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Some geometrical characteristics of a class involving analytic functions of third derivative

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Abstract

This paper introduces univalent functions defined on a unit disc $Ų = \{ꙁ \in \mathbb{C} : |ꙁ| < 1\}$ of analytic functions. Coefficient estimates are present, which belong to the class $Ş^{*}Ç(ς,σ,δ)$ and $ŢŞ^{*}Ç(ς,σ,δ)$, and other properties like distortion are derived for the class $Ş^{*}Ç(ς,σ,δ)$. Additionally, extreme points for the class $Ş^{*}Ç(ς,σ,δ)$ are also given.

Subject Classification: *30C45.*

Keywords: *Univalent functions, Negative coefficients, Starlike functions, Convex functions, Convolution.*

1. Introduction

Let us consider the class of analytic functions $Ƒ$ denoted by $\bar{A}$ on the unit disk $Ų = \{ ꙁ \in \mathbb{C}: |ꙁ| < 1\}$, normalized by:

$$Ƒ(0) = 0 = Ƒ_{ꙁ}(0) - 1, \text{ such that}$$

$$Ƒ(ꙁ) = ꙁ + a_2ꙁ^2 + a_3ꙁ^3 + \cdots + a_kꙁ^k + \cdots = ꙁ + \sum_{k=2}^{\infty} a_kꙁ^k, \ a_k \in \mathbb{C} \quad (1)$$

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Let Ş denote the family of all univalent functions in Ā.

Let Ţ represent the subclass of Ā in the following form:

$$F(z) = z - a_2 z^2 - a_3 z^3 - \cdots - a_k z^k - \cdots = z - \sum_{k=2}^{\infty} a_k z^k, \; a_k \geq 0 \quad (2)$$

Several authors [1], [2] and [3] have investigated classes similar to these and described a wide variety of properties for analytic functions.

The classes $Ş^*(\varsigma)$ and $Ç(\varsigma)$ are two subclasses of the univalent functions in class Ş that have been researched extensively. These classes describe starlike and convex functions of order ς $(0 \leq \varsigma < 1)$ in the open unit disk Ų. Both of these subclasses are considered to be significant and important. More information can be found in [4], [5], [6] and [7] as well as [8], according to the definition.

$$Ş^*(\varsigma) = \left\{F \in Ş : \operatorname{Re}\left(\frac{zF'(z)}{F(z)}\right) > \varsigma, z \in Ų\right\}, \quad 0 \leq \varsigma < 1, \text{ and}$$

$$Ç(\varsigma) = \left\{F \in Ş : \operatorname{Re}\left(1 + \frac{zF''(z)}{F'(z)}\right) > \varsigma, z \in Ų\right\}, \quad 0 \leq \varsigma < 1.$$

Now, we are going to use $ŢŞ^*(\varsigma) = Ş^*(\varsigma) \cap Ţ$ and $ŢÇ(\varsigma) = Ç(\varsigma) \cap Ţ$.

Definition 1.1: The function classes $Ş^*(\varsigma)$ and $Ç(\varsigma)$, have interesting generalizations to classes $Ş^*(\varsigma, \sigma)$ and $Ç(\varsigma, \sigma)$, defined as follows:

$$Ş^*(\varsigma, \sigma) = \left\{F \in Ş : \operatorname{Re}\left(\frac{\sigma z^2 F''(z) + zF'(z)}{\sigma^2 z^2 F''(z) + \sigma(2-\sigma) zF'(z) + (1-\sigma)^2 F(z)}\right) > \varsigma, z \in Ų\right\}, (0 \leq \sigma < 1)$$

and $Ç(\varsigma, \sigma) = \left\{F \in Ş : \operatorname{Re}\left(\frac{\sigma z^3 F'''(z) + (2\sigma+1) z^2 F''(z) + zF'(z)}{\sigma^2 z^3 F'''(z) + \sigma(1+\sigma) z^2 F''(z) + ((\sigma-1)^2 + \sigma(2-\sigma)) zF'(z)}\right) > \varsigma, z \in Ų\right\}$, $(0 \leq \sigma < 1)$. Consequently, Now, we are going to symbolize

$$ŢŞ^*(\varsigma, \sigma) = Ş^*(\varsigma, \sigma) \cap Ţ \text{ and } ŢÇ(\varsigma, \sigma) = Ç(\varsigma, \sigma) \cap Ţ.$$

