\mathfrak{V})$ contain a number of renowned subclasses of starlike and convex functions.

Quintuple subordination was first introduced by Robertson [2, 3]. When there are two holomorphic functions, $\zeta(t)$ as well as $\Delta(t)$ with $|\Delta(t)| < 1$, and $\zeta(t) = \Delta(t)h\big(\iota(t)\big)$, we declare that the function $\zeta(t)$ is quasi-subordination to the function $h(t)$, composed as $\zeta(t) \prec_q h(t)$.

If two analytic functions $\zeta(t)$ and $\Delta(t)$ exist, the result is that $\zeta(t) = \Delta(t)h(k(t)$, and $|\Delta(t)| < 1$. $\zeta(t) = h(k(t)$ when $\Delta(t) = 1$, meaning that $\zeta(t) \prec h(t)$ in D^*. Furthermore, if $k(t) = t$, then $\zeta(t) = \Delta(t)h(t)$, and $\zeta(t)$ is said to be majorized by $h(t)$ in D^*, where it is expressed as $\zeta(t) \ll h(t)$ (refer to Goyal and Goswami [4]). Thus, it is evident that both majorization and quasi-subordination are generalizations of subordination.

Definition 1.1: The multiplier operator $M_{\sigma,m}^{v}: \mathcal{A} \to \mathcal{A}$ Currently investigated in [5- 8] has been defined for $v \in \mathbb{Z}, m > -1, \sigma > 0$, and can be obtained by

$$M_{\sigma,m}^{v}\zeta(t) = t + \sum_{\iota=2}^{\infty}\left(1 + \frac{\sigma(\iota-1)}{m+1}\right)^{v} e_\iota t^\iota. \tag{2}$$

Definition 1.2: If function $\zeta \in \mathcal{A}$ satiating the quasi-subordination constitutes the class $S_q^*(\tau, \gamma, \mathfrak{V})$ as well as the class $\mathcal{J}_q(\vartheta, \alpha, \mathfrak{V})$ respectively:

$$\left[\frac{t(M_{\sigma,m}^{v}\zeta)'(t)}{(M_{\sigma,m}^{v}\zeta)(t)}\right]^{\alpha}\left(\frac{t(M_{\sigma,m}^{v}\zeta)'(t)}{(M_{\sigma,m}^{v}\zeta)(t)}+\tau\frac{t^{2}(M_{\sigma,m}^{v}\zeta)''(t)}{(M_{\sigma,m}^{v}\zeta)(t)}\right)-1\prec_{q}\mho(t)-1,$$

$$\left[1+\frac{t(M_{\sigma,m}^{v}\zeta)''(t)}{(M_{\sigma,m}^{v}\zeta)'(t)}\right]^{\alpha}\left[\vartheta\left(\frac{t(M_{\sigma,m}^{v}\zeta)'(t)}{(M_{\sigma,m}^{v}\zeta)(t)}\right)+\left(\frac{t(M_{\sigma,m}^{v}\zeta)''(t)}{(M_{\sigma,m}^{v}\zeta)'(t)}\right)\right]-1\prec_{q}\mho(t)-1,$$

where $0\leq\gamma\leq1;0\leq\tau\leq1,0\leq\alpha\leq1;0\leq\vartheta$.

Lemma 1.1: [9] *If f be a holomorphic function in $D^{*}$and $\Delta(0)=0,|\Delta(t)|<1,\iota(t)=\iota_{1}t+\iota_{2}t^{2}+\cdots$. Then, $|\iota_{2}-f\iota_{1}^{2}|\leq max\{1;|f|\}$, the outcome is acute for functions by $\iota(t)=t$ or $\iota(t)=t^{2}$.*

2. Main results

In this part, we investigate some Fekete-Szego related inequalities for the classes $\mathcal{J}_{q}(\vartheta,\alpha,\mho)$ and $S_{q}^{*}(\tau,\gamma,\mho)$. Allow $\Delta(t)=d_{0}+d_{1}t+d_{2}t^{2}+\cdots,P\in\mathbb{R}$ and $P>0$, as well as let $\mho(t)=1+P_{1}t+P_{2}t^{2}+\cdots$.

