Journal of Interdisciplinary Mathematics
ISSN: 0972-0502 (Print), ISSN: 2169-012X (Online)
Vol. 28 (2025), No. 3-A, pp. 781–785
DOI : 10.47974/JIM-1973

Results on metrizable bornological algebra

Zahraa Qais Ghazi *
Z. D. Al-Nafie †
Department of Mathematics
College of Education of Pure Sciences
University of Babylon
Hilla
Iraq

Abstract

This work studied the concept of a bornological algebra using a metric focus. The theory of bornological algebra generalizes from the metric space structures to the bornological set setting. The construction of this set allows us to expand the frame of the metrizability of algebras by developing these algebras using tools of the bornological and metrical language simultaneously. In this work, we improve the notion of metrizable bornological algebra and introduce many essential constructions in the class of bornological algebra.

Subject Classification: *Primary 93A30, Secondary 49K15.*

Keywords: *Bornological spaces, Metrizable spaces, Bornology, Topological algebra, Metric bornology.*

Introduction

The bornology on a set was introduced in 1977 by H. Hogbe-Nlend [4], and he studied some bornological constructions. The theory of bornological sets and spaces has excellent properties and can be expressed as a good instrument for developing more general cases to solve complex and difficult problems in functional analysis, differential topology, and differential geometry. Furthermore, a bornological algebra is an object that mixes both bornological and algebraic structures and to conclude As a result of this research, new vital results were found depending on the metrizability of spaces [5-9] and depending on metrically and biologically defined [2], [3], [10-13]. In this paper, we introduce many new structures on a metrizable bornological algebra using

* *E-mail:* **zahraa.qhazy.pure441@student.uobabylon.edu.iq (Correspondig Author)**
† *E-mail:* **pure.zahir.dobeas@uobabylon.edu.iq**

the metric structures introduced in the work of G, Beer [1]. In algebra as a vector space, the local base term means a local base at 0. A local base of a metrizable bornological algebra *X* is thus a collection of N of neighborhoods of 0 such that every neighborhood of 0 contains a member of N. Since the open sets of *X* are unions of members of N, In this case, a metric ρ must be consistent with X's bornology and translation invariant, meaning that $\rho(x+z, y+z) = \rho(x,y)$ for every $x, y, z \in X$. This is known as metric bornology on X, or ρ-bornology on X. This indicates that X has a countable local base. In this instance, a local base is designed at z by the balls‖ $B_r(z) = \{y \in X, \rho(y,z) < \frac{1}{n}$ with radius $\frac{1}{n}$ and center z. This creates a sufficient necessary condition for metrizability for a bornological algebra X.

1. Preliminaries

A bornology on a non-empty set *X* is a collection Ƀ of subsets of *X* satisfying the following axioms

(a) Ƀ covers *X*,

(b) If $\beta_1, \beta_2, \ldots, \beta_n \in$Ƀ, then $\bigcup_{k=1}^{n} \beta_k \in$Ƀ, $n \in \mathbb{N}$

(c) If $\beta \in$Ƀand $\beta' \in$ β, then $\beta' \in$ Ƀ

A pair (*X*,Ƀ) is said to be a bornological space, if, in addition, the following conditions hold, then (*X*, Ƀ) is called a bornological algebra over a field *K*

(d) If $\beta_1, \beta_2 \in$Ƀ, then $\beta_1 + \beta_2 \in$ Ƀ

(e) If $\alpha \in K$ and $\beta \in$Ƀ, then $\alpha\beta \in$ Ƀ and $\bigcup_{|\alpha| \le 1} \alpha\beta \in$ Ƀ

(f) If $x \in X$ and β∈Ƀ, then $x\beta, \beta x \in$Ƀ. (The multiplication is separately bounded),

Definition 1.1: A subfamily $\mathscr{B}$ of a bornology Ƀ is a base of $\mathscr{B}$ if every element of Ƀ is contained in an element of $\mathscr{B}$ such that $\mathscr{B}$ covers *X* and every finite union of elements of $\mathscr{B}$ is contained in a member of $\mathscr{B}$.

Definition 1.2: A bornological algebra (*X*, Ƀ) is said to be metrizable if the bornology of *X* is compatible with some metric ρ on *X*

2. Main results

Theorem 2.1: *Assume* $(X, \rho,$Ƀ$)$ *be a bornological algebra with bornology* Ƀ *and metrizable. If and only if the collection* Ƀ *has a countable local basis and for every* $\beta \in B$ *there exists a* $\exists\beta_0 \in$ Ƀ *such that* $cl(\beta) \subseteq int(\beta_0)$*, then a bornology* Ƀ *on X is a met-rick bornology.*

Proof: Assume 0,1] ∋ $f_n\left(cl(\beta_n)\right) = 0$. $\forall\, n \in \mathbb{N}$, assume f_n be a continuous function between [0,1] and $f_n\ (X/int(\beta_{n+1})) = 1$ and $\exists\ x \in \beta_{n(x)}$. Assume that $n(x)$ is an integer ∃x∈β_(n(x)) for every $x \in X$. applies to values in the region of x int(beta n(x)+1) that are nil zero. Consequently, f_1+f_2+f_3 … is a continuous and well-defined function on X. It should be noted here that S∈ ∫

always holds, meaning that f is limited on a subset $s \in x \leftrightarrow \beta_n$, i.e., $S \in Ƀ$ As defined by the metric d

$$d(x, y) \coloneqq \rho(x, y) + |f(x) - f(y)|$$

is equal to ρ, a set Sis d-bounded and a compatible metric if and only if f(S) is bounded, that is, S∈ Ƀ The forcing function [1] for the$(X, \rho, Ƀ)$.