Such classes $ŢŞ^*(\varsigma, \sigma)$ and $ŢÇ(\varsigma, \sigma)$ were extensively investigated by Altintaş and Owa [9]

Based on the preceding study, we define unification $Ş^*(\varsigma, \sigma)$ and $Ç(\varsigma, \sigma)$ as follows:

Definition 1.2: Assume that F is a function of form (1); if F it satisfies the condition

$$\operatorname{Re}\left(\frac{\delta\sigma z^3 F'''(z) + (\sigma + \delta\sigma + \delta) z^2 F''(z) + zF'(z)}{\delta\sigma^2 z^3 F'''(z) + (2\delta\sigma + \sigma^2) z^2 F''(z) + (\sigma(2-\sigma) + \delta(\sigma-1)^2) zF'(z) + (1-\sigma)^2(1-\delta) F(z)}\right) > \varsigma$$

For $z \in Ų$, $0 \leq \varsigma < 1$; $0 \leq \sigma < 1$, then it is contained in $Ş^*Ç(\varsigma, \sigma, \delta)$. We will denote $ŢŞ^*Ç(\varsigma, \sigma, \delta) = Ş^*Ç(\varsigma, \sigma, \delta) \cap Ţ$. Specifically, we have:

$$Ş^*Ç(\varsigma, \sigma, 0) = Ş^*(\varsigma, \sigma), \; Ş^*Ç(\varsigma, \sigma, 1) = Ç(\varsigma, \sigma), \; Ş^*Ç(\varsigma, 0, 0) = Ş^*(\varsigma)$$

$$Ş^*Ç(\varsigma, 0, 1) = Ç(\varsigma), \; ŢŞ^*Ç(\varsigma, \sigma, 0) = ŢŞ^*(\varsigma, \sigma), \; ŢŞ^*Ç(\varsigma, \sigma, 1) = ŢÇ(\varsigma, \sigma)$$

$ŞŢ^*Ç(\varsigma, 0, 0) = ŢŞ^*(\varsigma)$, $ŢŞ^*Ç(\varsigma, 0, 1) = ŢÇ(\varsigma)$. Specifically, we observe the following: $Ş^*Ç(\varsigma, 0, 0) = Ş^*(\varsigma)$ [10], $Ş^*Ç(\varsigma, 0, 1) = Ç(\varsigma)$ [10],

The objective of this study is to examine the unique characteristics that distinguish the classes $Ş^*Ç(\varsigma, \sigma, \delta)$ and $ŢŞ^*Ç(\varsigma, \sigma, \delta)$.

In this part, we will investigate a few features that are distinctive of the classes $Ş^*Ç(ς, σ, δ)$ and $ŢŞ^*Ç(ς, σ, δ)$.

2. Sufficient conditions for the class $Ş^*Ç(ς, \boldsymbol{σ}, \boldsymbol{δ})$and the class $ŢŞ^*Ç(ς, \boldsymbol{σ}, \boldsymbol{δ})$

The first theorem establishes an adequate requirement for the functions in the class $Ş^*Ç(ς, σ, δ)$ which is given by the following:

Theorem 2.1: *Let,* $0 \le ς < 1;\ 0 \le \sigma < 1$ *,* $ƶ \in Ų$ *and* $Ƒ \in Ā$ *, if the following condition*

$$\sum_{ķ=2}^{\infty}\left[ķ(ķ-1)(ķ-2)\left(\delta\sigma\left(1-\sigma(2-ς)\right)\right)+ķ(ķ-1)\left((1-\sigma)(\delta+\sigma)-(1-ς)(2\delta\sigma+\sigma^2\ )\right)+ķ\left(1-\left(\sigma(2-\sigma)+\delta(\sigma-1)^2\ \right)(2-ς)\right)-(1-\sigma)^2\ (1-\delta)(2-ς)\right]\left|a_ķ\right| \le 1-ς \tag{3}$$

is satisfied, then $Ƒ$ belong to the class $Ş^*Ç(ς, σ, δ)$.Regarding the function, the outcome is really sharp.

$$Ƒ_ķ(ƶ) = ƶ + \left[(1-ς)\ /\ \left(ķ(ķ-1)(ķ-2)\left(δσ\left(1-σ(2-ς)\right)\right)+ķ(ķ-1)\left((1-σ)(δ+σ)-(1-ς)(2δσ+σ^2\ )\right)+ķ\left(1-\left(σ(2-σ)+δ(σ-1)^2\ \right)(2-ς)\right)-(1-σ)^2\ (1-δ)(2-ς)\right)\right];\ ƶ \in Ũ,\quad ķ = 2, 3, \ldots \tag{4}$$