Theorem 2.1: *If $\zeta(t)$ in the class $S_{q}^{*}(\tau,\gamma,\mho)$, then*

$$|e_{2}|\leq\frac{P_{1}(m+1)^{v}}{4(2\tau+\gamma)(m+\sigma)^{v}},$$

$$|e_{3}-\mu e_{2}^{2}|\leq\frac{P_{1}(m+1)^{v}}{8(3\tau+\gamma)(m+2\sigma)^{v}}$$

$$\times max\left\{1,\left|\frac{1}{4}\left(\frac{2\mu P_{1}(3\tau+\gamma)(m+1)^{v}}{(2\tau+\gamma)^{2}(m+\sigma)^{v}}+\left(\begin{matrix}\frac{P_{1}(2\tau-2\tau\gamma)}{(2\tau+\gamma)^{2}}\\+\frac{P_{1}}{(2\tau+\gamma)^{2}}\end{matrix}\right)\right)\right|\right\}. \tag{3}$$

Proof: Given holomorphic functions $\Delta(t),\iota(t)$, and $\zeta(t)\in S_{q}^{*}(\tau,\gamma,\mho)$, let $|\Delta(t)|<1,\iota(t)=0$, and $\iota(t)<1$, so that

$$\left[\frac{t(M_{\sigma,m}^{v}\zeta)'(t)}{(M_{\sigma,m}^{v}\zeta)(t)}\right]^{\alpha}\left(\frac{t(M_{\sigma,m}^{v}\zeta)'(t)}{(M_{\sigma,m}^{v}\zeta)(t)}+\tau\frac{t^{2}(M_{\sigma,m}^{v}\zeta)''(t)}{(M_{\sigma,m}^{v}\zeta)(t)}\right)-1$$

$$=\Delta(t)\left(\mho\left(\frac{\Phi(\iota(t)-1)}{\Phi(\iota(t)+1)}\right)-1\right). \tag{4}$$

By using the first and second derivatives of (4) in the formula above left side, we obtain

$$\left[\frac{t(M_{\sigma,m}^{v}\zeta)'(t)}{(M_{\sigma,m}^{v}\zeta)(t)}\right]^{\alpha}\left(\frac{t(M_{\sigma,m}^{v}\zeta)'(t)}{(M_{\sigma,m}^{v}\zeta)(t)}+\tau\frac{t^{2}(M_{\sigma,m}^{v}\zeta)''(t)}{(M_{\sigma,m}^{v}\zeta)(t)}\right)-1$$

$$=\left(\begin{matrix}\gamma\left(\frac{m+\sigma}{m+1}\right)^{v}e_{2}t+\gamma\left(2\left(\frac{m+2\sigma}{m+1}\right)^{v}e_{3}-\left(\frac{m+\sigma}{m+1}\right)^{2v}e_{2}^{2}\right)t^{2}\\+\frac{\gamma(\gamma-1)}{2!}\left(\frac{m+\sigma}{m+1}\right)^{2v}e_{2}^{2}t^{2}\end{matrix}\right)$$

$$\times\left(\begin{matrix}\left(\frac{m+\sigma}{m+1}\right)^{v}e_{2}t+\left(2\left(\frac{m+2\sigma}{m+1}\right)^{v}e_{3}-\left(\frac{m+\sigma}{m+1}\right)^{2v}e_{2}^{2}\right)t^{2}\\+2\tau\left(\frac{m+\sigma}{m+1}\right)^{v}e_{2}t+\tau\left(6\left(\frac{m+2\sigma}{m+1}\right)^{v}e_{3}-2\left(\frac{m+\sigma}{m+1}\right)^{2v}e_{2}^{2}\right)t^{2}\end{matrix}\right)-1$$

$$= \begin{pmatrix} (2\tau+\gamma)\left(\frac{m+\sigma}{m+1}\right)^{v} e_2 w + 2(3\tau+\gamma)\left(\frac{m+2\sigma]}{m+1}\right)^{v} e_3 w^2 \\ +\left(-2\tau+2\gamma\tau+\frac{\gamma(\gamma-1)}{2!}\right)\left(\frac{m+\sigma}{m+1}\right)^{2v} e_2^2 w^2 \end{pmatrix}$$

$$= \frac{d_0 \iota_1 P_1}{4} t + \left(\frac{d_0 \iota_1^2 (P_2 - P_1)}{16} + \frac{d_0 \iota_2 P_1}{4} + \frac{d_1 \iota_1 P_1}{4}\right) t^2 + \cdots. \quad (5)$$