Corollary 2.2: *Assume Ƀ Then Ƀ is a metric-bornology ↔ (X,τ,Ƀ) contains a (necessarily unique) continuous real-valued nonnegative function f with forcing property.,τ,Ƀ)such that f is forcing.*

Proof: Assume Ƀ be a metric bornology i.e. Ƀ $=Ƀ_\rho(X)$, then $x \longmapsto \rho(y, x)$is in other words, Whenever $y \in X$ is fixed, a forcing function is used. On the other hand, $\{f^{-1}([0, n]) : n \in \mathbb{N}\}$ is a countable local basis for the bornology if f is a forcing function for Ƀ. We may use $f^{-1}([0, \beta + 1])$ to find a closed interval [0,ʃ] if β∈σ, and H can represent 0 ≤ β < Ƀ as [0,ʃ].

After that, $f^{-1}([0, \beta + 1])$ is open in ʃ by continuity, containing cl(ʃ). Utilize Hu's theorem.

Theorem 2.3: *An uncountable family of metrics $\{\rho_i : i \in I\}$ in the space $(X, \rho, Ƀ)$can be consistent with Ƀ in such a way that $Ƀ_{\rho_i}(X) \neq Ƀ_{\rho_j}(X)$.*

Proof: Assume ρ is a bounded compatible metric.A sequence(xn) in X, in which the elements should not be grouped in pairs except when they are distinct. elements are not to be clustered.$\forall B \leq X$ and closed $\forall n \in \mathbb{N}$ we set $\delta_n = \frac{1}{3}\rho(x_n, x_j) > 0, j \neq n$ Now suppose ϑ = {a2 : |a| = ∞} are the uncountable family of infinite subsets of ℕ, and let ϑ be the equivalence relation over ϑ such that $E \sim F$ if and only if $|E \ \Delta \ F|$ is finite, a finite set of positive integers.he family of balls $\{B_\rho(x_n, \delta_n) : n \in \mathbb{N}$. Every equivalence class is countable, not all the equivalence classes are countable, hence the countably many equivalence classes must be uncountably many. Assume$\{Ei$:Then exactly one of $i \in I\}$ for each equivalence class contains one representative. For each $i \in I$, assume fi:Suppose that $X \rightarrow R$ is a continuous function with $n \in Ei$ mapping xn to n and mapping $X / \cup_{n \in E_i} B_\rho(x_n, \delta_n)$. Choose now a metric ρi on X such that $\rho_i(x, y) \coloneqq \rho(x, y) + |f_i(x) - f_i(y)$.

Whereas n∈E_i\E_(j)}. In the second, f_i is unbounded on {x_n: $f_i$is such that $\{x_n : n \in E_i \backslash E_j\} \rightarrow \{0\}$, whereas $\{x_n : n \in E_i \backslash E_j\}$ does.

Theorem 2.4: *In the space $(X, \rho, Ƀ)$. The following conditions are equivalent:*

(1) *a metric ρ is uniformly equivalent to a metric $d \ni Ƀ_\rho(X) = Ƀ$;*

(2) Ƀ *has a countable local base and $\exists \varepsilon > 0 \ \forall M \in Ƀ, B_d(M, \varepsilon_n) \in Ƀ$;*

(3) $(\mathrm{X}, \tau_\mathrm{d}, Ƀ)$ has a d-uniformly continuous forcing function.

Proof: (1) ⇒ (2) The fact that the bornology is metric implies that it has a countable local basis. According to the uniform equivalence of the metrics, select ε>0 so that $d(x,y) < \varepsilon \to \rho(x,y) < 1$. Assume that $M \in Ƀ = Ƀ_\rho(X)$. For some γ>0, we have $M \subseteq B_d(z,\gamma)$when z∈X is fixed.

It follows that $B_d(B_d(z,\gamma)) \subseteq B_d(z,\gamma+1)$, since $B_d(M,\varepsilon) \subseteq B_d(B_\rho(z,\gamma)$, $\varepsilon) \subseteq B_d(z,\gamma+1)) \in Ƀ$.

(2) ⇒ (3). We may choose a base$\{U: U_n: n \in \mathbb{N}\}$ for the bornology such that $\forall n$, $B_d(U_n,\varepsilon) \subseteq U_{n+1}$ from condition (2). Use $f_n: X \to [0,\varepsilon]$ by $f_n(x) = \min\{d(x,U_n), \varepsilon\}$. to define f_n:X→[0,ε] for all n∈N. It is evident that f_n translates U_n to zero and is nonexpansive. As previously stated, write $f = \sum_{n=1}^{\infty} f_n$, for each $x \in X$, f_n finally disappears on $B_d(x,\varepsilon)$ indicating that f is continuous and finite-valued.

After setting $h = \max\{j(x), j(y)\};\}$, suppose t that $j(x) = h$. Since $d(x,y) < \varepsilon$, because $h - 1 \leq j(y) \leq h$

$|f(x) - f(y) \leq |f_h - 2(x) - f_h - 2(y)| + |f_h - 1(x) - f_h - 1(y)| \leq 2d(x,y).|$ as required. Finally, if $x \in U_n$, then $f(x) \leq (n-1)\varepsilon$, whereas if $x \notin U_n$, then $f(x) \geq (n-1)\varepsilon$.

From these facts, it follows that f is a forcing function for Ƀ.

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Received May, 2024