Proof: By definition (1.1), a function $Ƒ$ is a member of the class $Ş^*Ç(ς, σ, δ)$, if it satisfies the condition

$$\operatorname{Re}\left(\frac{δσƶ^3Ƒ'''(ƶ)+(σ+δσ+δ)ƶ^2Ƒ''(ƶ)+ƶƑ'(ƶ)}{δσ^2ƶ^3Ƒ'''(ƶ)+(2δσ+σ^2)ƶ^2Ƒ''(ƶ)+((σ(2-σ)+δ(σ-1)^2)ƶƑ'(ƶ)+(1-σ)^2(1-δ)Ƒ(ƶ)}\right) > ς \tag{5}$$

for $ƶ \in Ų, 0 \le ς < 1,\ 0 \le σ < 1$. Also, it is easy to demonstrate that requirement (5) is true if

$$\left|\frac{δσƶ^3Ƒ'''(ƶ)+(σ+δσ+δ)ƶ^2Ƒ''(ƶ)+ƶƑ'(ƶ))}{δσ^2ƶ^3Ƒ'''(ƶ)+(2δσ+σ^2)ƶ^2Ƒ''(ƶ)+((σ(2-σ)+δ(σ-1)^2)ƶƑ'(ƶ)+(1-σ)^2(1-δ)Ƒ(ƶ)}-1\right| \le 1-ς \tag{6}$$

At this point, let us prove that the assumption of the theorem meets the condition. Through basic computation, we obtain

$$\left|\frac{δσƶ^3Ƒ'''(ƶ)+(σ+δσ+δ)ƶ^2Ƒ''(ƶ)+ƶƑ'(ƶ)}{δσ^2ƶ^3Ƒ'''(ƶ)+(2δσ+σ^2)ƶ^2Ƒ''(ƶ)+(σ(2-σ)+δ(σ-1)^2)ƶƑ'(ƶ))+(1-σ)^2(1-δ)Ƒ(ƶ)}-1\right|$$

$$\le \left(\left(\sum_{ķ=2}^{\infty}[ķ(ķ-1)(ķ-2)(1-\sigma)\delta\sigma+ķ(ķ-1)(1-\sigma)(\delta+\sigma)+ķ(1-\sigma(2-\sigma)-\delta(\sigma-1)^2)-(1-\sigma)^2(1-\delta)]\left|a_ķ\right|+[1-σ(2-σ)-δ(σ-1)^2]-(1-σ)^2(1-δ)\right)\Big/\left(\sum_{ķ=2}^{\infty}[ķ(ķ-1)(ķ-2)\delta\sigma^2+ķ(ķ-1)(2\delta\sigma+\sigma^2)+ķ(\sigma(2-\sigma)+\delta(\sigma-1)^2)+(1-\sigma)^2(1-\delta)]\left|a_ķ\right|+σ(2-σ)+δ(σ-1)^2+(1-σ)^2(1-δ)\right)\right)$$

The final formulation of the inequality which is equivalent to the following:

$\sum_{k=2}^{\infty}\big[k(k-1)(k-2)\big(\delta\sigma-\delta\sigma^2(2-\varsigma)\big)+k(k-1)\big((1-\sigma)(\delta+\sigma)-(1-\varsigma)(2\delta\sigma+\sigma^2)\big)+k\big(1-\sigma(2-\sigma)-\delta(\sigma-1)^2-(1-\varsigma)(\sigma(2-\sigma)+\delta(\sigma-1)^2)\big)-(1-\sigma)^2(1-\delta)-(1-\varsigma)(1-\sigma)^2(1-\delta)\big]|a_k|\leq 1-\varsigma$

$\sum_{k=2}^{\infty}\big[k(k-1)(k-2)\big(\delta\sigma-\delta\sigma^2(2-\varsigma)\big)+k(k-1)\big((1-\sigma)(\delta+\sigma)-(1-\varsigma)(2\delta\sigma+\sigma^2)\big)+k\big(1-(\sigma(2-\sigma)+\delta(\sigma-1)^2)(2-\varsigma)\big)-(1-\sigma)^2(1-\delta)(2-\varsigma)\big]|a_k|\leq 1-\varsigma$

The inequality (6) is satisfied if the previous inequality is considered satisfied.