By contrasting the like coefficient of t and t^2 in equation (5), we obtain

$$e_2 = \frac{d_0 \iota_1 P_1 (m+1)^{v}}{4(2\tau+\gamma)(m+\sigma)^{v}},$$

$$e_3 = \frac{P_1 (m+1)^{v}}{8(3\tau+\gamma)(m+2\sigma)^{v}} \cdot \left[\iota_2 d_0 + \iota_1 d_1 + \frac{1}{4}\iota_1^2 \left(\frac{{d_0}^2 P_1 (2\tau - 2\tau\gamma)}{(2\tau+\gamma)^2} - \frac{{d_0}^2 P_1}{(2\tau+\gamma)^2}\right)\right].$$

Consequently,

$$e_3 - \mu e_2^2 = \frac{P_1 (m+1)^{v}}{8(3\tau+\gamma)(m+2\sigma)^{v}} \times$$

$$\left[\iota_2 d_0 + \iota_1 d_1 - \frac{1}{4}\iota_1^2 \begin{pmatrix} \frac{2\mu {d_0}^2 P_1 (3\tau+\gamma)(m+1)^{v}}{(2\tau+\gamma)^2 (m+\sigma)^{v}} - \begin{pmatrix} \frac{{d_0}^2 P_1 (2\tau - 2\tau\gamma)}{(2\tau+\gamma)^2} \\ -\frac{{d_0}^2 P_1}{(2\tau+\gamma)^2} \end{pmatrix} \end{pmatrix}\right]. \quad (6)$$

As a result, using [10,11], we obtain $|d_0| \leq 1$ and $d_1 = (1 - d_0^2)\Psi$ for some $\Psi(|\Psi| \leq 1)$, when we replace the value of d_1 in equation (6), we have

$$e_3 - \mu e_2^2 = \frac{P_1 (m+1)^{v}}{8(3\tau+\gamma)(m+2\sigma)^{v}} \times \iota_2 d_0 + \iota_1 \Psi + \frac{(P_2 - P_1) d_0}{4P_1}\iota_1^2$$

$$\left[-\left(\frac{1}{4}\begin{pmatrix} \frac{2\mu {d_0}^2 P_1 (3\tau+\gamma)(m+1)^{v}}{(2\tau+\gamma)^2 (m+\sigma)^{v}} - \frac{P_1 (2\tau - 2\tau\gamma)}{(2\tau+\gamma)^2} \\ + \frac{P_1}{(2\tau+\gamma)^2} \end{pmatrix}\iota_1^2 + \iota_1 \Psi\right) {d_0}^2\right]. \quad (7)$$

In this case $d_0 = 0$ in equation (7), we have

$$|e_3 - \mu e_2^2| \leq \frac{P_1 (m+1)^{v}}{8(3\tau+\gamma)(m+2\sigma)^{v}}.$$

Alternatively, assuming $d_0 \neq 0$ in (7), let's say that

$$\Gamma(d_0) = \iota_1 \Psi + \iota_2 d_0 + \frac{(P_2 - P_1) d_0}{4P_1}\iota_1^2$$

$$-\left(\frac{1}{4}\begin{pmatrix} \frac{2\mu {d_0}^2 P_1 (3\tau+\gamma)(m+1)^{v}}{(2\tau+\gamma)^2 (m+\sigma)^{v}} - \frac{P_1 (2\tau - 2\tau\gamma)}{(2\tau+\gamma)^2} \\ + \frac{P_1}{(1+2\tau+\gamma)^2} \end{pmatrix}\iota_1^2 + \iota_1 \Psi\right) {d_0}^2.$$

$$|e_3 - \mu e_2^2| \leq \left|\frac{P_1 (m+1)^{v}}{8(3\tau+\gamma)(m+2\sigma)^{v}}\right| \times$$

$$\left[\left|\iota_2 - \frac{1}{4}\iota_1^2 \begin{pmatrix} \frac{2\mu P_1 (3\tau+\gamma)(m+1)^{v}}{(2\tau+\gamma)^2 (m+\sigma)^{v}} - \begin{pmatrix} \frac{(P_2 - P_1)}{P_1} + \frac{P_1 (2\tau - 2\tau\gamma)}{(2\tau+\gamma)^2} \\ -\frac{P_1}{(2\tau+\gamma)^2} \end{pmatrix} \end{pmatrix}\right|\right].$$