Consequently, proof of theorem (2.1) has been completed.

Now, for the function$F \in TS^*C(\varsigma,\sigma,\delta)$, this theorem establishes the necessary condition.

Theorem 2.2: *Let* $0\leq\varsigma<1$, $0\leq\sigma<1$, $z\in U$ *and* F *given by (2), then* F *is a member of the class* $TS^*C(\varsigma,\sigma,\delta)$ *if and only if the subsequent requirement is met.*

$$\sum_{k=2}^{\infty}\Big[k(k-1)(k-2)\Big(\delta\sigma\big(1-\sigma(2-\varsigma)\big)\Big)+k(k-1)((1-\sigma)(\delta+\sigma)-(1-\varsigma)(2\delta\sigma+\sigma^2))+k(1-(\sigma(2-\sigma)+\delta(\sigma-1)^2)(2-\varsigma))-(1-\sigma)^2(1-\delta)(2-\varsigma)\Big]|a_k|\leq 1-\varsigma \tag{7}$$

is satisfied. Regarding the functions, the outcome is really sharp:

$$F_k(z)=z-\Big((1-\varsigma)\Big/\Big(k(k-1)(k-2)\big(\delta\sigma(1-\sigma(2-\varsigma)\big)+k(k-1)\big((1-\sigma)(\delta+\sigma)-(1-\varsigma)(2\delta\sigma-\sigma^2)\big)+k\big(1-(\sigma(2-\sigma)+\delta(\sigma-1)^2)(2-\varsigma)\big)-(1-\sigma)^2(1-\delta)(2-\varsigma)\Big)\Big)z^k \tag{8}$$

$$z\in U,\quad k=2,3,\ldots.$$

Proof: The proof is very similar to the proof that was established for theorem (2.1) Let $0\leq\varsigma<1$, $0\leq\sigma<1$, $0\leq\delta\leq1$, $F\in TS^*C(\varsigma,\boldsymbol{\sigma},\boldsymbol{\delta})$, $z\in U$ such that,

$$\mathrm{Re}\left(\frac{\delta\sigma z^3F'''(z)+(\sigma+\delta\sigma+\delta)z^2F''(z)+zF'(z)}{\delta\sigma^2z^3F'''(z)+(2\delta\sigma+\sigma^2)z^2F''(z)+(\sigma(2-\sigma)+\delta(\sigma-1)^2)zF'(z)+(1-\sigma)^2(1-\delta)F(z)}\right)>\varsigma\quad ,z\in U$$

From basic calculations, it can write as

$$=\Big|-k(k-1)(k-2)(\delta\sigma-\delta\sigma^2)\sum_{k=2}^{\infty}a_kz-k(k-1)(\sigma-\delta\sigma+\delta-\sigma^2)\sum_{k=2}^{\infty}a_kz^k+(1-\sigma(2-\sigma)-\delta(\sigma-1)^2)\big(z-k\sum_{k=2}^{\infty}a_kz^k\big)-(\sigma-1)^2(1-\delta)(z-\sum_{k=2}^{\infty}a_kz^k)\Big|\Big/\Big|-k(k-1)(k-2)\delta\sigma^2\sum_{k=2}^{\infty}a_kz^k-k(k-1)(2\delta\sigma+\sigma^2)\sum_{k=2}^{\infty}a_kz^k+(\sigma(2-\sigma)+\delta(\sigma-1)^2)\big(z-k\sum_{k=2}^{\infty}a_kz^k\big)+(\sigma-1)^2(1-\delta)(z-\sum_{k=2}^{\infty}a_kz^k)\Big|$$

The final term within the brackets of the inequality is a real number when z is selected as a real number and also is limited by$(1-\varsigma)$.Which is equivalent to the following:

$\sum_{ķ=2}^{\infty}\left[ķ(ķ-1)(ķ-2)\left(\delta\sigma\left(1-\sigma(2-ς)\right)\right)+(ķ(ķ-1)(1-\sigma)(\delta+\sigma)-(1-ς)(2\delta\sigma+\sigma^2\ ))+ķ(1-(\sigma(2-\sigma)+\delta(\sigma-1)^2\ )(2-ς))-(1-\sigma)^2\ (1-\delta)(2-ς)\right]\left|a_ķ\right|\leq 1-ς$, which same as (8).