So, using Lemma 1.1, we get

$$|e_3 - \mu e_2^2| \leq \frac{P_1 (m+1)^{v}}{8(3\tau+\gamma)(m+2\sigma)^{v}} \times$$

$$max\left\{1,\left|\frac{1}{4}\left(\frac{2\mu P_1(3\tau+\gamma)(m+1)^{\upsilon}}{(2\tau+\gamma)^2(m+\sigma)^{\upsilon}}+\left(\begin{matrix}-\frac{(P_2-P_1)}{P_1}-\frac{P_1(2\tau-2\tau\gamma)}{(2\tau+\gamma)^2}\\+\frac{P_1}{(2\tau+\gamma)^2}\end{matrix}\right)\right)\right|\right\}.$$

Corollary 2.1: [12] *Let* $\zeta(t)$ *belongs to the class* $S_q^*(\tau,\mho)$ *and* $\gamma=0,\upsilon=0$, *then*

$$|e_2|\leq\frac{P_1}{4(1+2\tau)}$$

$$|e_3-\mu e_2^2|\leq\frac{P_1}{8(1+3\tau)}.max\left\{1,\left|\frac{1}{4}\left(\frac{2\mu P_1(1+3\tau)}{(1+2\tau)^2}-\frac{(P_2-P_1)}{P_1}-\frac{P_1}{1+2\tau}\right)\right|\right\}.$$

Theorem 2.2: *Let* $\zeta(t)$ *belong to the class* $\mathcal{G}_q(\vartheta,\alpha,\mho)$, *then*

$$|e_2|\leq\frac{P_1(m+1)^{\upsilon}}{4(1+\vartheta+2\alpha)(m+\sigma+1)^{\upsilon}},$$

and, for any kind of complex number μ,

$$|e_3-\mu e_2^2|\leq\frac{P_1(m+1)^{\upsilon}}{8(2\vartheta+3\alpha)(m+2\sigma)^{\upsilon}}\times$$

$$max\left\{1,\left|\frac{1}{4}\left(\frac{2\mu P_1(2\vartheta+3\alpha)(m+1)^{\upsilon}}{(\vartheta+2\alpha)^2(m+2\sigma)^{\upsilon}}-\left(\begin{matrix}\frac{(P_2-P_1)}{P_1}+\frac{P_1(3\vartheta-2\alpha\vartheta)}{(1+\vartheta+2\alpha)^2}\\-\frac{\gamma(\gamma-1)}{2!}\frac{P_1}{(\vartheta+2\alpha)^2}\end{matrix}\right)\right)\right|\right\}. \tag{12}$$

Proof: If $\zeta(t)\in\mathcal{J}_q(\vartheta,\alpha,\mho)$, then analytic functions $\Delta(t)$ and $\iota(t)$ exist, with $|\Delta(t)|<1,\iota(t)=0$, as well as $\iota(t)<1$, and we obtain by Lemma 1.1

$$|e_3-\mu e_2^2|\leq\frac{P_1(m+1)^{\upsilon}}{8(2\vartheta+3\alpha)(m+2\sigma)^{\upsilon}}\times$$

$$max\left\{1,\left|\frac{1}{4}\left(\frac{2\mu P_1(2\vartheta+3\alpha)(m+1)^{\upsilon}}{(\vartheta+2\alpha)^2(m+2\sigma)^{\upsilon}}-\left(\begin{matrix}\frac{(P_2-P_1)}{P_1}+\frac{P_1(3\vartheta-2\alpha\vartheta)}{(1+\vartheta+2\alpha)^2}\\-\frac{P_1}{(1+\vartheta+2\alpha)^2}\end{matrix}\right)\right)\right|\right\}.$$

Corollary 2.2: [12,13] *If* $\zeta(t)$ *in the class* $\mathcal{J}_q(\vartheta,\mho)$ *and* $\alpha=0,\upsilon=0$, *then*

$$|e_2|\leq\frac{P_1}{4(1+\vartheta)},$$

$$|e_3-\mu e_2^2|\leq\frac{P_1}{8(1+2\vartheta)}max.\left\{1,\left|\frac{1}{4}\left(\frac{2\mu P_1(1+2\vartheta)}{(1+\vartheta)^2}-\frac{P_1(1+3\vartheta)}{(1+\vartheta)^2}\right)\right|\right\}.$$

3. Conclusions

This paper investigated a subclass of analytic-univalent functions specified by the unit disk D^*. The starlike function and convex function classes were among the most popular subclasses of analytic-univalent functions that this novel class generalized. Taylor's series and quasi-subordination were used in the class definition. It is well-established that the concepts of majorization and subordination can be applied more broadly to quasi-subordination. However, estimations for some co-efficient bounds and the Fekete-Szegö solution to the problem were among the properties of the function that were investigated.

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