Consequently, proof of Theorem (2.1) has been completed.

3. Distortion theorem and Extreme Points for the class $Ş^*Ç(ς,\boldsymbol{\sigma},\boldsymbol{\delta})$

The distortion bounds for functions in $Ş^*Ç(ς,\sigma,\delta)$ are given by the following theorem.

Theorem 3.1: *For* $0\leq ς<1,\ 0\leq\sigma<1, ꙁ\in Ụ$ *.If* $Ғ$ *belong to the class* $\in Ş^*Ç(ς,\sigma,\delta)$ *, then*

$$|ꙁ|-\frac{(1-ς)}{2((1-\sigma)(\delta+\sigma)-(1-ς)(2\delta\sigma-\sigma^2))+2\left(1-(\sigma(2-\sigma)+\delta(\sigma-1)^2)(2-ς)\right)-(1-\sigma)^2(1-\delta)(2-ς)}|ꙁ|^2\leq Ғ(ꙁ)\leq|ꙁ|+\frac{(1-ς)}{2((1-\sigma)(\delta+\sigma)-(1-ς)(2\delta\sigma-\sigma^2))+2\left(1-(\sigma(2-\sigma)+\delta(\sigma-1)^2)(2-ς)\right)-(1-\sigma)^2(1-\delta)(2-ς)}|ꙁ|^2$$

Regarding the functions, the outcome is really sharp:

$$Ғ(ꙁ)=|ꙁ|+\frac{(1-ς)}{2((1-\sigma)(\delta+\sigma)-(1-ς)(2\delta\sigma-\sigma^2))+2\left(1-(\sigma(2-\sigma)+\delta(\sigma-1)^2)(2-ς)\right)-(1-\sigma)^2(1-\delta)(2-ς)}ꙁ^2$$

Proof: $|Ғ(ꙁ)|=\left|ꙁ+\sum_{ķ=2}^{\infty}a_ķ ꙁ^n\right|\leq|ꙁ|+\sum_{ķ=2}^{\infty}a_ķ|ꙁ^n|\leq|ꙁ|+|ꙁ|^2\sum_{ķ=2}^{\infty}a_ķ$

By theorem 2.1, we get:

$$\sum_{ķ=2}^{\infty}a_ķ\leq\ \frac{(1-ς}{2((1-\sigma)(\delta+\sigma)-(1-ς)(2\delta\sigma-\sigma^2))+2\left(1-(\sigma(2-\sigma)+\delta(\sigma-1)^2)(2-ς)\right)-(1-\sigma)^2(1-\delta)(2-ς)}\tag{10}$$

So, $|Ғ(ꙁ)|\leq|ꙁ|+((1-ς)/\left(2\left((1-\sigma)(\delta+\sigma)-(1-ς)(2\delta\sigma-\sigma^2)\right)+2\left(1-(\sigma(2-\sigma)+\delta(\sigma-1)^2)(2-ς)\right)-(1-\sigma)^2(1-\delta)(2-ς)\right))|ꙁ|^2$

Also, $|Ғ(ꙁ)|=\left|ꙁ-\sum_{ķ=2}^{\infty}a_ķ ꙁ^ķ\right|\geq|ꙁ|-\sum_{ķ=2}^{\infty}a_ķ\left|ꙁ^ķ\right|\geq|ꙁ|-|ꙁ|^2\sum_{ķ=2}^{\infty}a_ķ$

Thus, $|Ғ(ꙁ)|\geq|ꙁ|-((1-ς)/\left(2\left((1-\sigma)(\delta+\sigma)-(1-ς)(2\delta\sigma-\sigma^2)\right)+2\left(1-(\sigma(2-\sigma)+\delta(\sigma-1)^2)(2-ς)\right)-(1-\sigma)^2(1-\delta)(2-ς)\right))\ |ꙁ|^2$ finally, it proves theorem (3.1)

4. Conclusion

We have discovered new classes of univalent functions, denoted as $Ş^*Ç(ς,\sigma,\delta)$ and $ŢŞ^*Ç(ς,\sigma,\delta)$, within the open unit disk $Ụ=\{ꙁ\in\mathbb{C}:|ꙁ|<1\}$ using our method. We identified the necessary requirements for these classes. We invite scholars to present new types of analytic functions, including harmonic and multivalent, for future study.

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